Lyapunov Abstractions for Inevitability of Hybrid Systems

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Inevitability Property

- **Definition.** A set of states **S** of system **A** is inevitable if every execution starting from arbitrary state reaches S in bounded time

- Examples:
  - Autonomous vehicle reaches destination
  - Routing protocol recovers from failures
  - Traffic control protocol **does not** deadlock

in bounded time

Inevitability of Hybrid Systems

If S is inevitable for each of the individual dynamical subsystems, S may not be inevitable for combined hybrid system

Goal: Design algorithm for verifying inevitability of HA. Given

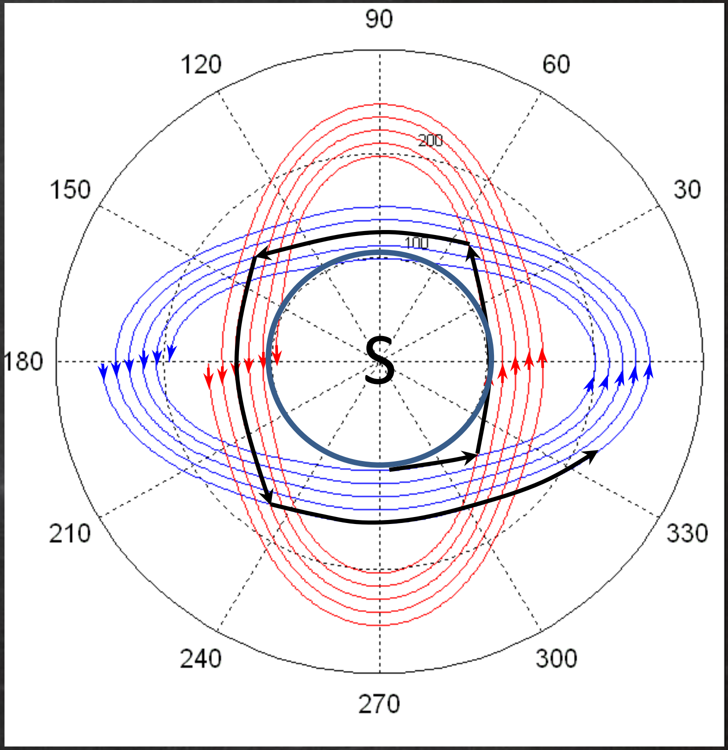
(a) HA *A* and a set S, it should either produce

(b) a proof that S is inevitable OR

(c) a counter-example behavior of A that does not ever reach S

What is a proof?

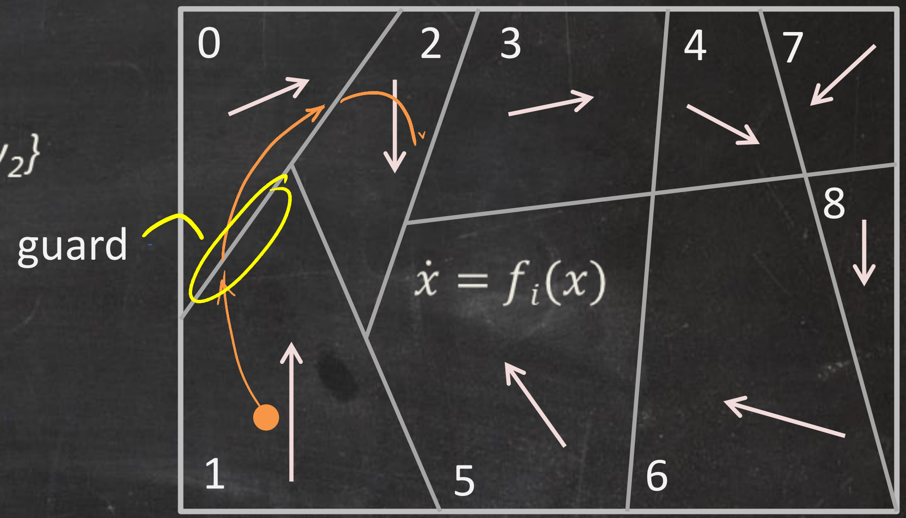
What is a counter-example ?



Hybrid Automata (HA)

- *$A = < X, L, Q_0, D, T>$*
- *L: set of locations*
- *X: set of continuous variables $\{x_1, x_2, v_1, v_2\}$*
- *Q: state space = $\mathbb{R}^4$ x L*

- *$D \subseteq Q \times Q$ discrete transitions*
- *T: trajectories each $\tau \in T$, $\tau : [0, t] \longrightarrow Q$*
- *over which continuous variables flow according to $\dot{x} = f_i(x)$*
  - *Rectangular HA: $\dot{x} \in [a_i, b_i]$*
  - *Linear HA: : $\dot{x} = A_i x + b_i$*
- An *execution* of **A** is a sequence $\tau_0, \tau_1, \tau_2, \ldots$
- **Assume A** is ***non-blocking***, *i.e,* if time diverges along **every** execution



Outline

- Background ✓
- Hybrid Step Relation
- Well-Foundedness and Inevitability
- Relational Abstractions
- Conclusions

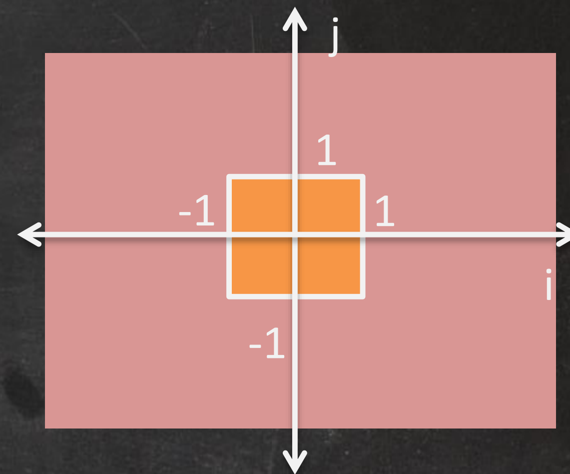
Termination and Inevitability

- Similarity to Program Termination (Halting state inevitability)
- Well-founded relations
- Dense time model vs Well-foundedness
- Hybrid Step Relation

Lets talk about Termination

Termination of Programs: An Example

```
integer i,j;  /* initially arbitrary */
while (|i| > 1 or |j| > 1)
        { i= i + j; j = j - 1; }
```



- Program terminates if transiton relation T_A is ***well-founded***
- ***Transition relation***
- T_A : If (|i| > 1 OR |j| > 1) then (i' = i+j AND j' = j-1)
- For above program T_A is not well-founded
- (4,2) → (6,1) → (7, 0) → (7,-1) → (6, -2) → (4,-3) ...
- (-4,2) → (-2,1) → (-1, 1) stops
- But, $I \land T_A$ is, where $I \triangleq |i + j(j+1)/2| <= 1 \land j \leq 1$

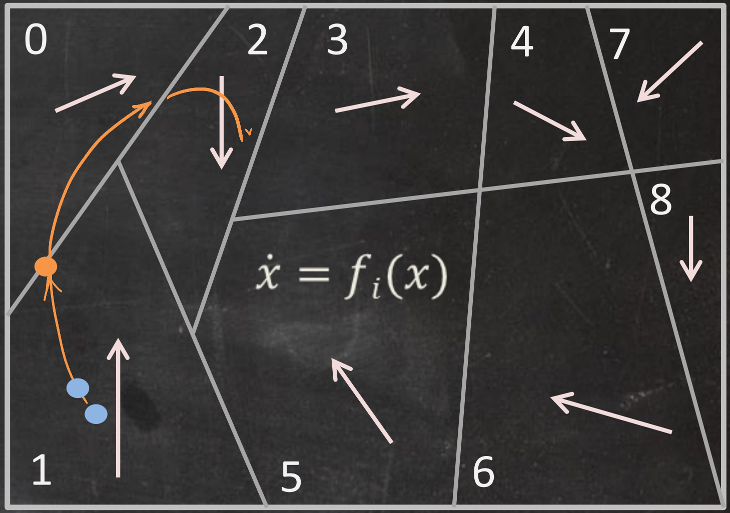
Does not have infinite chains $q_0 q_1 \ldots$ where $q_i T_A q_{i+1}$

$T_1 = \{ < q, q' > \mid q' = q + 1 \}$

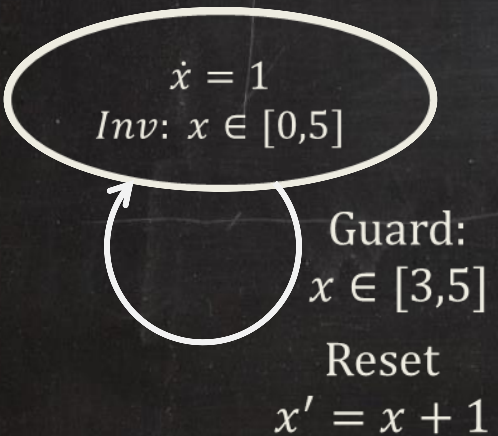
$T_2 = \{ < q, q' > \mid q' = q + 1 \land q' <$ [illegible] $\}$

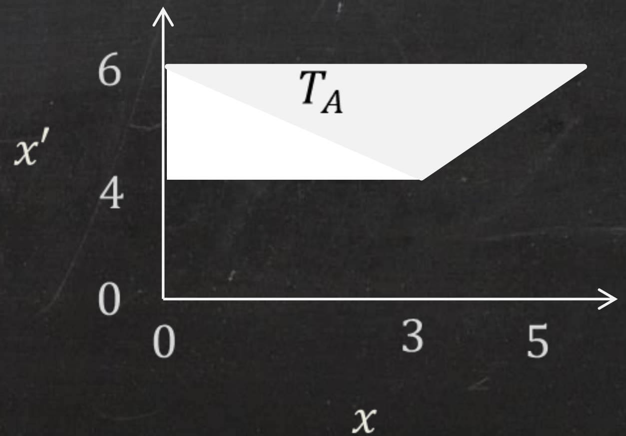
Hybrid Step Relation

Definition. $T_A \subseteq Q \times Q$ **hybrid step relation (HSR)**
$(q, q') \in T_A \iff$ *there exists* q'' such that there exists a trajectory from q to q'' and a transition from q'' to q'



Example:

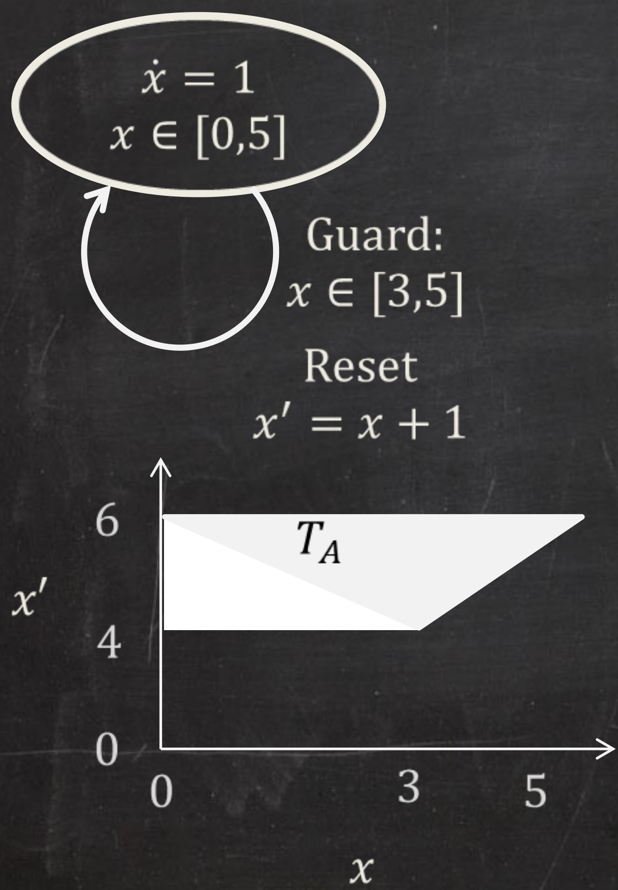




$0 \le x \le 5$ AND
$\exists t\colon 3 \le x + t \le 5$ AND
$x + t + 1 = x'$
After quantifier elimination

$0 \le x \le 5$ AND
$x + 1 \le x'$ AND
$4 \le x' \le 6$

Is it possible to perform this self-loop infinitely many times ?



$0 \leq x \leq 5$ AND $x + 1 \leq x'$ AND $4 \leq x' \leq 6$

- (0,4) (4,5) (5,6) stop
- All finite sequences

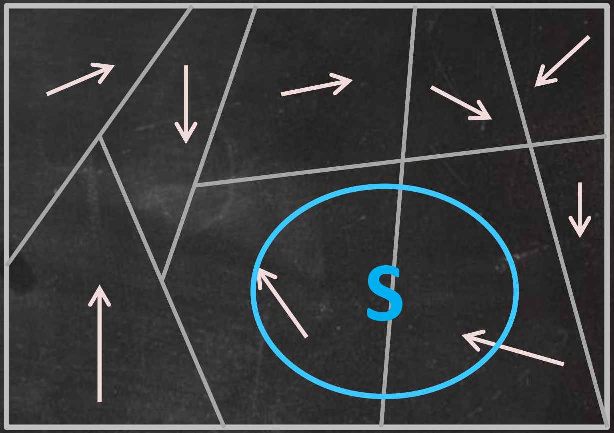
Inevitability and Well-foundedness

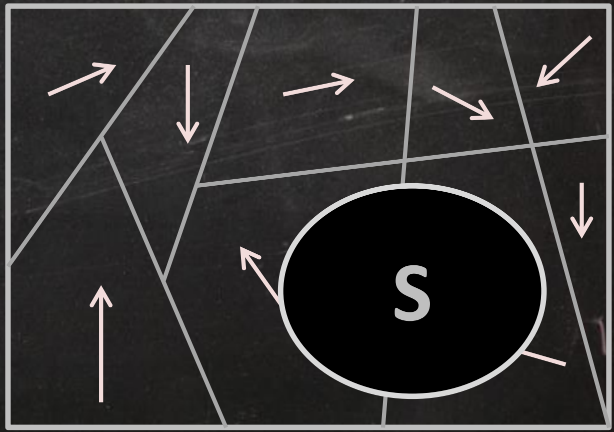
Theorem 1. S is inevitable for *A* iff hybrid-step relation $T_{A/S}$ for A/S is well-founded

Definition: *A/S* = obtained by removing **S** from ***A***

Remove transitions from S

All trajectories stop at S





Proof Sketch

- **Theorem 1. S is inevitable for *A*** iff hybrid-step relation $T_{A/S}$ for A/S is well-founded

- ($T_{A/S}$ Well-founded => S is inevitable for A)
  - If $T_{A/S}$ is well founded then there are no infinite chains outside **S**
  - Every execution outside S has finitely many transitions
  - Since, finite duration elapses between transitions (local nonblocking), total time outside S is also finite => Since, A is non-blocking, S is inevitable
- (S is inevitable for A => $T_{A/S}$ Well-founded)
  - Suppose there is an infinite decreasing chain $q_0 q_1$ … in $T_{A/S}$
  - Chain corresponds to an execution α with infinitely many transitions outside S
  - Time diverges in α (nonZeno) outside S, which contradicts inevitability of S

Hybrid Step Relations for Loops

Theorem 1. **S is inevitable** for ***A*** iff $T_{A/S} \subseteq R$, *R* is well-founded

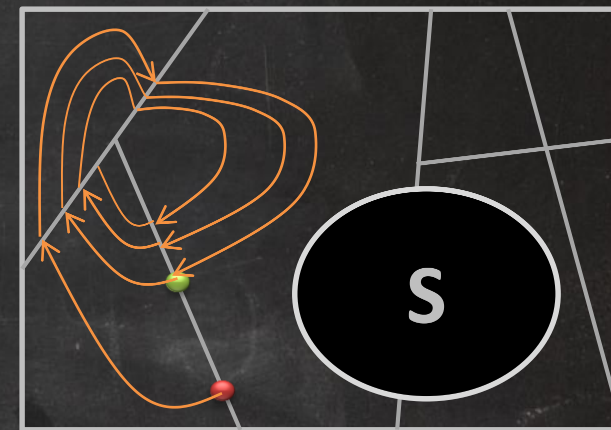
Using [Podelski & Rybalchenko 2004]

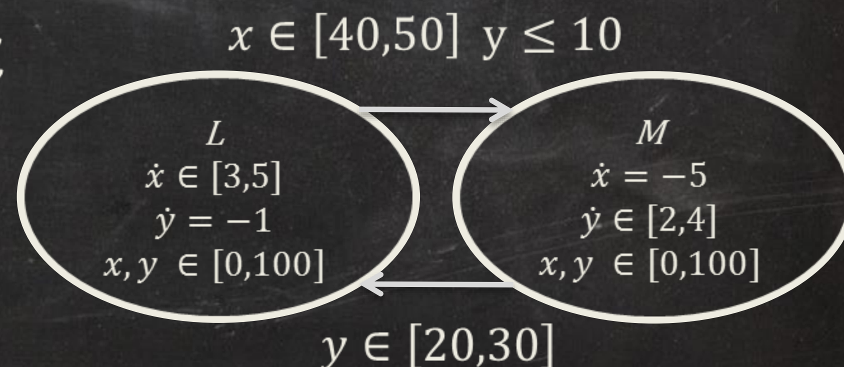
Theorem 2. **S is inevitable for *A*** iff $T_{A/S}{}^{+} \subseteq \bigcup_{i=1}^{n} R_i$, where $\{R_i\}$ is a collection of well-founded relations and $T_{A/S}{}^{+}$ is the transitive closure of $T_{A/S}$

- (a,c) $\in T_{A/S}{}^{+}$ iff a $T_{A/S}$ b_1 $T_{A/S}$ b_2 $T_{A/S}$ … $T_{A/S}$ c
- $(q, q') \in T_{A/S}{}^{+}$ iff there is execution α: *q* to *q'*
- Need to show that every execution is well-founded
- Suffices to consider loops, i.e., executions starting and ending at the

Using Disjoint Union of Well-founded Relations

- For every loop O, find a well-founded relation R_i containing T_O
- Example, Rectangular HA:
- T_{MLM} = $(x, y \in [0,100] \; AND \; x' \in [40,50] \; AND \; y' \leq 10 \; AND \; x' - x \in [-25,-1] \; AND \; y' \geq y + 2)$
- T_{MLM} can be computed and
- Well-foundedness of T_{MLM} can be checked using linear functions over *x, x′, y, y′ e.g. using* Rankfinder





For Linear Dynamical Systems computing HSR involves Matrix Exponentials

General Dynamics

- For a location $l \in L$ suppose we have a Lyapunov-like function $V_l\colon \mathbb{R}^4 \longrightarrow \mathbb{R}$ with
  - (stable) $\exists\ \lambda_l < 0$ and $B_l > 0$ such that for any trajectory τ in $l \epsilon L$, $V_l(\tau(t)) \leq B_l\, e^{\lambda_l t} V_l(\tau(0))$

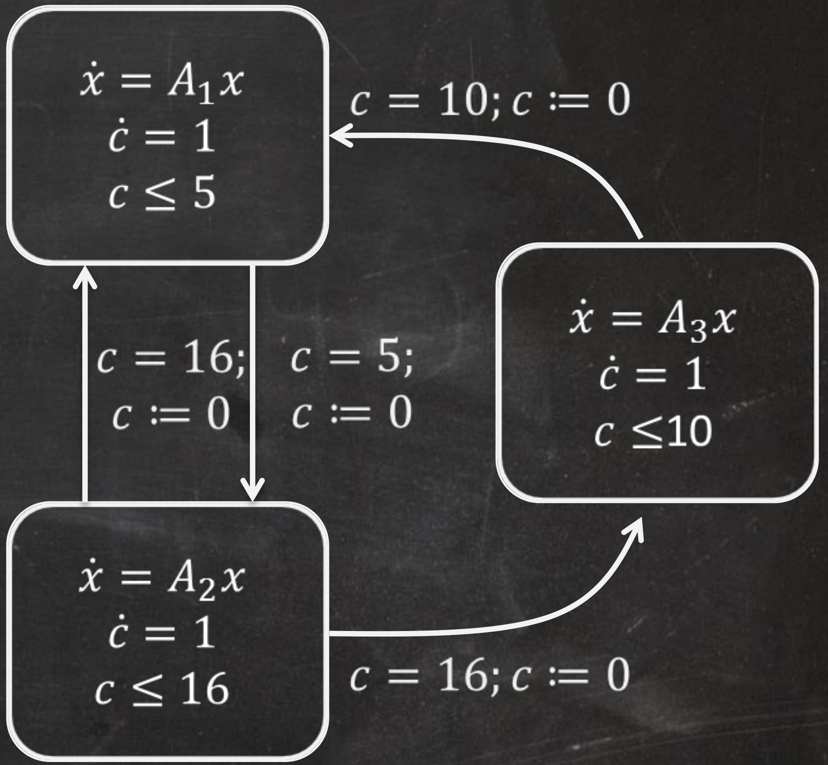
  OR
  - (unstable) $\exists\ \lambda_l > 0$ and $B_l > 0$ such that for any trajectory τ in $l \epsilon L$,$V_l(\tau(t)) \leq B_l\, e^{\lambda_l t} V_l(\tau(0))$
- We can over-approximate T_A^+ hybrid step relation if we know bounds on dwell time

Lyapunov Abstraction

- $\mathcal{V} = \{V_{l,i}\}_{i=1}^{k}$: Collection of k Lyapunov functions for location l
- Abstraction: $\beta\colon \mathbb{R}^n \rightarrow \mathbb{R}^k$
  - $\beta_{\mathcal{V}}(x) = V_{l,1}(x), \ldots, V_{l,k}(x)$ where $x.loc = l$
- Abstraction of HSR
  - $\beta_{\mathcal{V}}(\Gamma) = \{(y, y') | \exists\, x, x' : \beta(y) = x \ \wedge \beta(x') = y'\}$
- Theorem: If $\beta_{\mathcal{V}}(\Gamma)$ is well-founded then so is Γ.
- Next: Steps, Loops, and Gamma (Γ)

Example: Time Triggered Linear HA

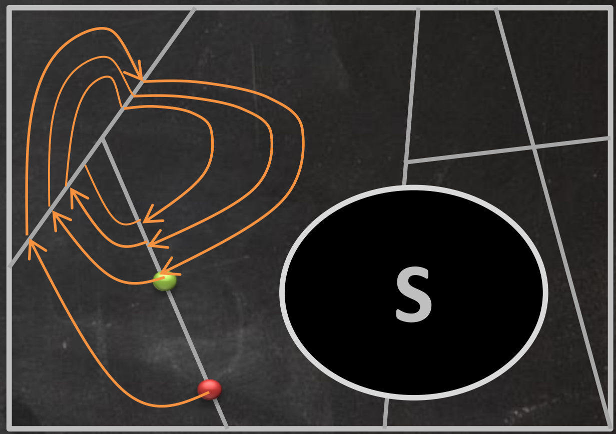
- Clock (c) constrains dwell time at each location
  - Unstable: upper bound
  - Stable: lower bound
- Guards overapproximated by level sets of $V_{i,l}$
- $\mu_{i,l,m}$:Bound on growth of $V_{i,l}(x) \leq \mu_{i,l,m} V_{i,m}(x')$
- $(y, y') \in \beta \;\Leftrightarrow\; \exists\, y''$such that
  - $y_i'' \leq B_l\, e^{\lambda_l D} y_i$ where D:lower bound
  - $G_{i,min} \leq y_i'' \leq G_{i,max}$
- $y_i' \leq \mu_{i,l,m}\, y_i'' \leq \mu_{i,l,m} B_l e^{\lambda_l D} y_i$
- $y_i' \leq \frac{y_i}{K} \wedge y_i \geq c_i$

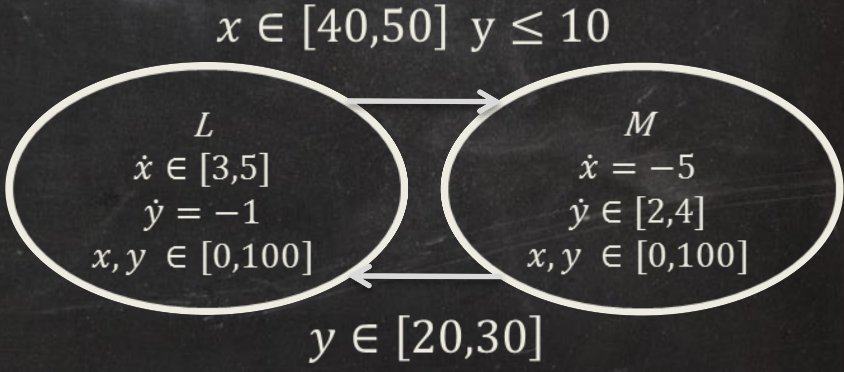


$A_1 = \begin{bmatrix} -1 & 0 \\ 5 & -3 \end{bmatrix}$ $A_2 = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ $A_3 = \begin{bmatrix} -4 & -2 \\ 0 & -9 \end{bmatrix}$

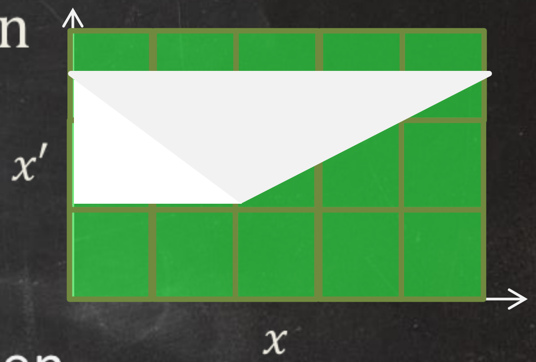
Using Disjoint Union of Well-founded Relations

- For every loop O, find a well-founded relation R_i containing T_O
- For Rectangular HA and TTLHA we can compute (approximate) T_O
- Well-foundedness of T_O can be checked using linear functions over *x, x′, y, y′ e.g. using* Rankfinder
- But there may be infinitely many loops to consider
- We will abstract each T_O with an abstract transition relation





Abstracting Loop HSRs with Transition Predicates



- Given $\mathcal{P} = \{P_1, \dots, P_m\}$ a collection of transition predicates, i.e., each $P_i \subseteq Q \times Q$
- $abs_{\mathcal{P}}(T_O) \supseteq T_O$ is the smallest superset of T_O constructed by intersecting P_i's
- **Observe.** If $\mathcal{P}$ is finite, $abs_{\mathcal{P}}$ has finite range; even with infinitely many loops there are a finite number of $abs_{\mathcal{P}}(T_o)$'s to check

- **Theorem 3.** **S** inevitable for ***A*** if there exist (1) predicates $\mathcal{P} = \{P_1, \dots, P_m\}$ and (2) well-formed relations $\mathcal{R} = \{R_1, \dots, R_n\}$ such that for every loop **O** of A/S, $abs_{\mathcal{P}}(T_O) \subseteq R$

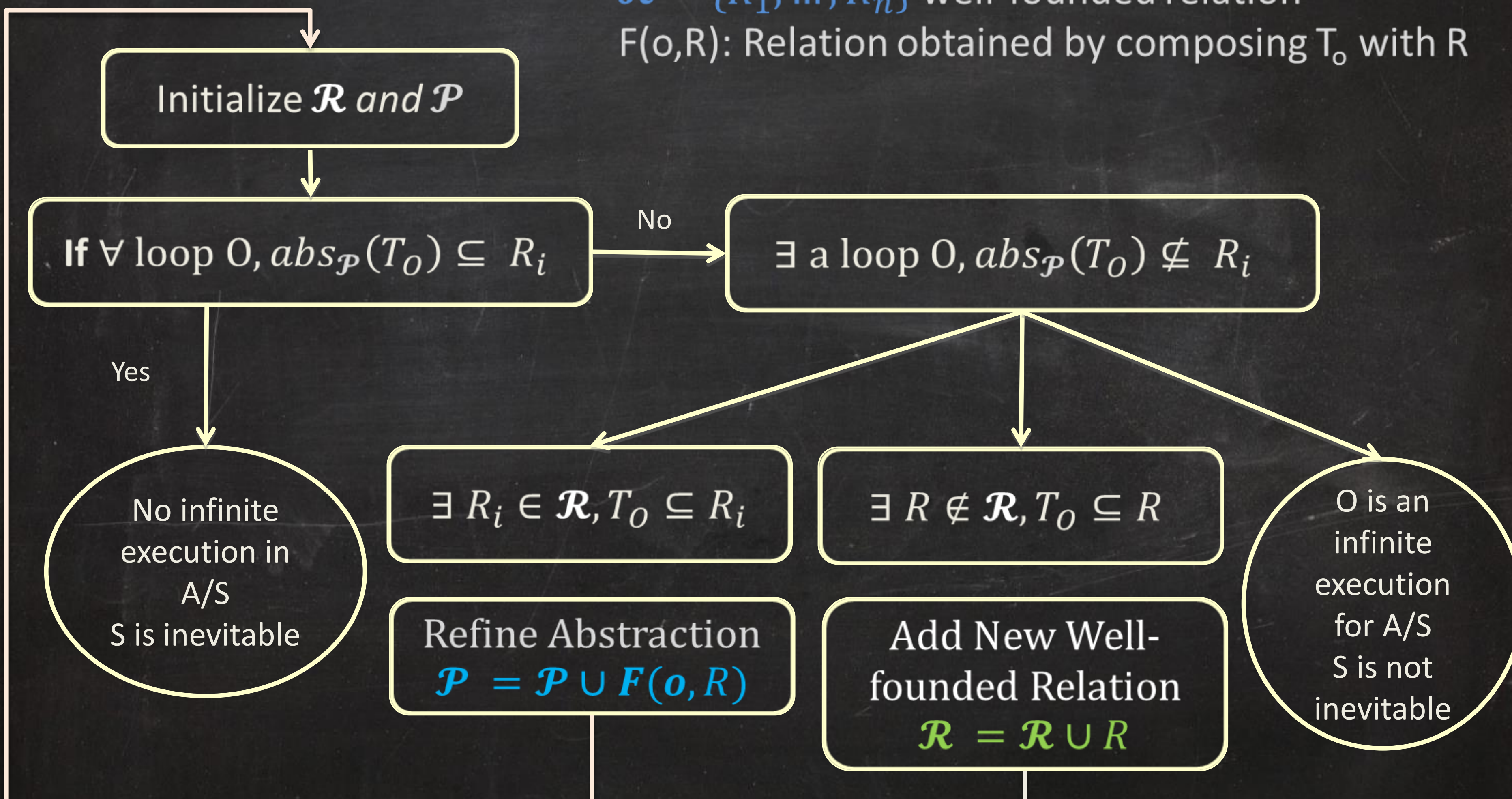
Abstraction-Refinement Algorithm

T_O : Transition relation for loop o

$\mathcal{P} = \{P_1, \dots, P_m\}$ transition predicates

$\mathcal{R} = \{R_1, \dots, R_n\}$ well-founded relation

F(o,R): Relation obtained by composing T_o with R



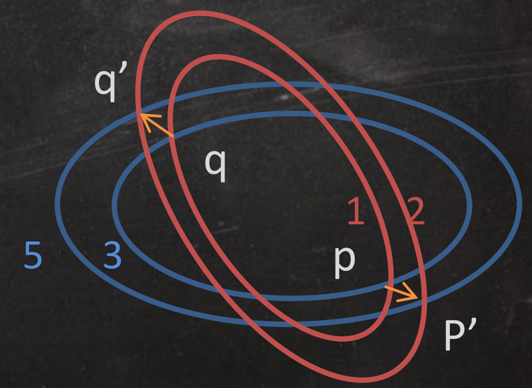
Bringing it all together

- Inevitability of HA A to set S
- Prove well-foundedness of $T_{A/S}$
- Prove well-foundedness of abstract loop transition relations $abs_P(T_o)$ that constitute $T_{A/S}$
- Completeness
  - For rectangular initialized HA, guaranteed to terminate
  - Linear TTHA symmetric with respect to the k Lyapunov functions: if $x\ T_L\ x'$, then for all $q \in \mathrm{Abs}^{-1}{}_{\nu}(x)$ there exists $q' \in \mathrm{Abs}^{-1}{}_{\nu}(x')$ such that $q\ T_A\ q'$

Problem (n, |L|)	Unstable locations	Time (sec)
(2,5)	2	0.01
(2,10)	3	0.14
(2,20)	5	1.88
(2,40)	8	88.94
(2,50)	9	392.85
(3,20)	5	2.02
(3,40)	8	38.11
(4,20)	5	100.49
(4,40)	8	110.34

$V_1(q) = 1\ V_2(q) = 3$
$V_1(q') = 2\ V_2(q) = 5$

$\langle(1,3), (2,5)\rangle \in T_L$



Ongoing and future directions

- What additional (robustness) assumption are needed for completeness of inevitability verification?
- Nonlinear Ranking Functions
- Invariant generation + Ranking
- Extension to networked and distributed hybrid systems

Questions ?

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