Rigorous Simulation-Based Analysis of Linear Hybrid Systems

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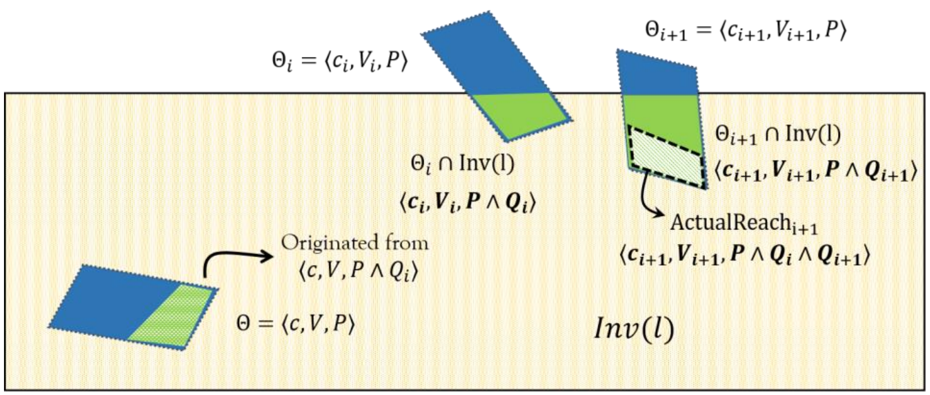
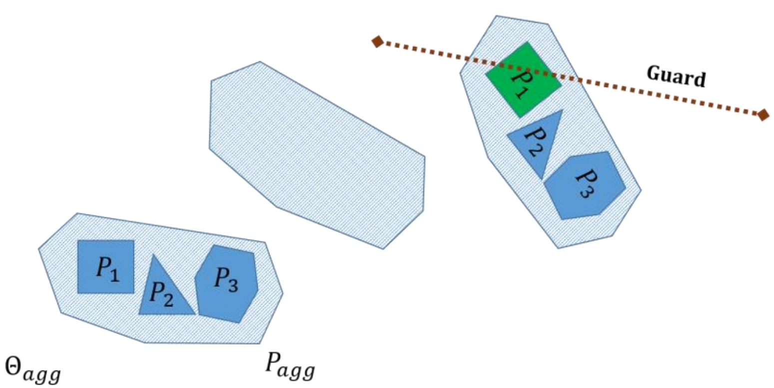
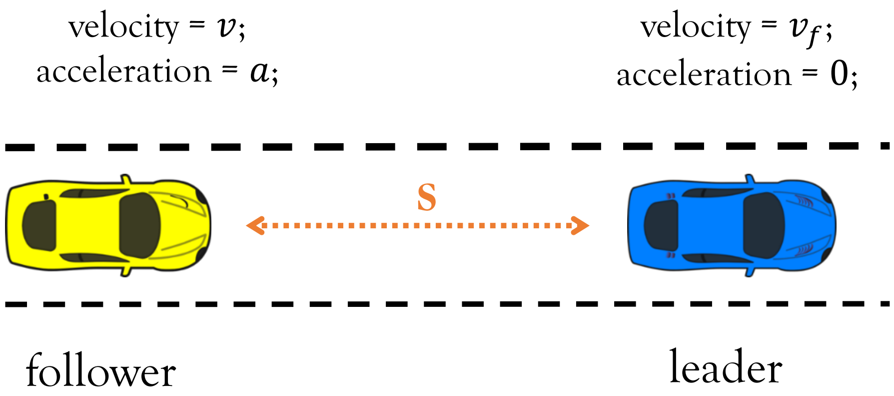




Motivating Example: Leader-Follower System



Dynamics of the system

$\dot{s} = v_f - v;$

$\dot{v} = a - k_{aero} v;$

$\dot{a} = u;$

k_{aero} is the air–drag

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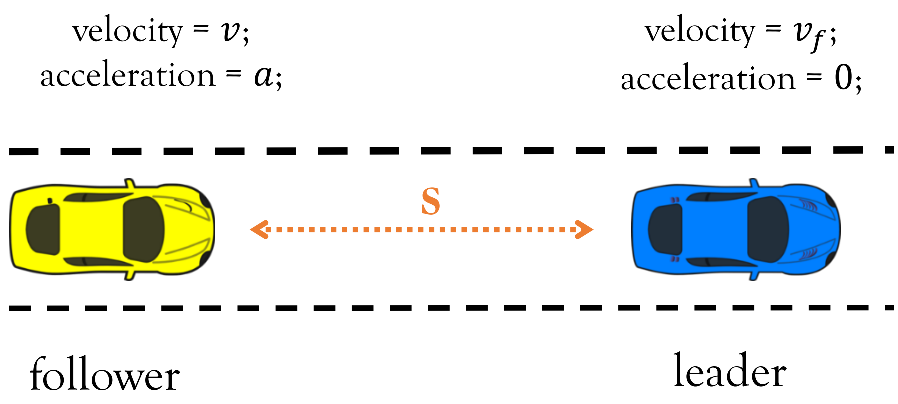

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if(cond1) then

$u = -2a - 2(v - v_f);$

if(cond2) then

$u = -3a - 2(v - v_f);$

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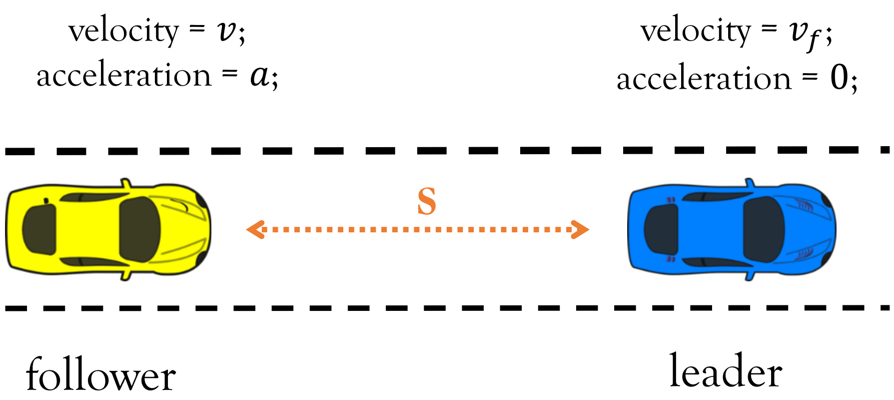

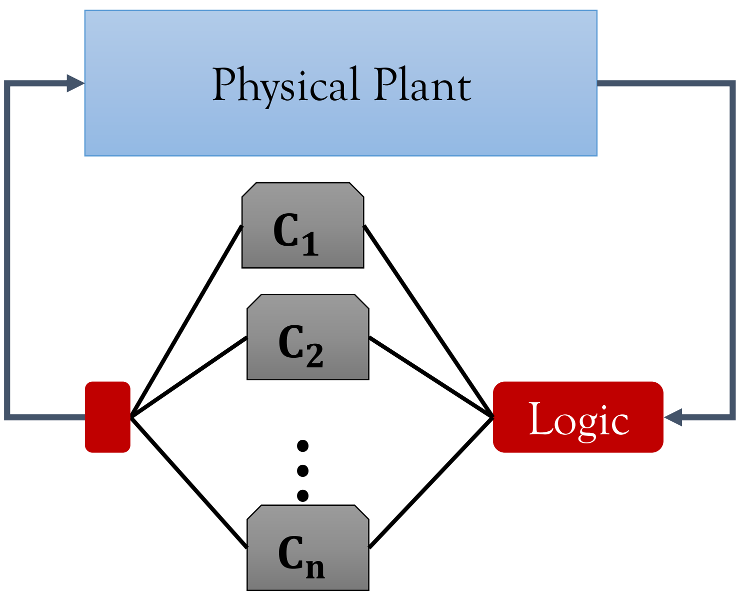

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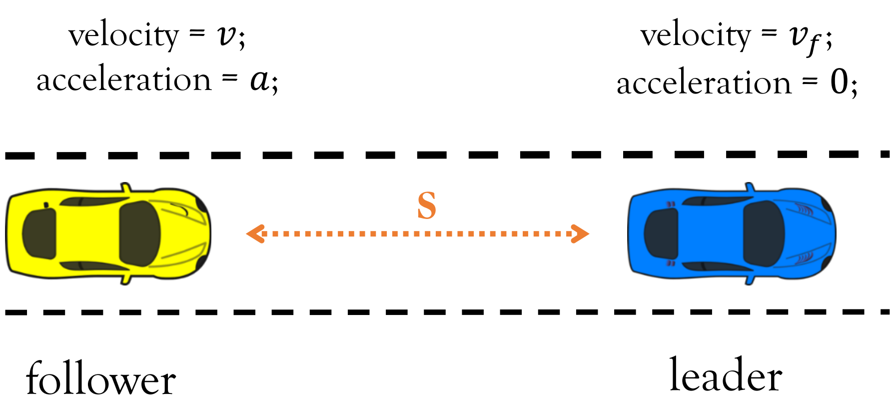

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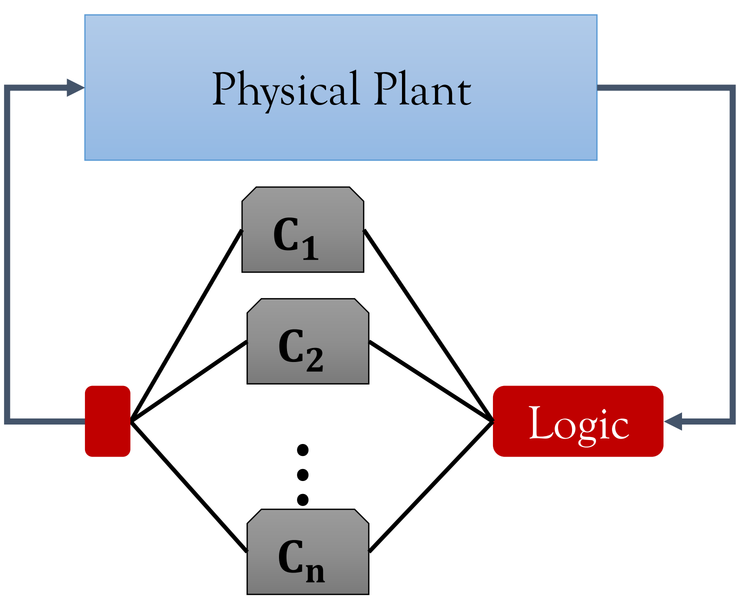
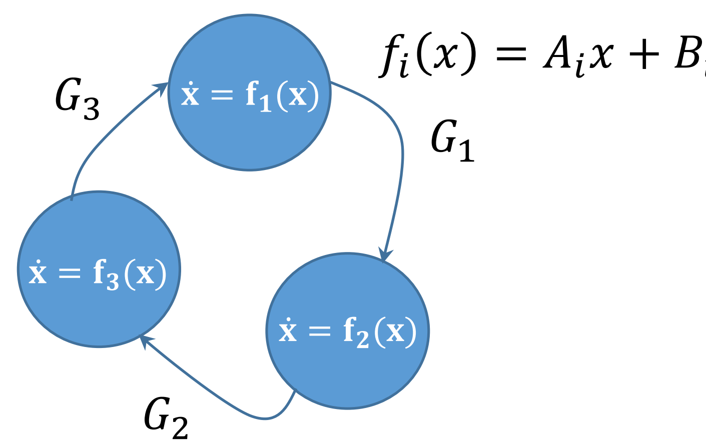
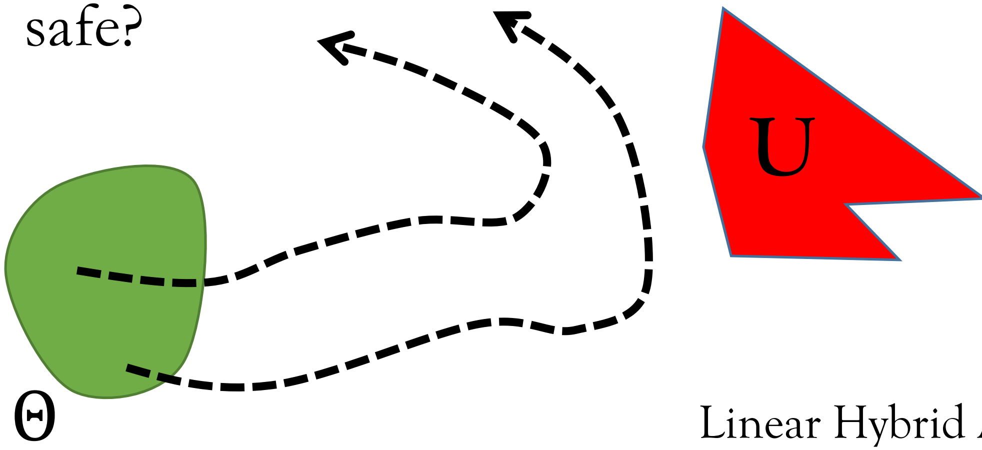


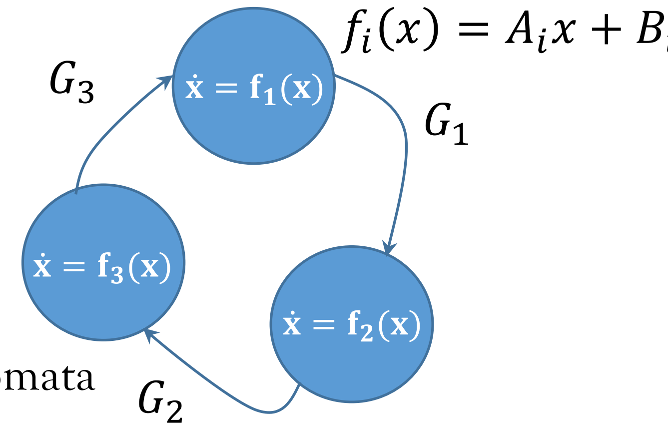


Linear Hybrid Automata

Safety Verification Problem

- Given a Linear Hybrid Automata H, with initial set Θ and unsafe set U, are all the behaviors starting from Θ for bounded time T_b are safe?





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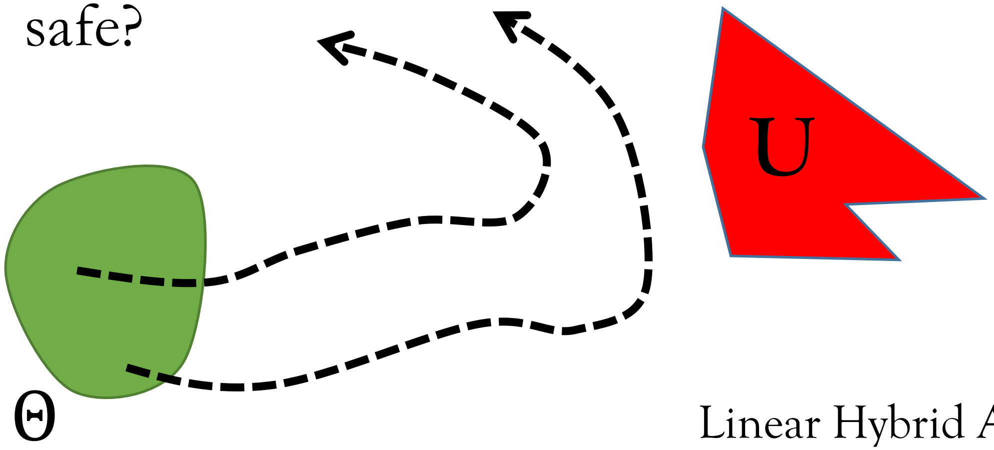

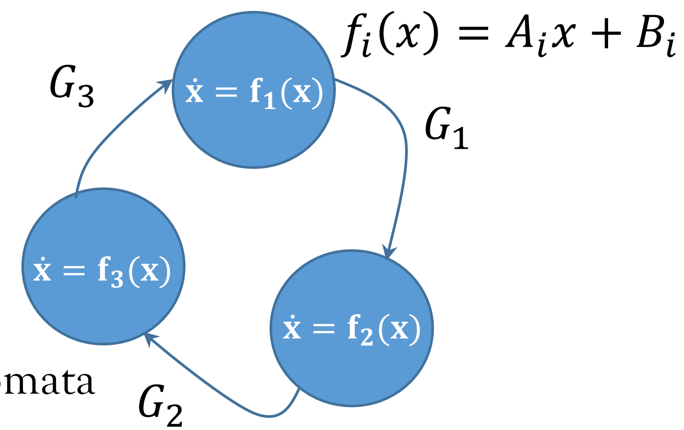

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- One technique: Use a safety verification tool such as SpaceEx, Flow*, or CORA, etc.

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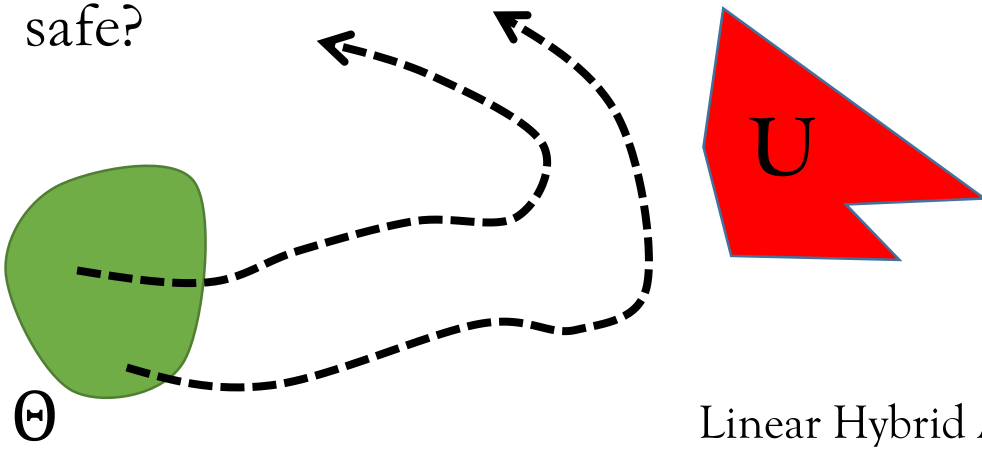

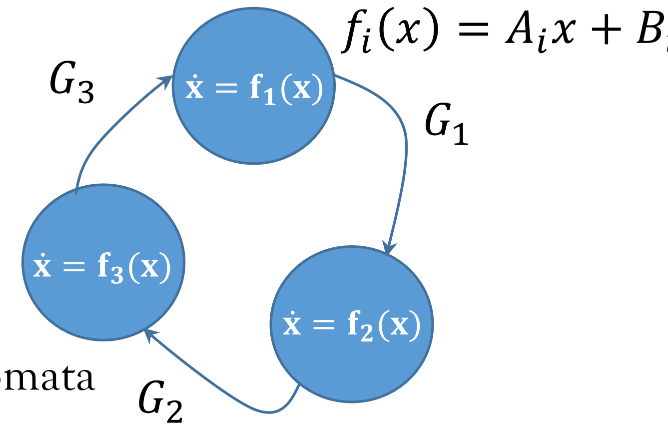

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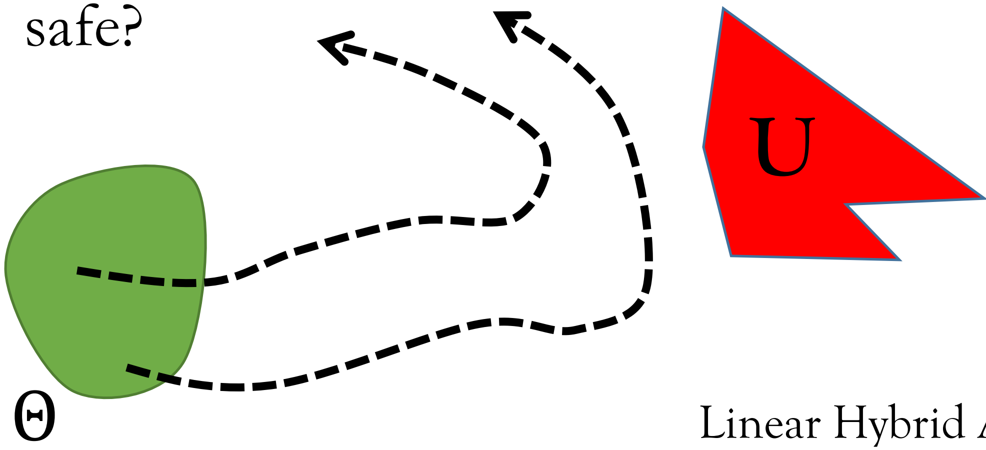

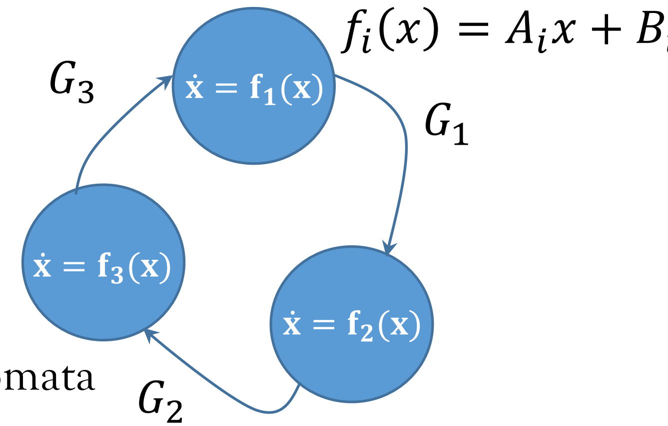

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This paper

Simulations ↔ Verification

Simulation-Equivalent Reachability (Safety)

<u>Assumptions</u>

1. We are provided with a simulation engine (oracle) that provides a discrete time simulation for a differential equation $\dot{x} = Ax + B$.
2. All the sets encountered such as invariants, guards, initial set, and unsafe set are all conjunctions of **linear predicates**.

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Contributions

1. Compute simulation-equivalent reachable set (safety verification).
2. New technique called forward constraint propagation for handling invariants.
3. New on-the-fly aggregation and deaggregation techniques.
4. Sound and complete with respect to the simulation engine provided.

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Overview

- ✓ Motivation and Contributions.
- Dynamic analysis technique for linear systems verification.
- Observations of the dynamic analysis technique.
- Invariant constraint propagation.
- Dynamic deaggregation.
- Experimental evaluation.
- Conclusions and Future work.

Dynamic Analysis Technique For Linear System

Dynamic Analysis Technique

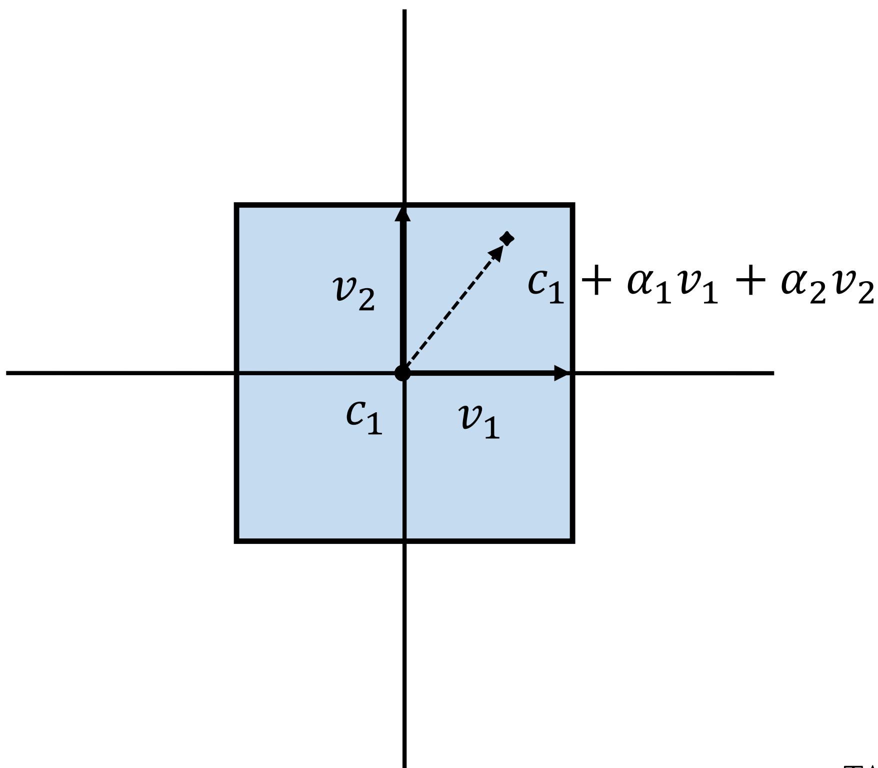
1. The representation: **Generalized stars**.
2. The property of linear systems: **Superposition principle**.
3. The reachable set computing technique: **Safety verification of an $\boldsymbol{n}$ dimensional system using $\boldsymbol{n+1}$ simulations**.

P.S.Duggirala, M.Viswanathan, *"Parsimonious, Simulation Based Verification of Linear Systems"*, International Conference on Computer Aided Verification (CAV) 2016.

Representation: Generalized Stars

- Generalized star is represented as $\langle c, V, P \rangle$
- c – center, V – set of vectors, P – predicate.

$$\langle c, V, P \rangle = \{\, x \mid \exists \bar{\alpha} = (\alpha_1, \ldots, \alpha_n), \mathrm{c} + \Sigma_i \alpha_i v_i = x, P(\bar{\alpha}) = \top \}$$



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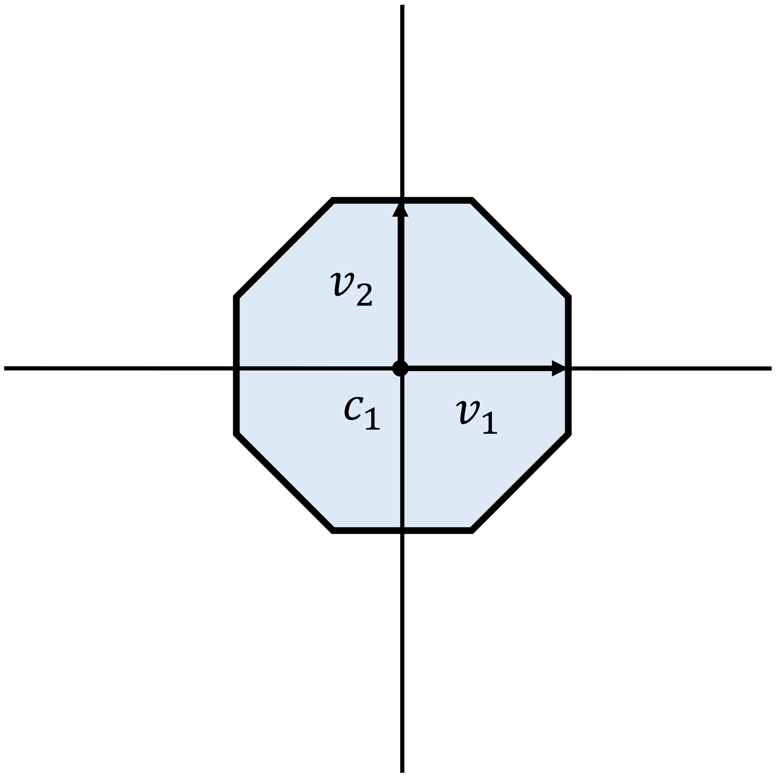
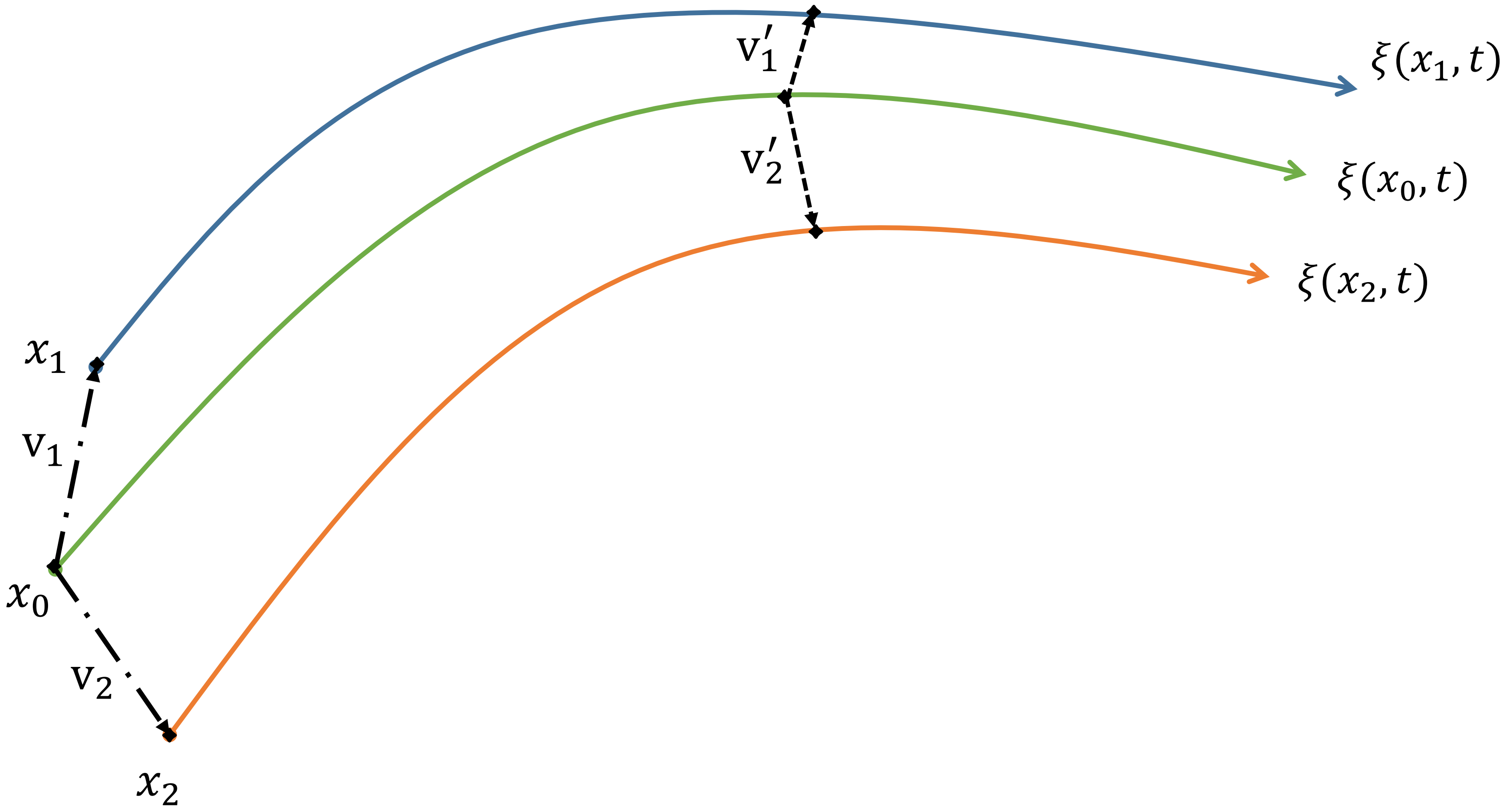


$$P(\langle \alpha_1, \alpha_2 \rangle) \triangleq |\alpha_1| \leq 1 \wedge |\alpha_2| \leq 1 \wedge |\alpha_1 + \alpha_2| \leq 1.5$$

Property: Superposition



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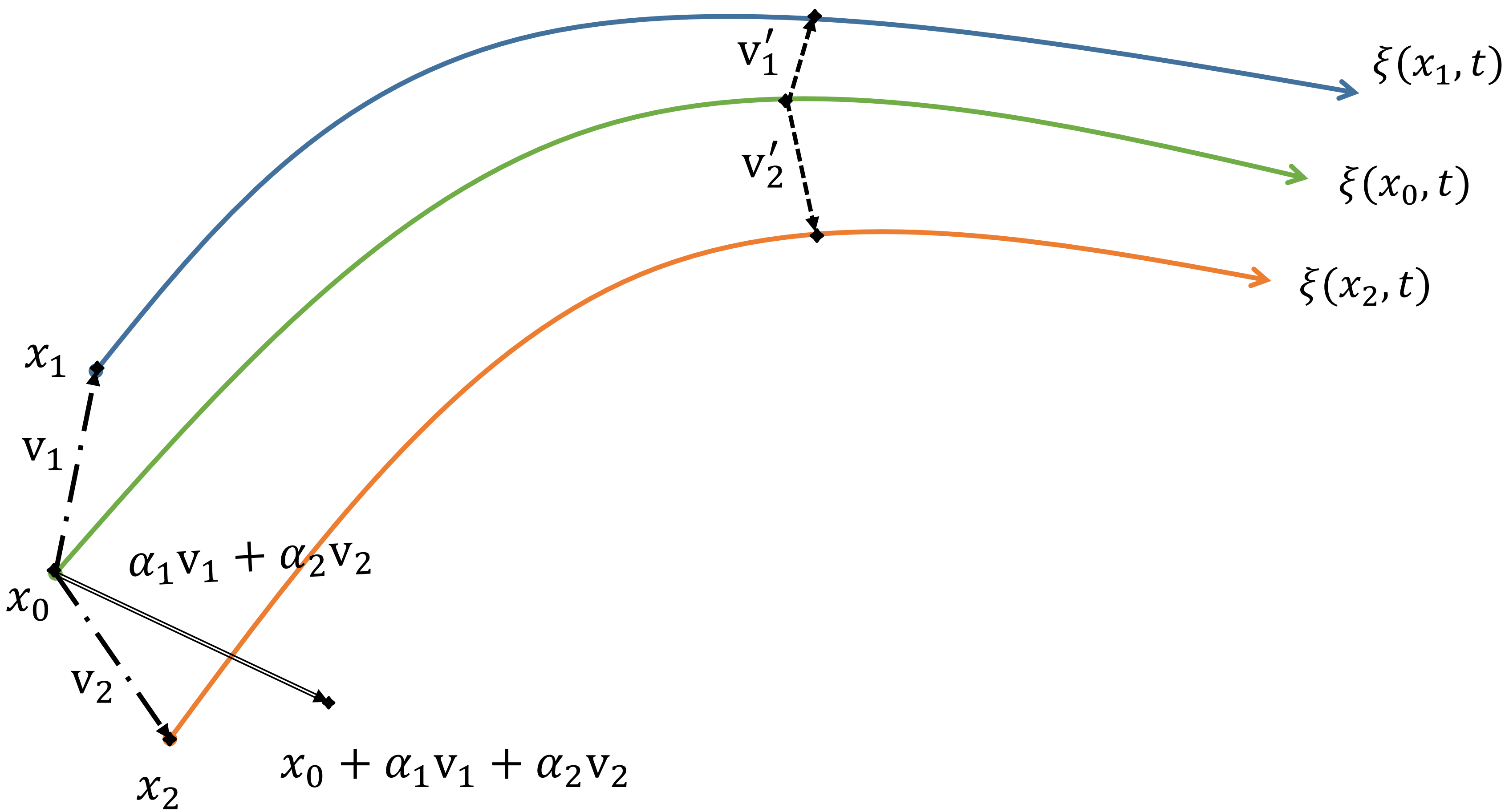

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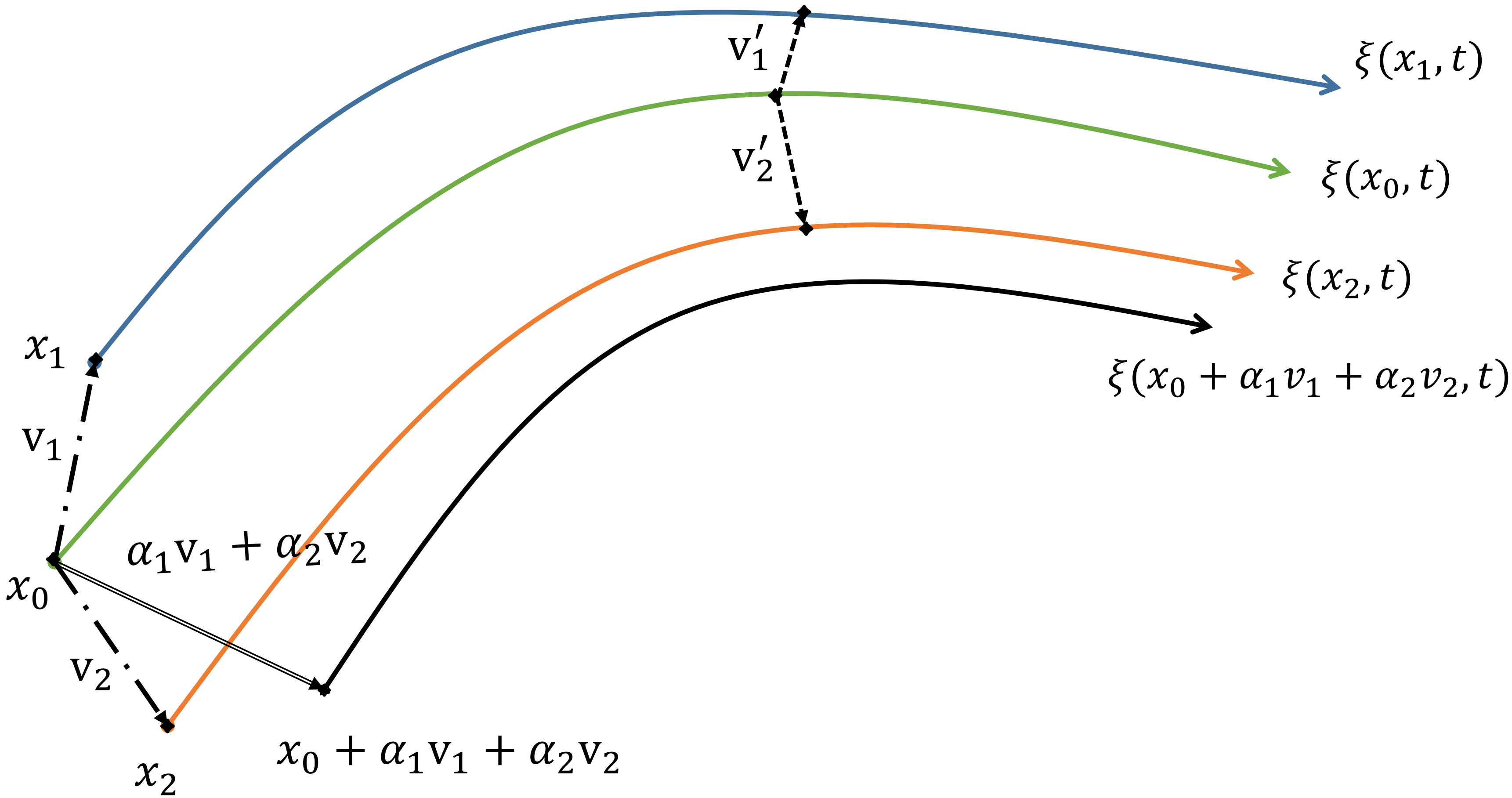

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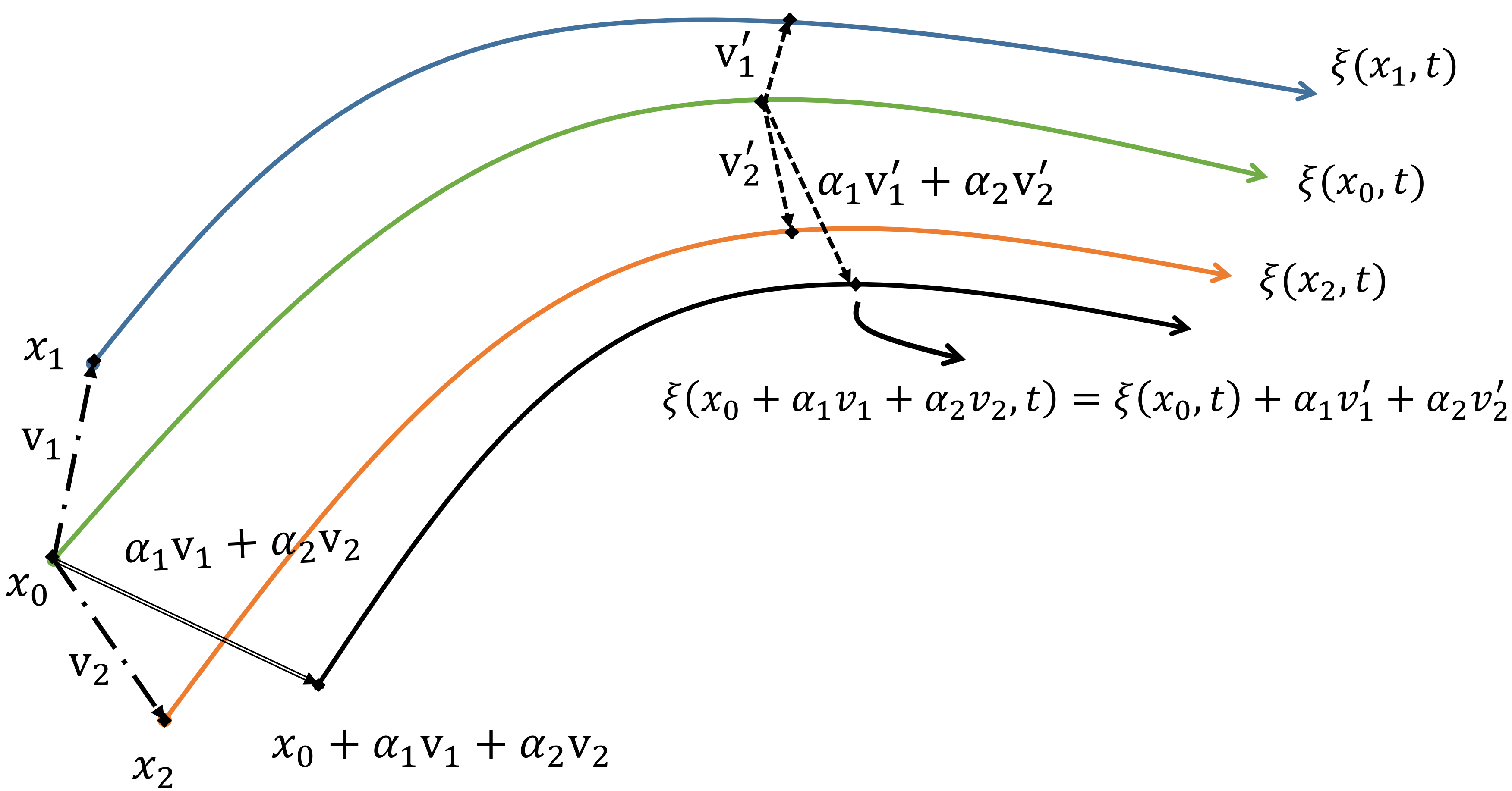

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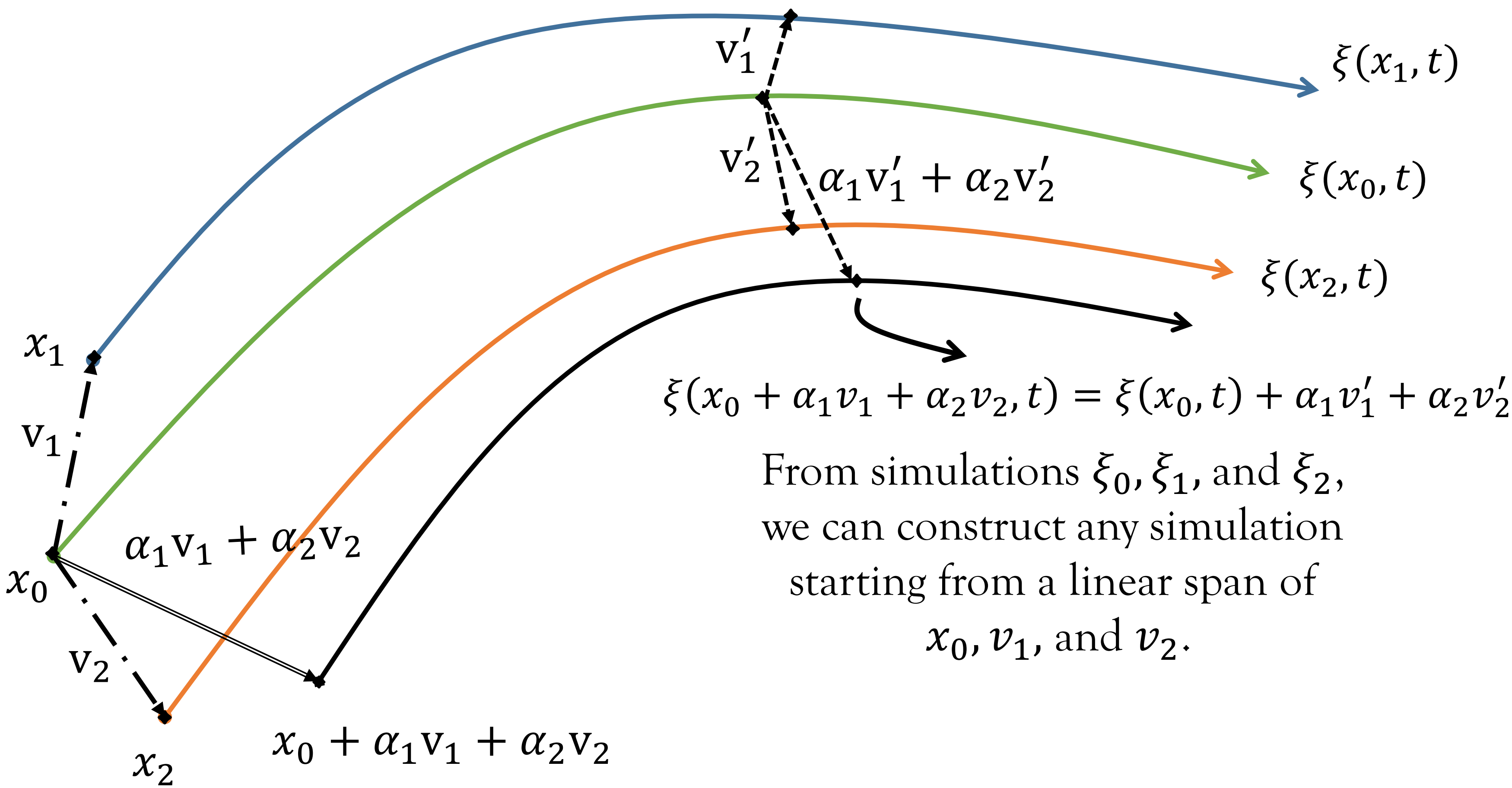
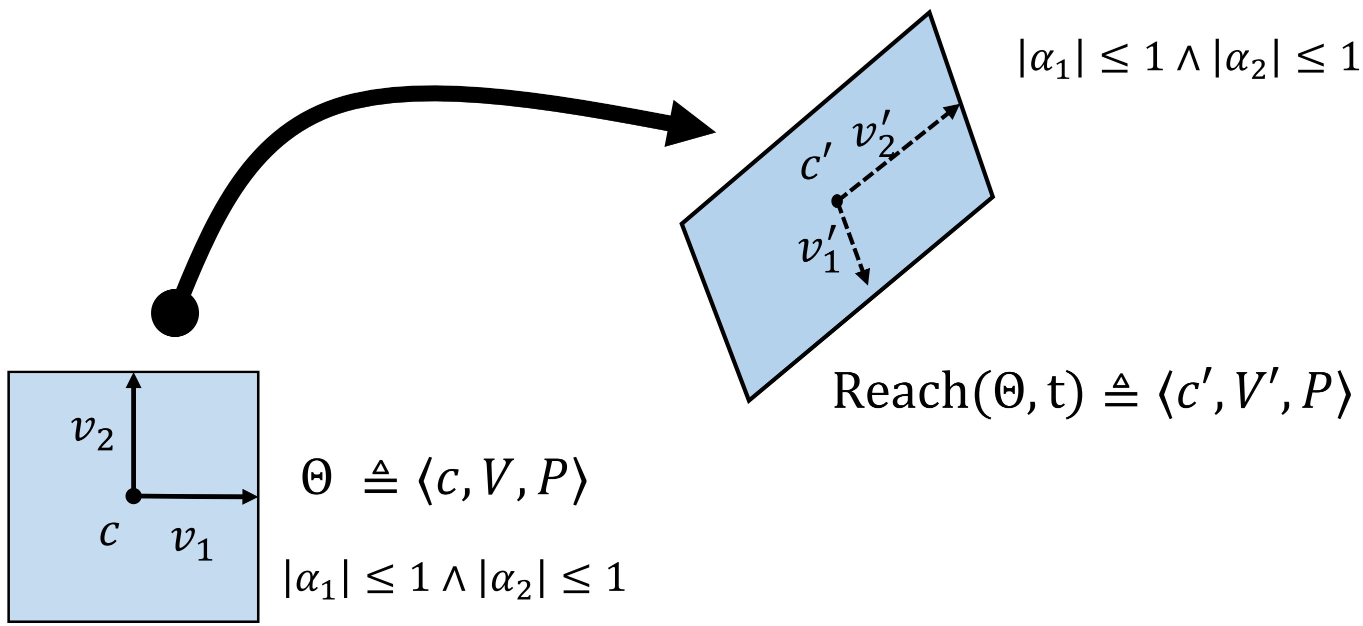


From simulations ξ_0, ξ_1, and ξ_2, we can construct any simulation starting from a linear span of x_0, v_1, and v_2.

Technique: Basic Idea

- Given initial set $\Theta = \langle c, V, P \rangle$, the **Reach** is computed not as new predicate, but is done by changing the *center* and the *basis* vectors.



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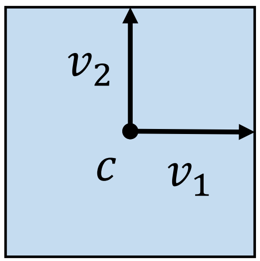
Technique
Representation + Superposition

Given $\Theta \triangleq \langle c, V, P \rangle$ to compute reachable set



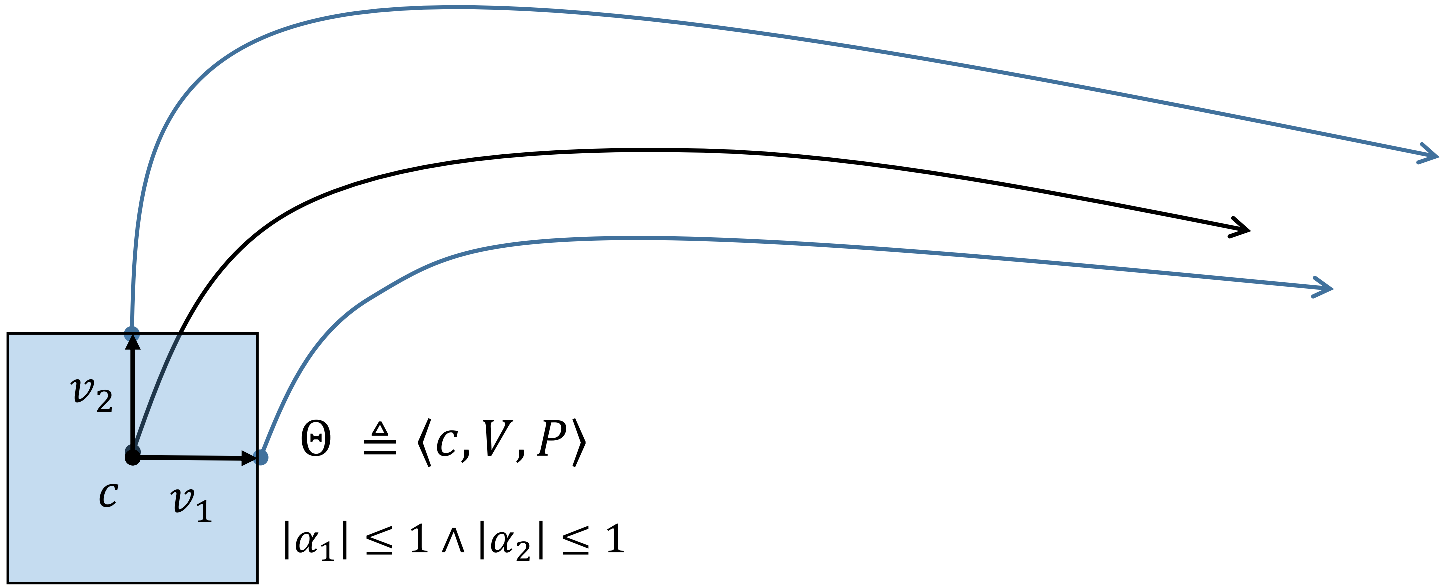
$\Theta \triangleq \langle c, V, P \rangle$

$|\alpha_1| \leq 1 \wedge |\alpha_2| \leq 1$

Technique
Representation + Superposition

Given $\Theta \triangleq \langle c, V, P \rangle$ to compute reachable set

1. Simulate from c
2. Simulate from $c + v_i$ for each i



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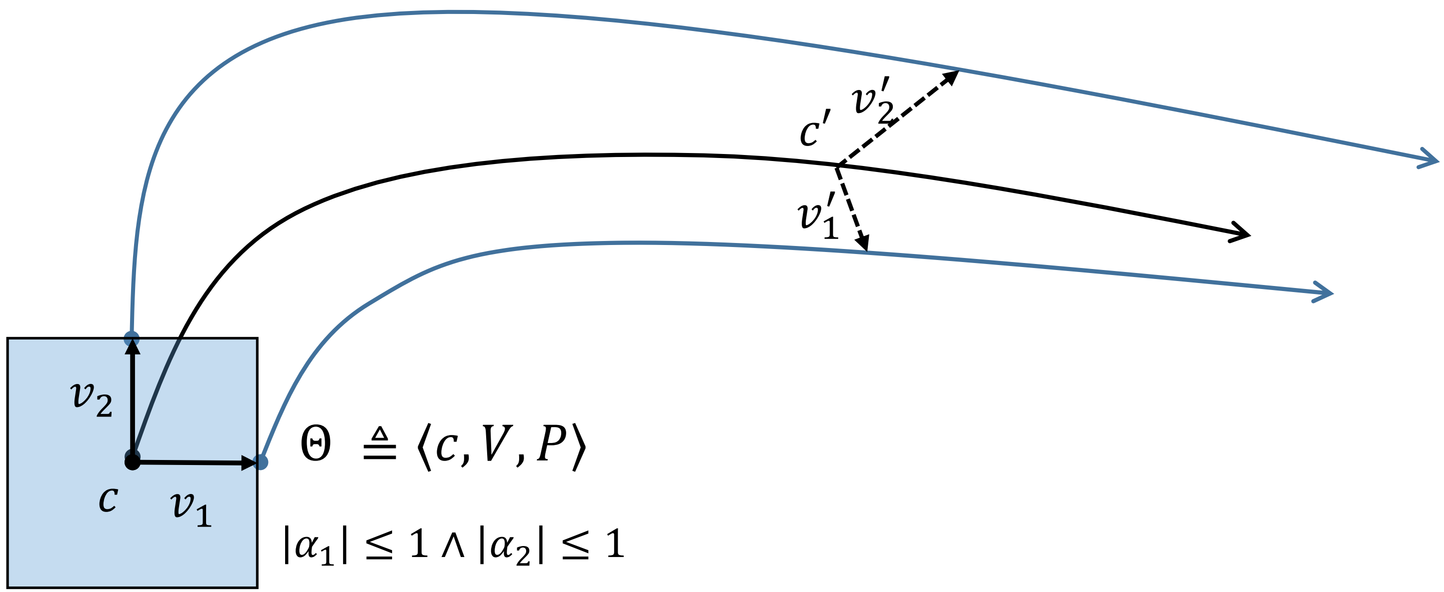

Reachable set at time t is given by $\langle c', V', P \rangle$ where

1. c' is the simulation corresponding to c

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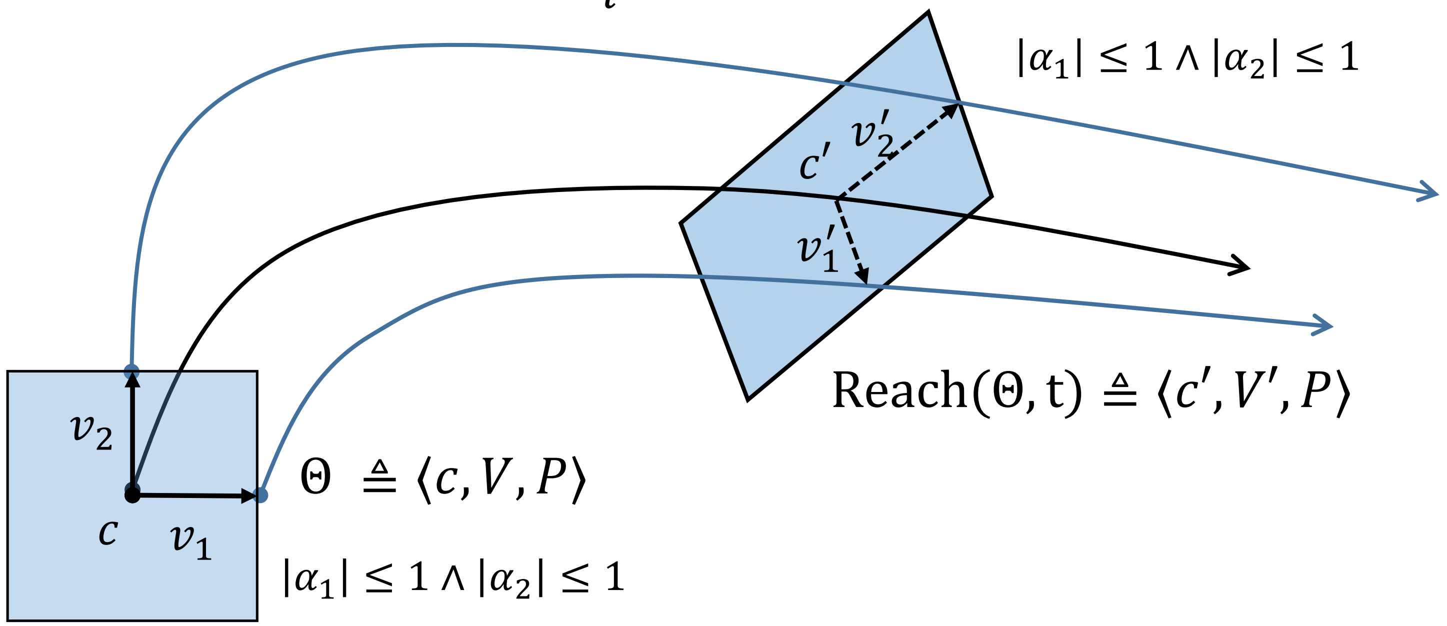

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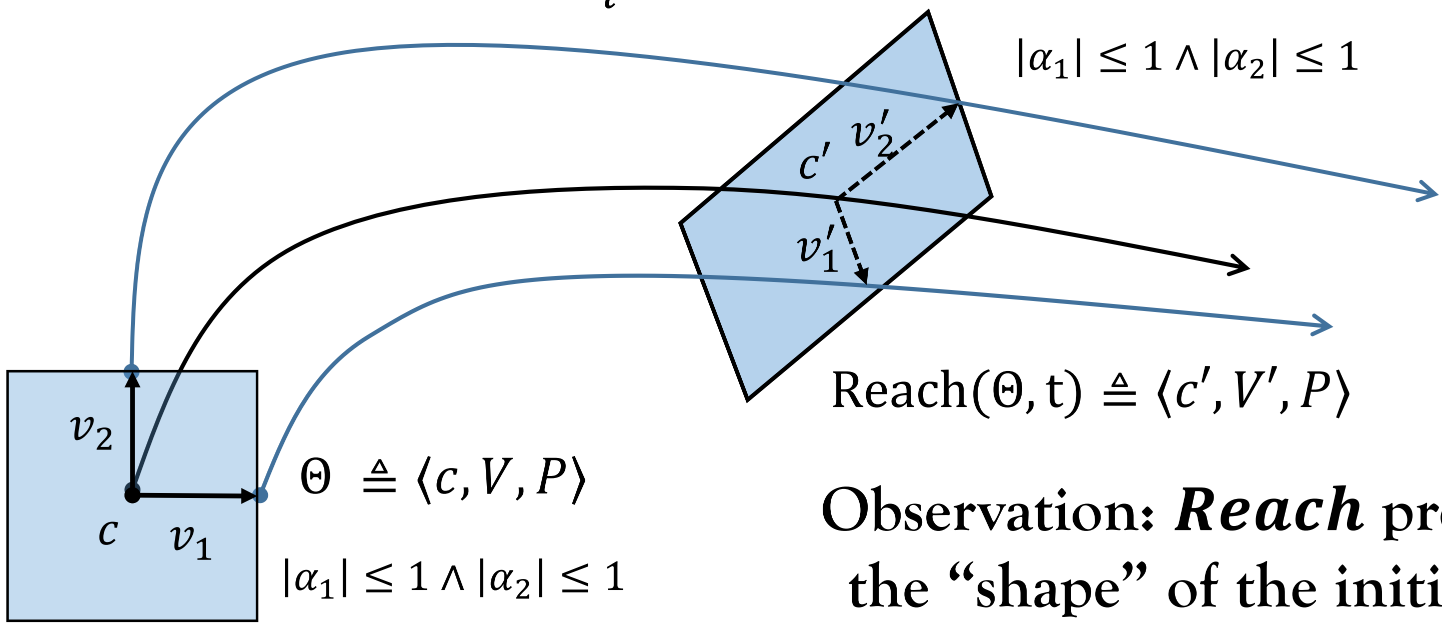

Observation: *Reach* preserves the "shape" of the initial set.

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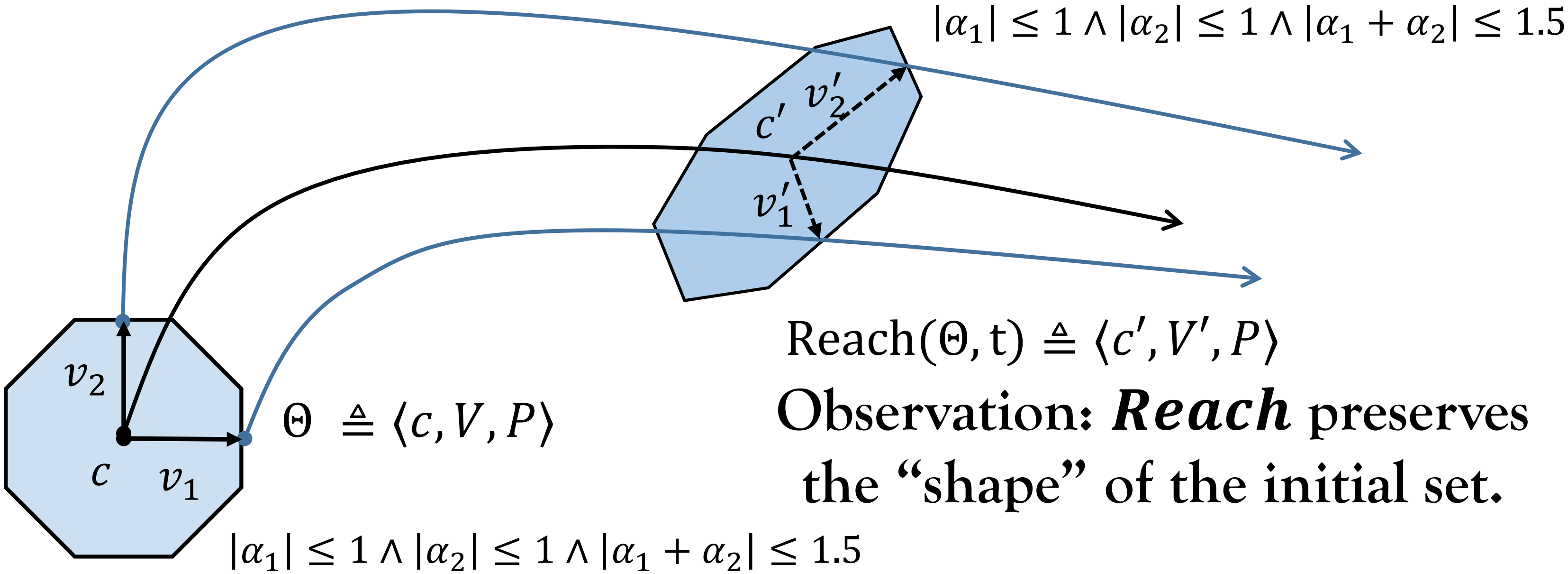

$\mathrm{Reach}(\Theta, \mathrm{t}) \triangleq \langle c', V', P \rangle$

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Using Discrete Time Simulation Engine

Initial set $\Theta \triangleq \langle c, V, P \rangle$; Simulation engine ρ; step size h;

For computing the reachable set at time $j \cdot h$ instant

1. Generate simulation $\rho(c, j \cdot h)$;
2. For each $v_i \in V$, generate simulation $\rho(c + v_i, j \cdot h)$;
3. Reachable set denoted as Θ_j is defined as $\langle c', V', P \rangle$ where
   1. $c' = \rho(c, j \cdot h)$;
   2. $v_i' = \rho(c + v_i,\ j \cdot h) - \rho(c, j \cdot h)$;

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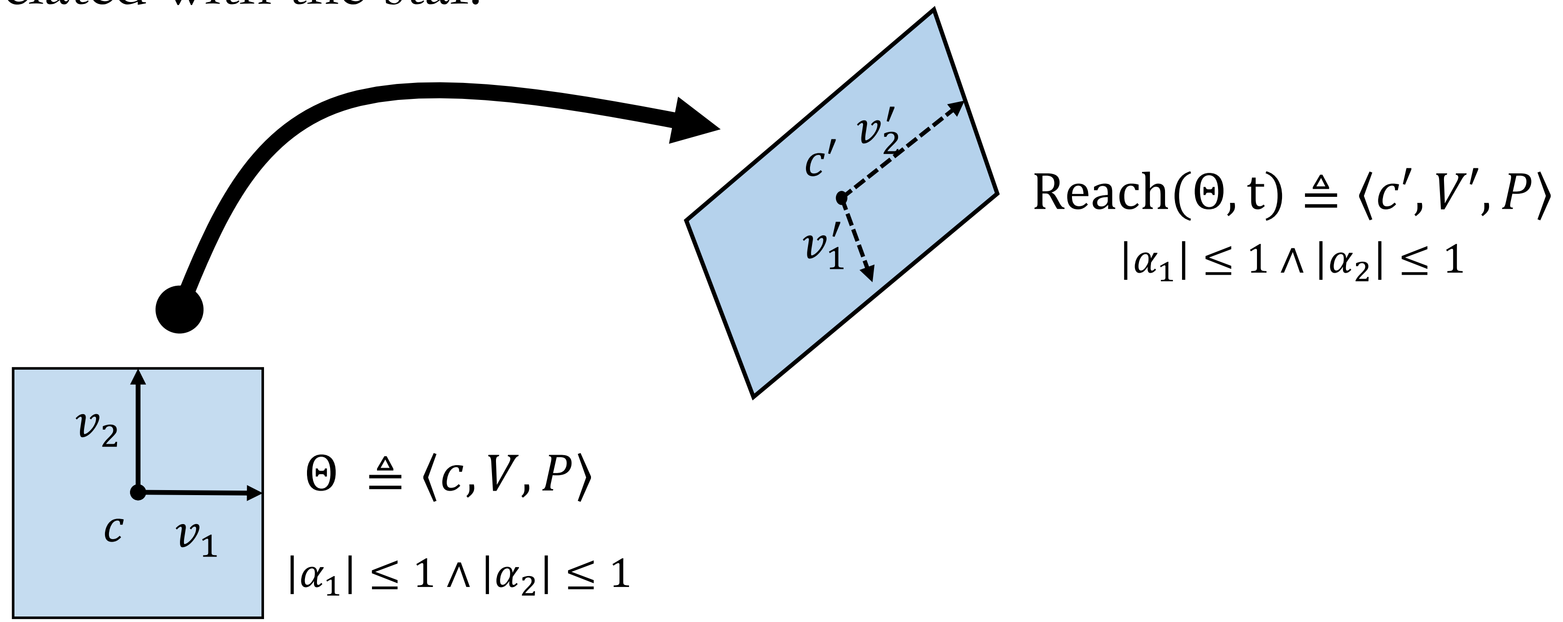
Given initial set $\boldsymbol{\Theta}$, procedure $\mathbf{Reach}(\boldsymbol{\Theta}, \mathbf{h}, \mathbf{k} \cdot \boldsymbol{h})$ returns $\boldsymbol{\Theta_1}, \boldsymbol{\Theta_2}, \ldots, \boldsymbol{\Theta_k}$ where $\boldsymbol{\Theta_j} = \langle \boldsymbol{c_j}, \boldsymbol{V_j}, \boldsymbol{P} \rangle$ is the reachable set from Θ at time instance $j \cdot h$.

Observations

1. The discrete time reachable set doesn't change the predicate associated with the star.



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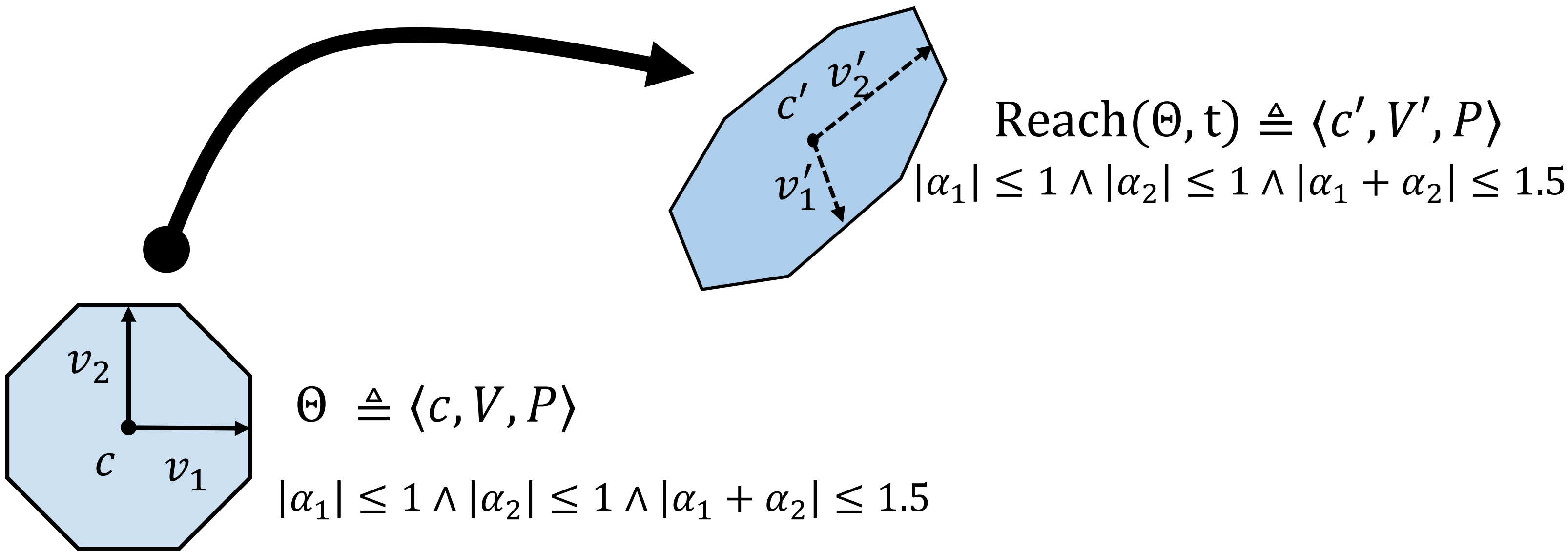
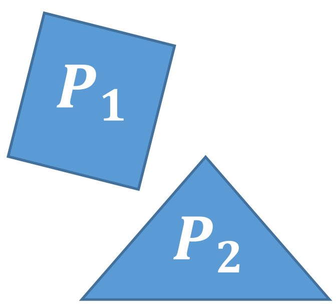


To compute reachable set of a new initial set, just changing the predicate suffices!

Observations

2. It is easy to aggregate and de-aggregate sets on-the-fly.

$\mathbf{\Theta_1} = \langle \boldsymbol{c}, \boldsymbol{V}, \boldsymbol{P_1} \rangle$



$\mathbf{\Theta_2} = \langle \boldsymbol{c}, \boldsymbol{V}, \boldsymbol{P_2} \rangle$

Notice: all have same center and basis in their representation

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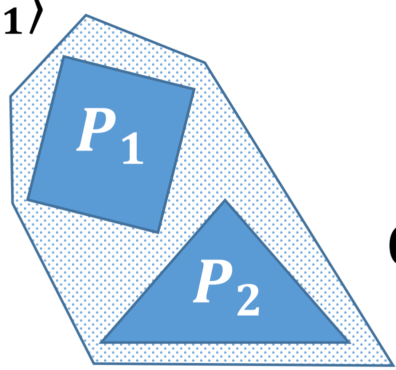
$\boldsymbol{\Theta_1 = \langle c, V, P_1 \rangle}$



$\boldsymbol{\Theta_{agg} = \langle c, V, P_{agg} \rangle}$

$\boldsymbol{(P_1 \vee P_2) \Rightarrow P_{agg}}$

$\boldsymbol{\Theta_2 = \langle c, V, P_2 \rangle}$

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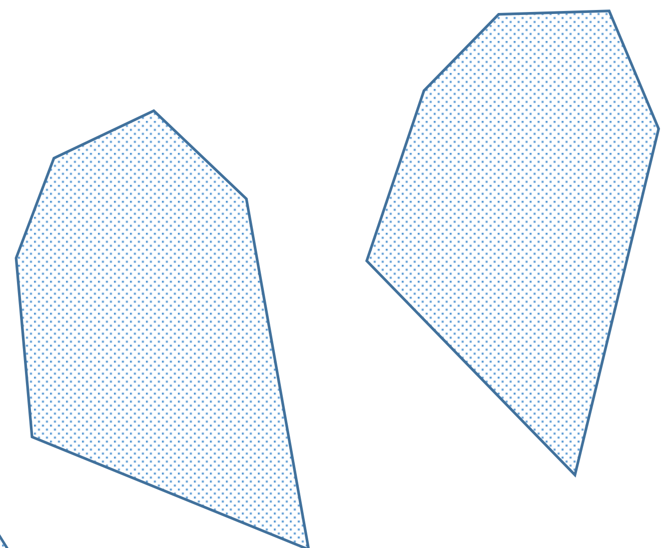
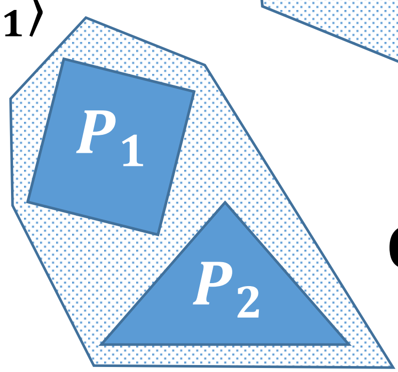


$$\Theta_1 = \langle c, V, P_1 \rangle$$



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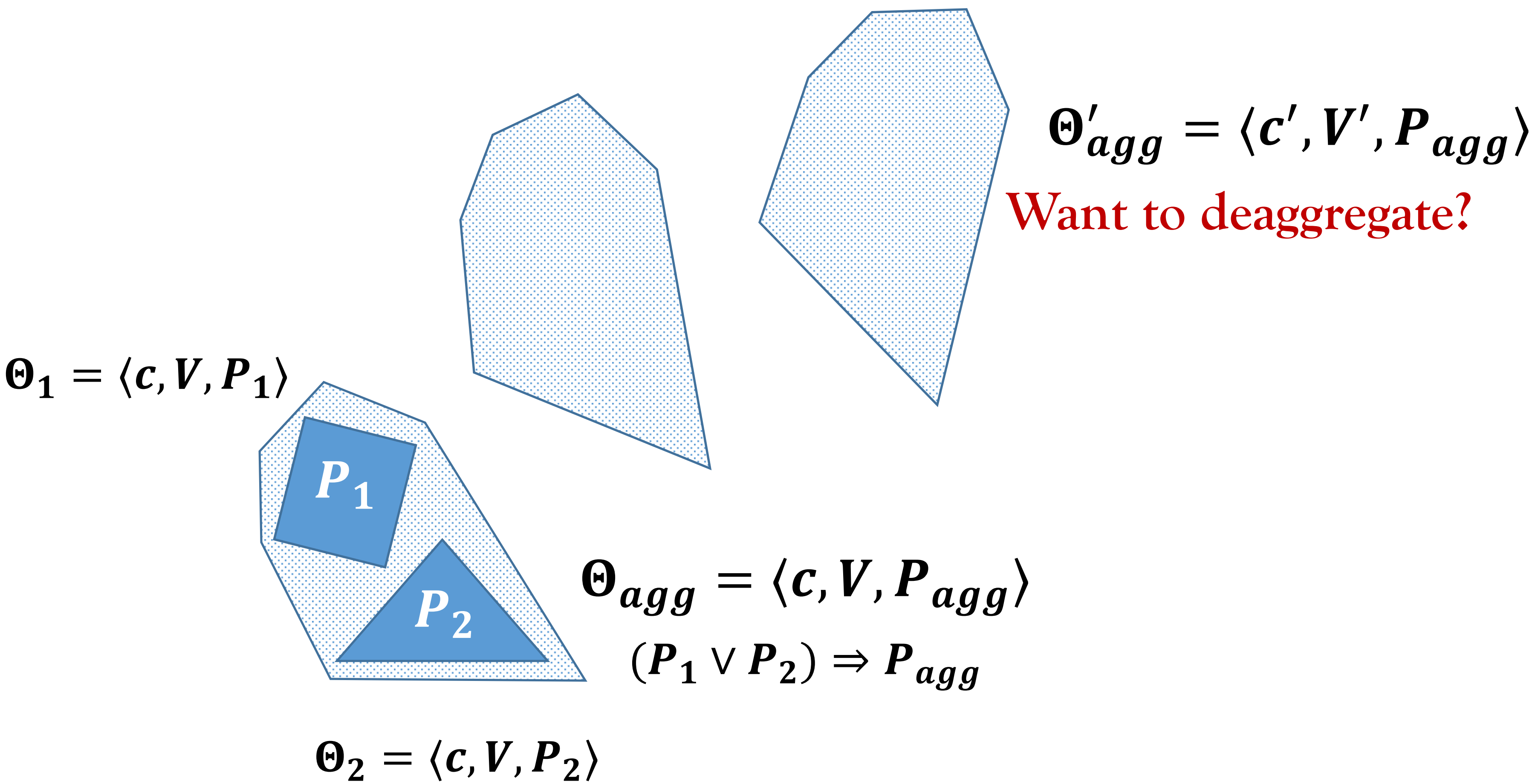

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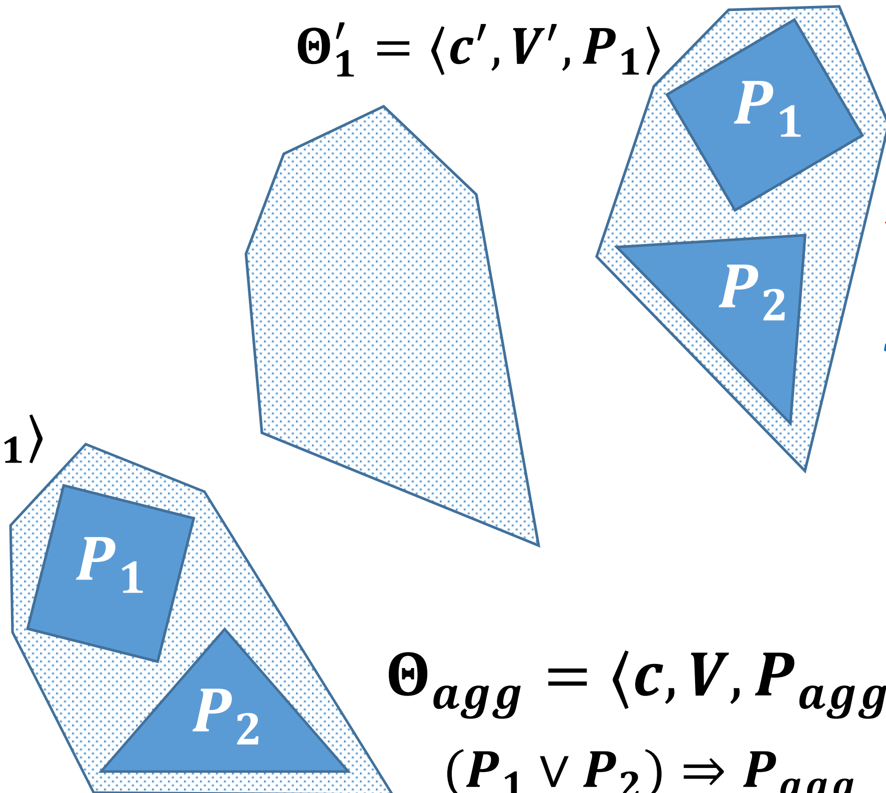
$\Theta_1' = \langle c', V', P_1 \rangle$



$\Theta_{agg}' = \langle c', V', P_{agg} \rangle$

Want to deaggregate?

Just change the predicates!

$\Theta_2' = \langle c', V', P_2 \rangle$

$\Theta_1 = \langle c, V, P_1 \rangle$

$\Theta_{agg} = \langle c, V, P_{agg} \rangle$

$(P_1 \vee P_2) \Rightarrow P_{agg}$

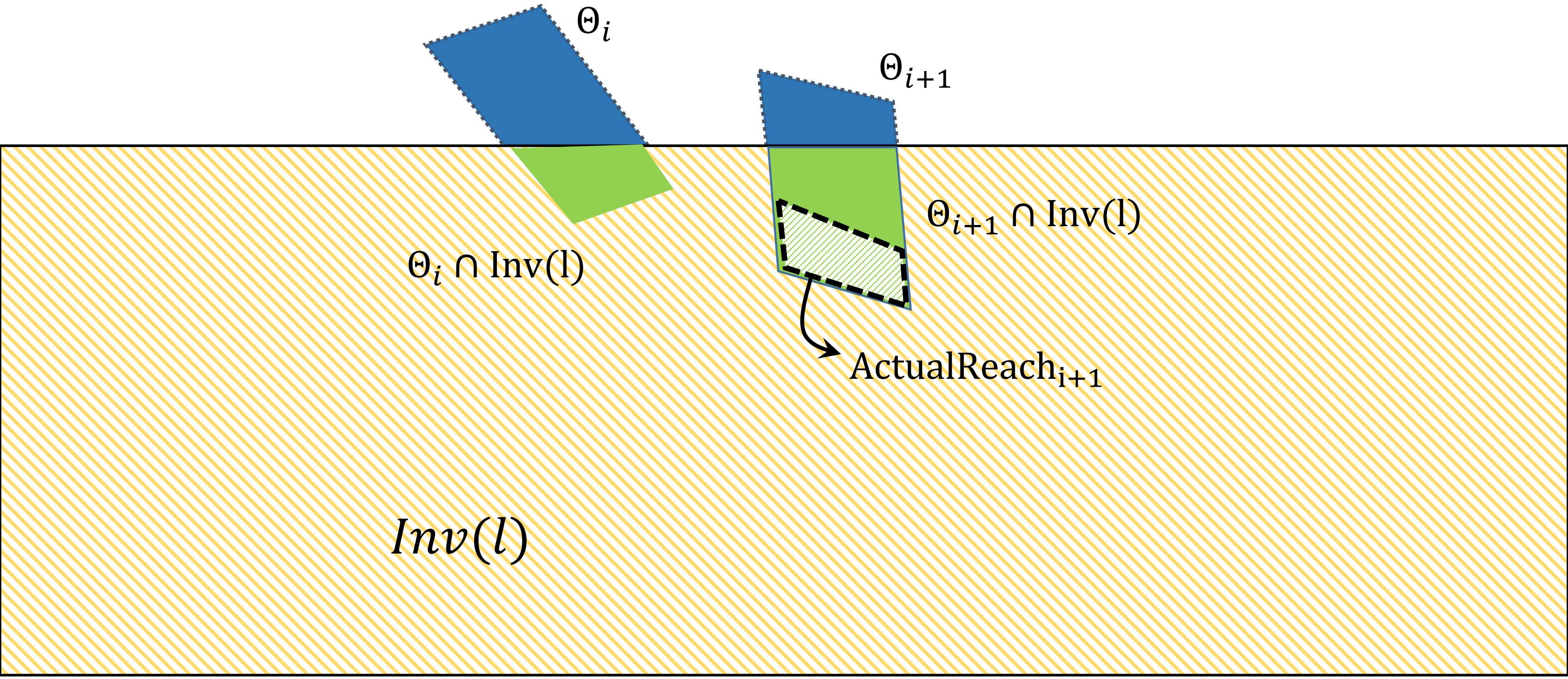
$\Theta_2 = \langle c, V, P_2 \rangle$

Notice: all have same center and basis in their representation

Handling Invariants and Discrete Transitions

The Problems With Invariants

- Given $\Theta_1, \Theta_2, \ldots, \Theta_k$ as discrete time reachable sets for a given mode, performing just $\Theta_j \cap Inv$ only gives an overapproximation.



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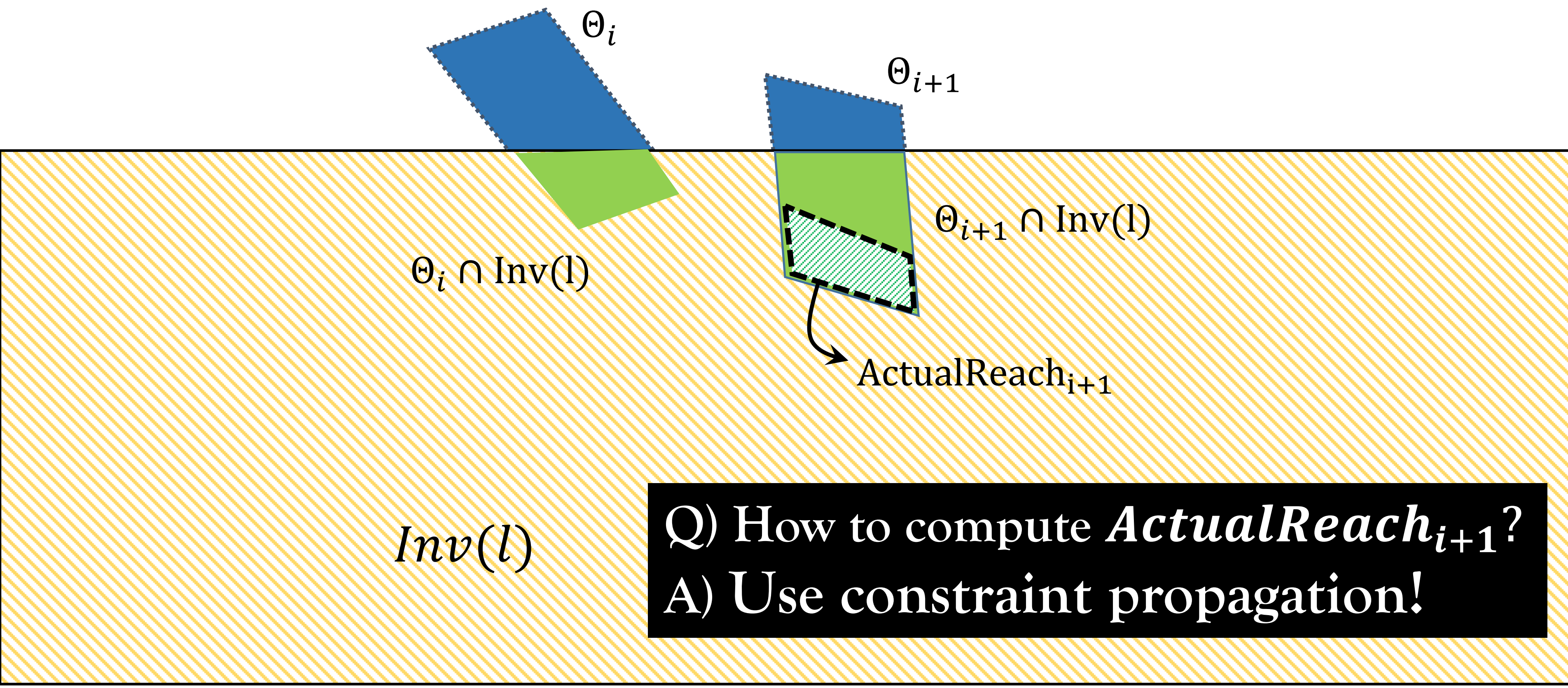
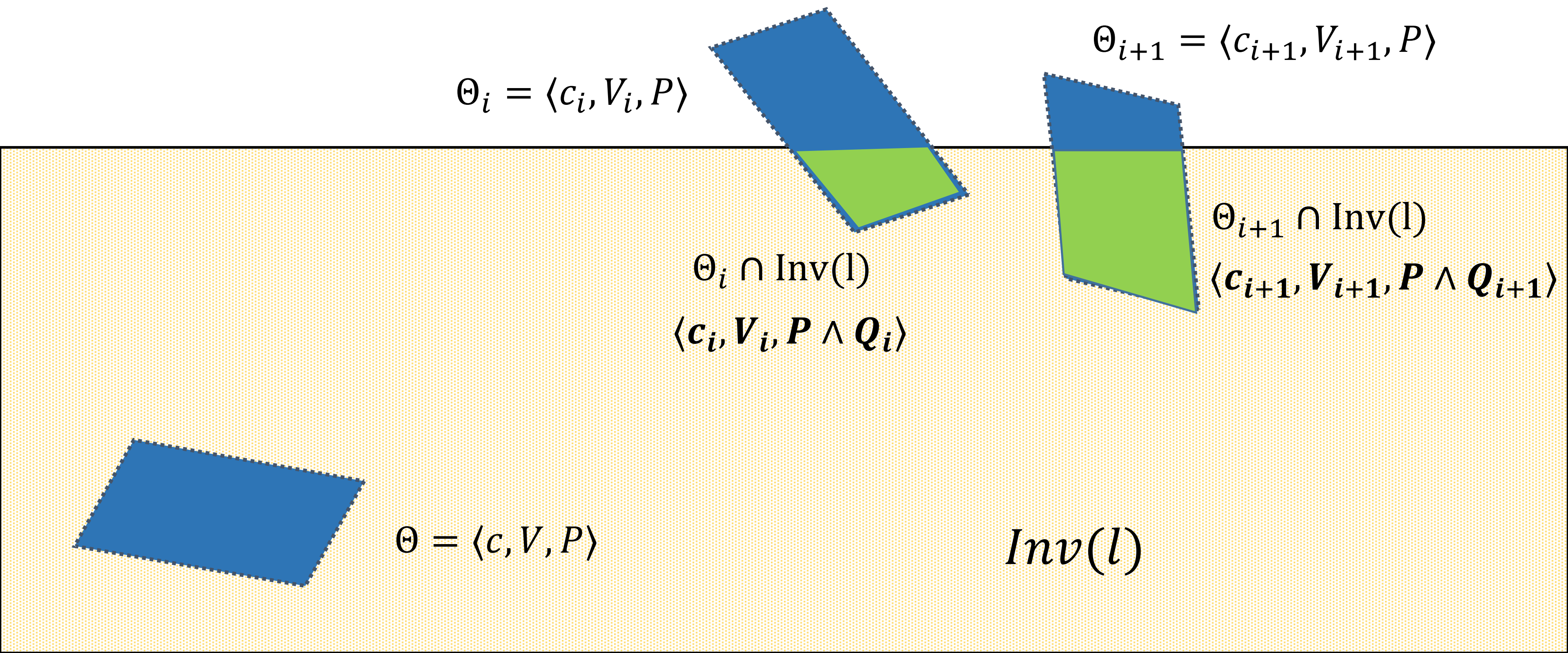


Forward Constraint Propagation

1. Convert Inv into the center and basis of i^{th} star as $\langle c_i, V_i, Q_i \rangle$.
2. $\Theta \cap Inv = \langle c_i, V_i, P \wedge Q_i \rangle$



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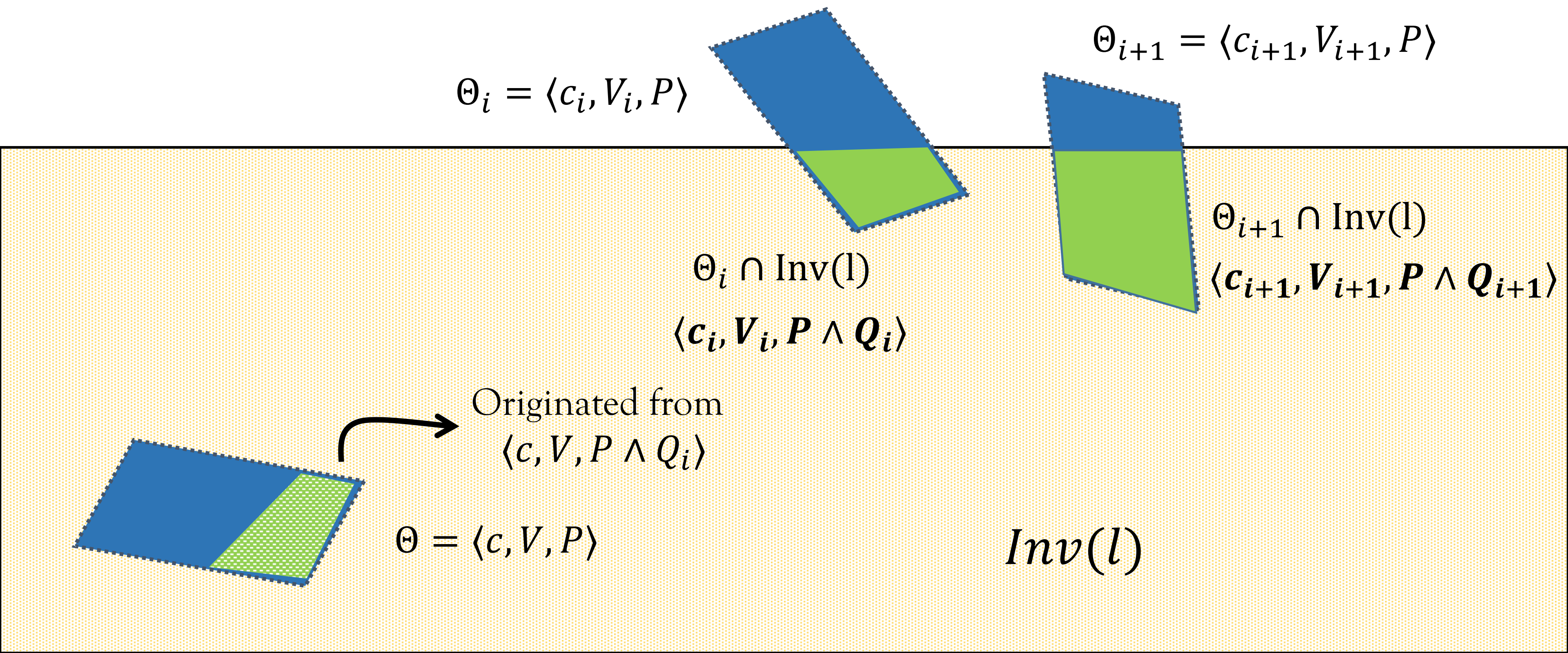

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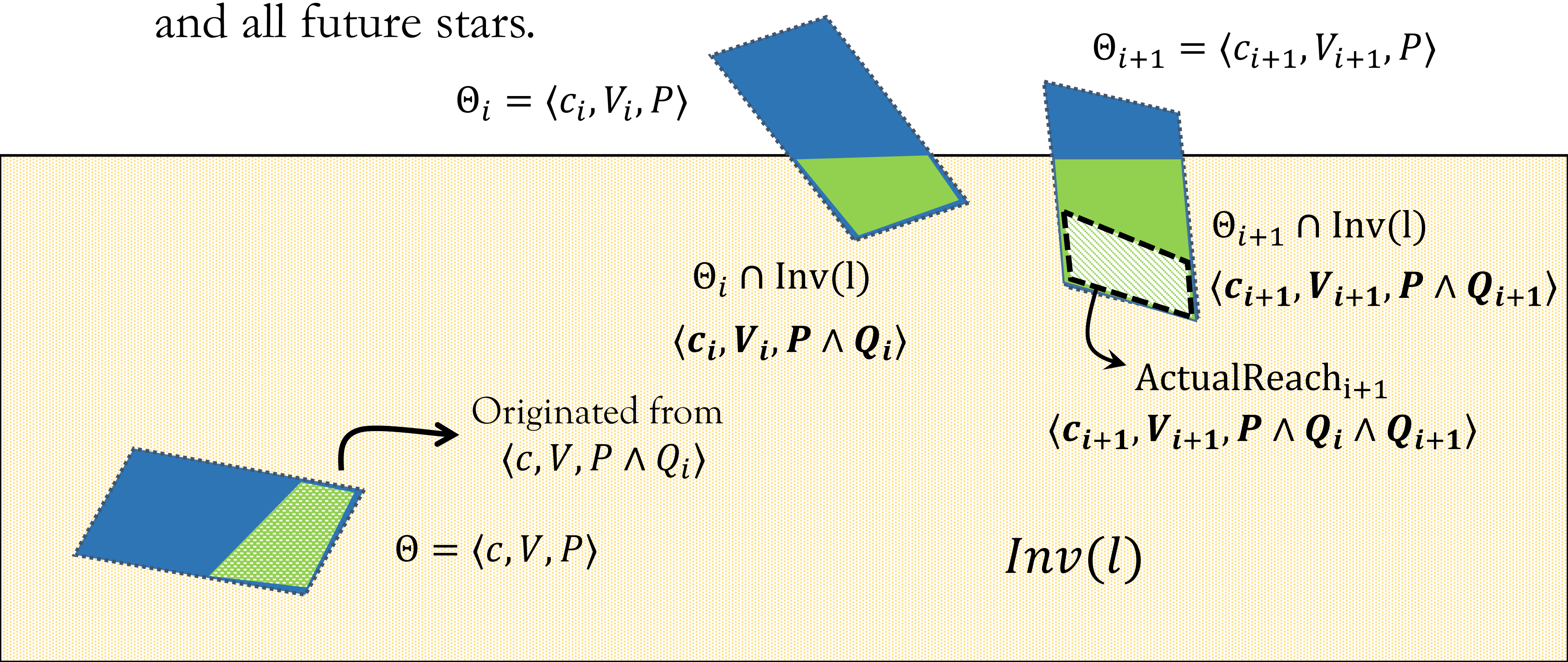
4. Propagate constraint Q_i forward --- add it to predicates of itself and all future stars.



Invariant Constraint Propagation

1. Compute reachable sets $\Theta_1, \Theta_2, \ldots, \Theta_k$.

2. Convert Inv into star representation of Θ_i as $\langle c_1, V_1, Q_1 \rangle, \langle c_2, V_2, Q_2 \rangle, \ldots, \langle c_k, V_k, Q_k \rangle$

3. For each Θ_i, add $Q_1 \wedge Q_2 \wedge \cdots \wedge Q_i$ into its predicate.

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No. of predicates increase linearly with time?

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Optimizations

1. If $\Theta_i \subseteq Inv$, then $P \wedge Q_i \equiv P$. Hence, no constraint is added.

2. If $\Theta_i \subseteq Inv^c$, then $P \wedge Q_i \equiv \bot$. Hence, no need to add Q_i.

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4. **[Empirical heuristic]:** Compare successive constraints Q_i and Q_{i+1} and if Q_{i+1} is stronger than Q_i, replace Q_i with Q_{i+1}.

Discrete Transitions

- Discrete transitions are enabled when the reachable set overlaps with the guard condition.
- If reachable set from Θ overlaps with guard G_i at $\Theta_{i,1}, \Theta_{i,2}, \dots, \Theta_{i,l}$. That is, Θ has l successor sets.
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Solution: Aggregation

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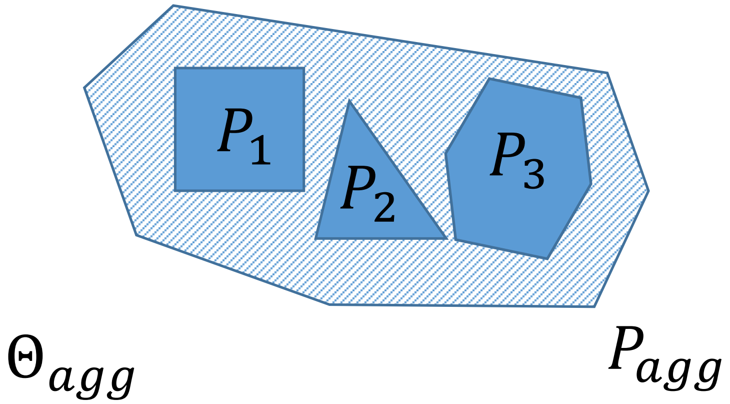
Damned if you do!
Damned if you don't!

Dynamic Aggregation Illustration

1. Aggregate all the sets by default and compute reachable set.



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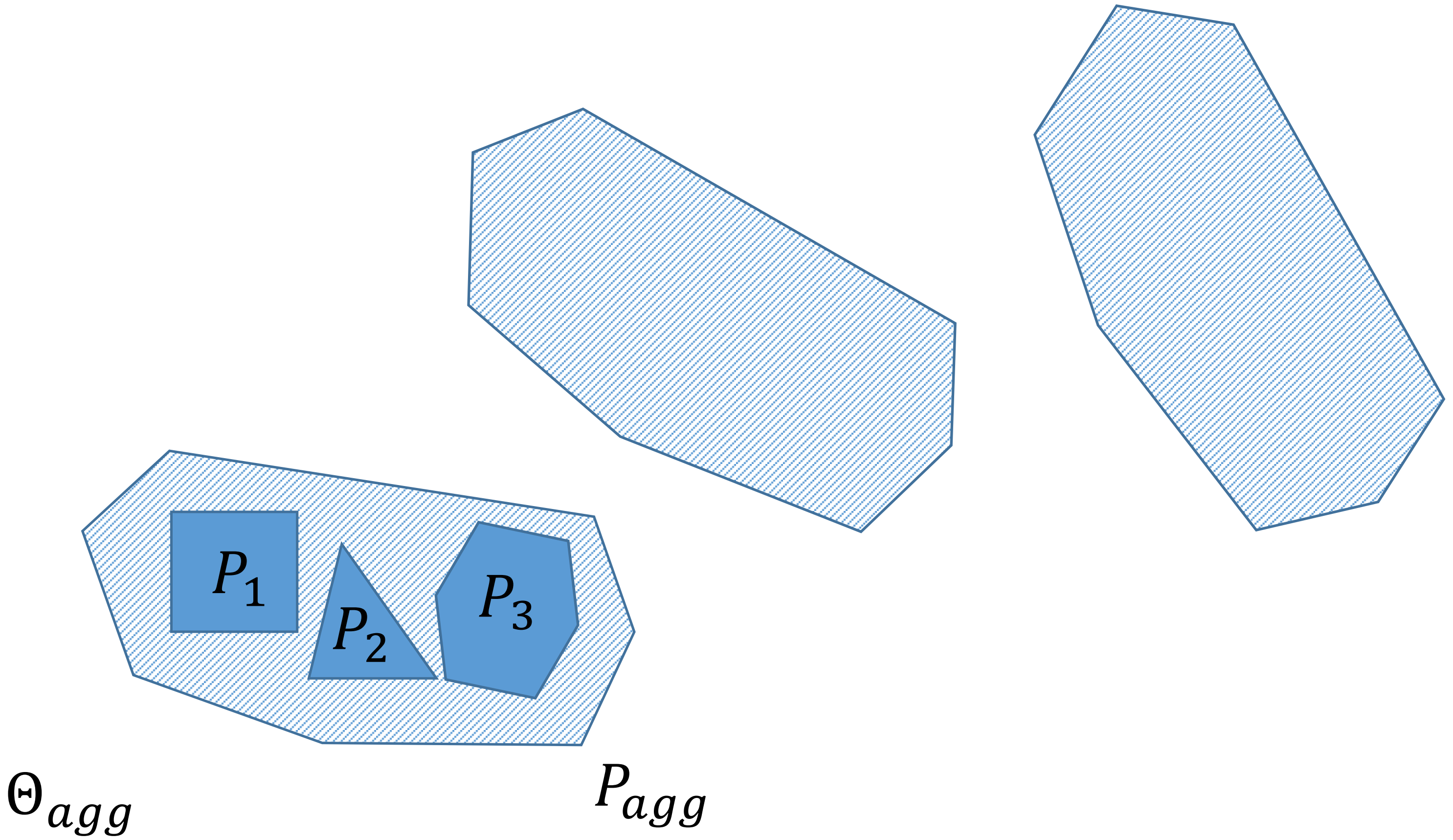
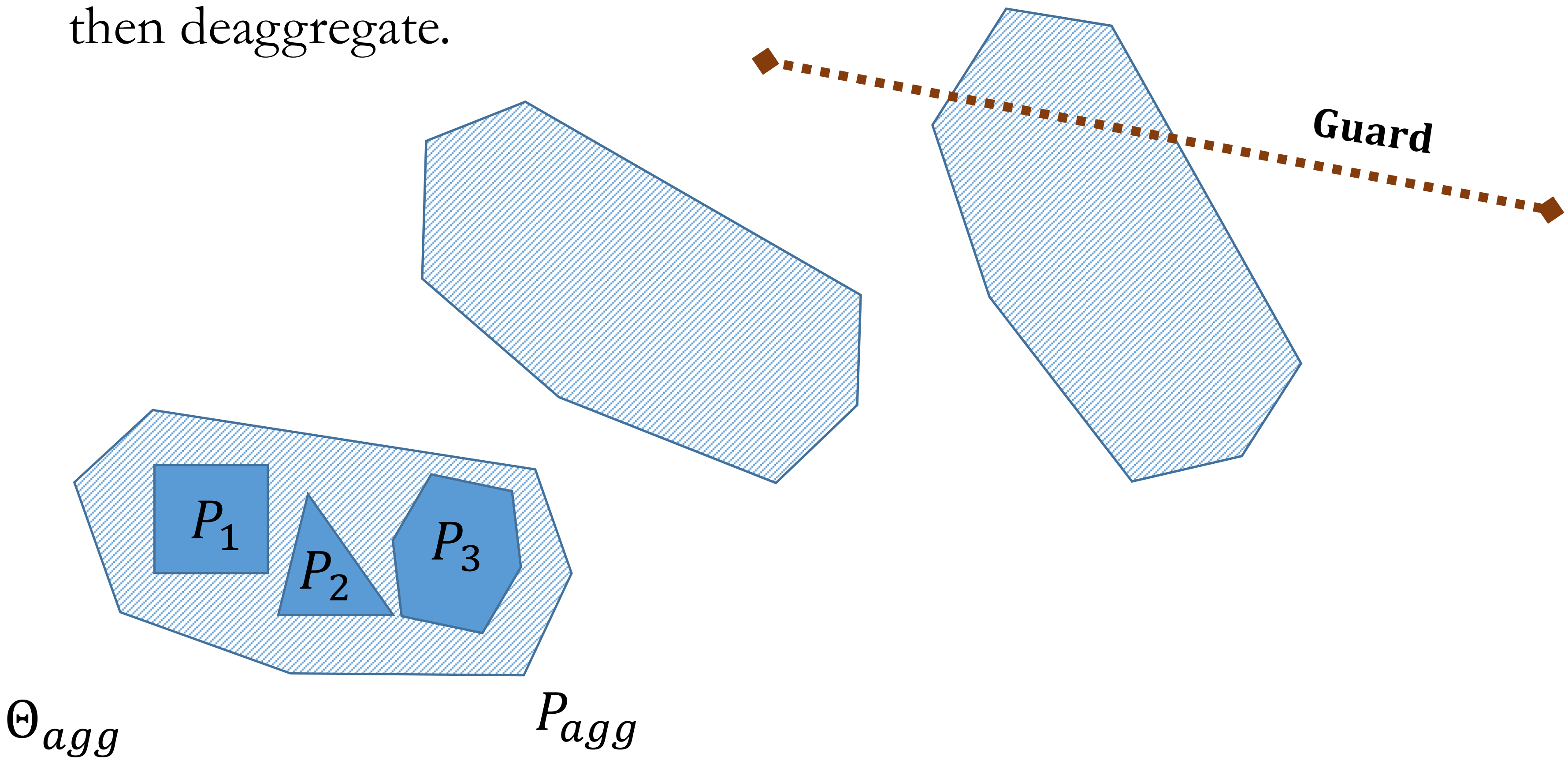


Dynamic Aggregation Illustration

1. Aggregate all the sets by default and compute reachable set.
2. When the aggregated set intersects with a guard or unsafe set, then deaggregate.



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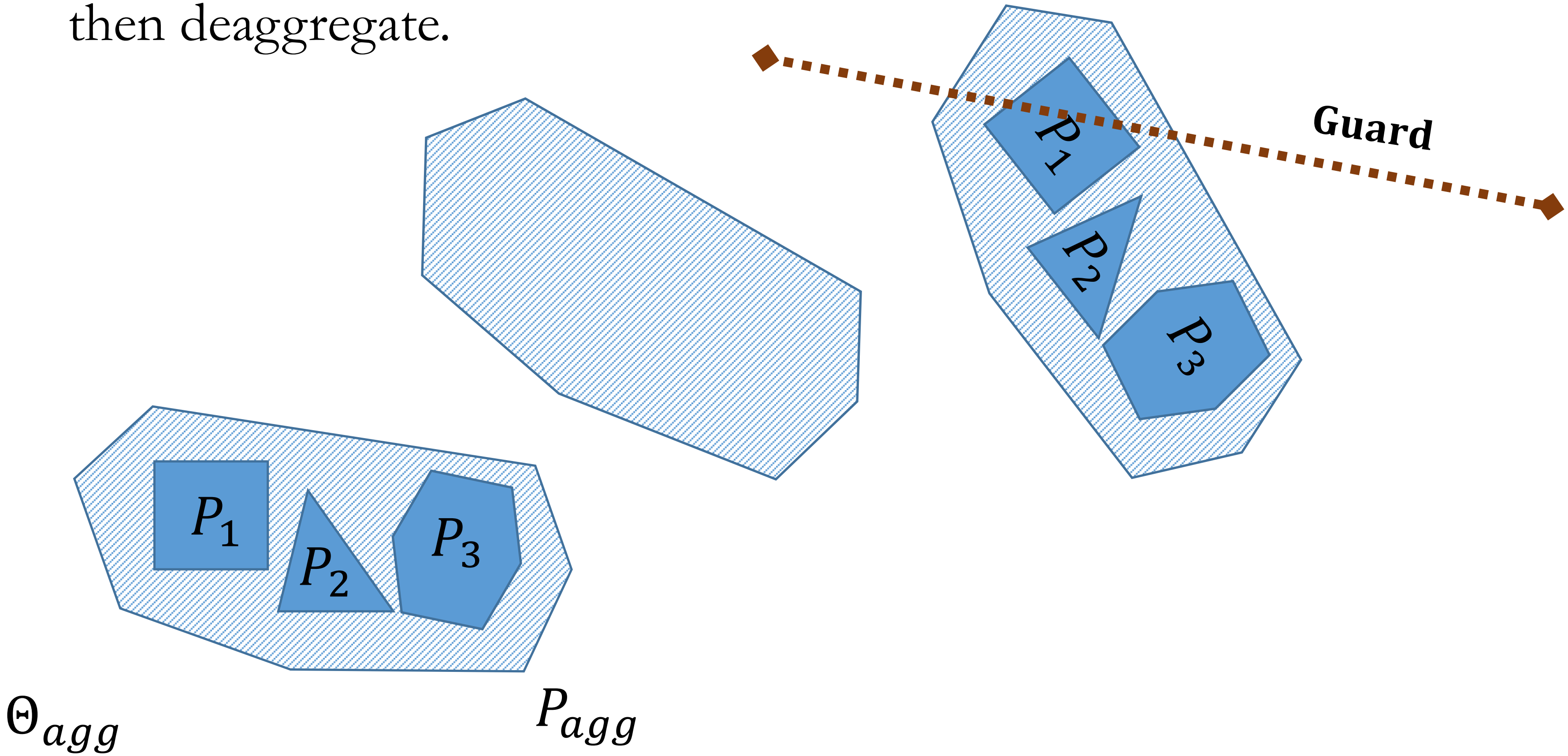

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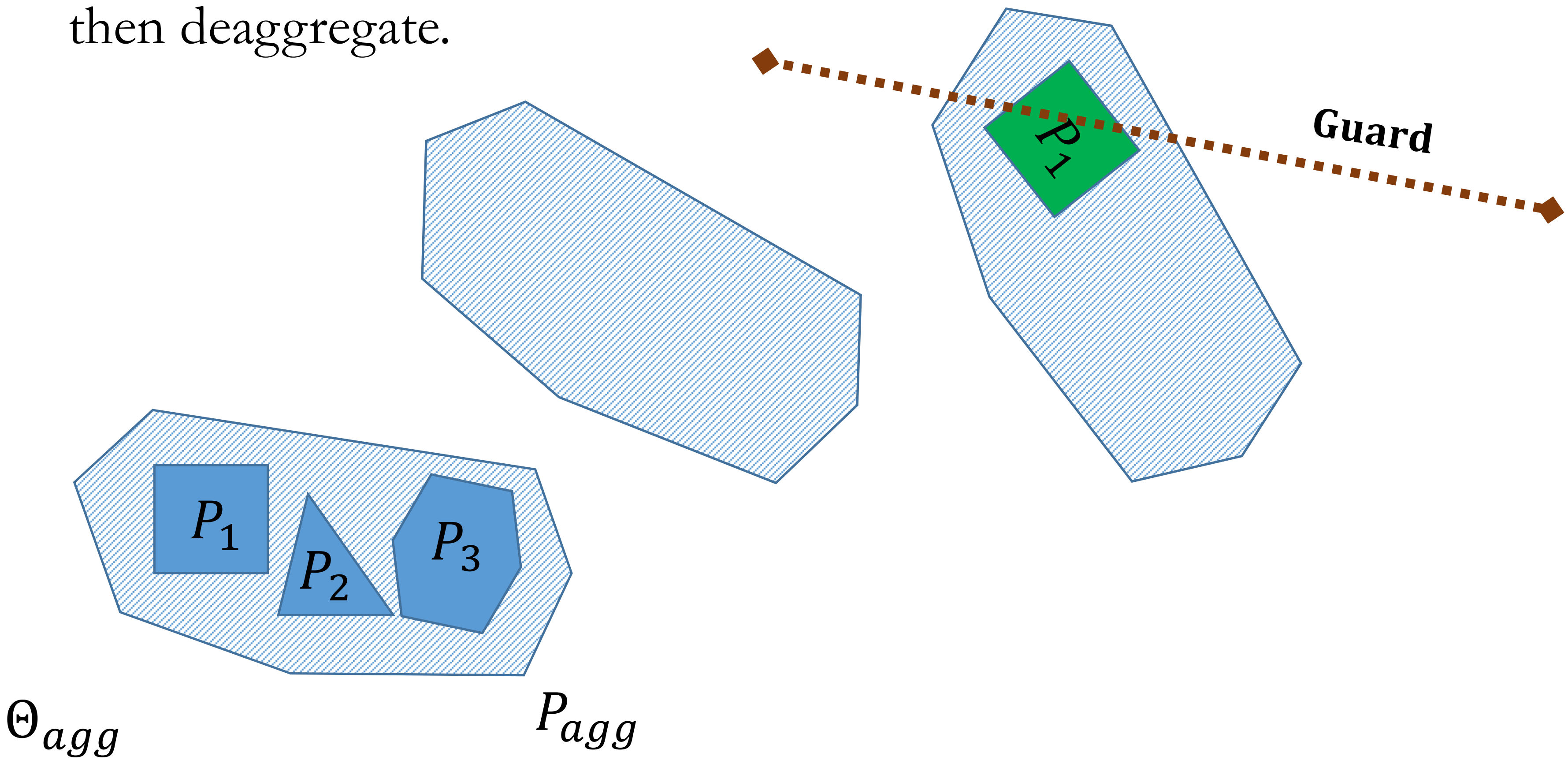
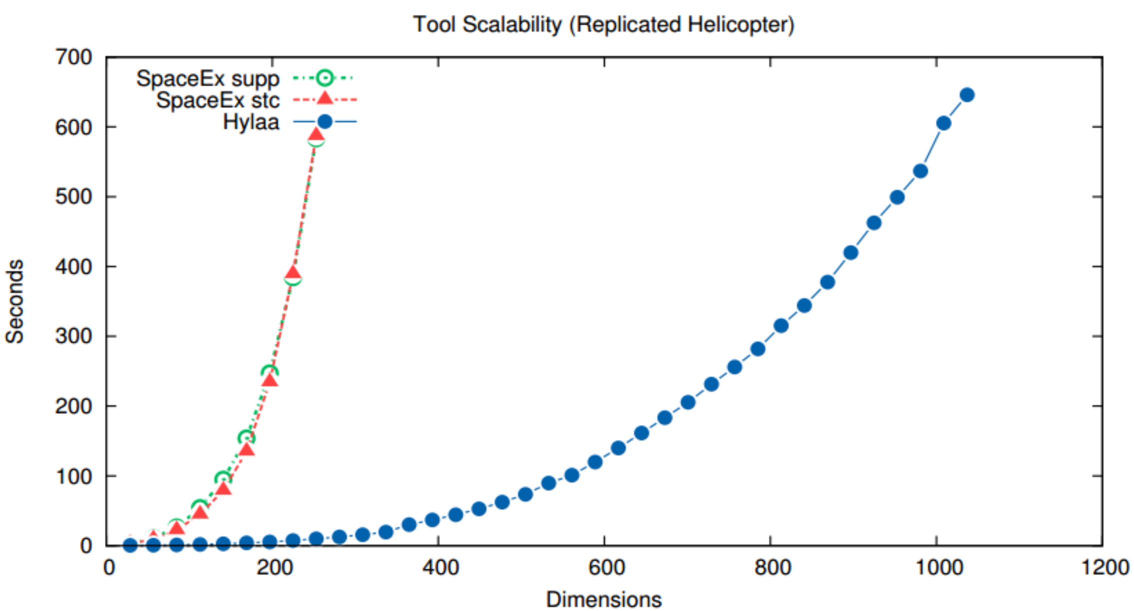


Overview

- ✓Motivation and Contributions.
- ✓Dynamic analysis technique for linear systems verification.
- ✓Observations of the dynamic analysis technique.
- ✓Invariant constraint propagation.
- ✓Dynamic deaggregation.
- ▪ Experimental evaluation.
- ▪ Conclusions and Future work.

Experimental Evaluation
HyLAA

Scalability with respect to number of dimensions.***

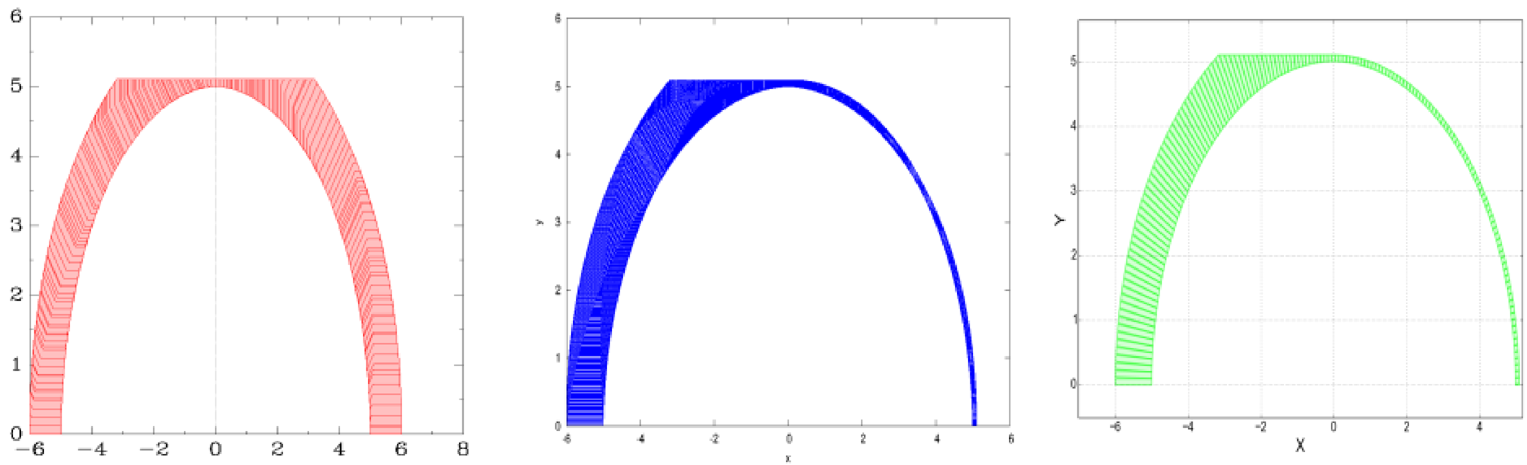


# Dims	supp	stc	HyLAA
29	2.98	2.60	0.42
57	10.93	9.48	0.67
141	94.83	79.23	2.65
253	583.27	587.42	9.79
449	-	-	52.67
1009	-	-	605.38

*** accurate comparison of tools is very hard owing to semantics and parameters during verification. HyPro might be a good solution.

http://stanleybak.com/hylaa/

HyLAA
Constraint Propagation



(a) SpaceEx `stc` (b) Flow* (c) HyLAA

Step	No Trim	Trim
0.05	16	5
0.005	119	9
0.001	576	25
0.0005	1148	45

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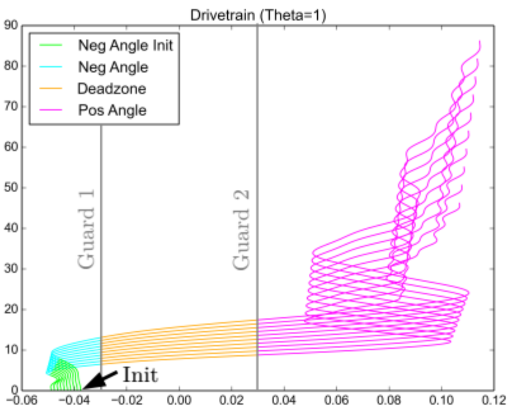
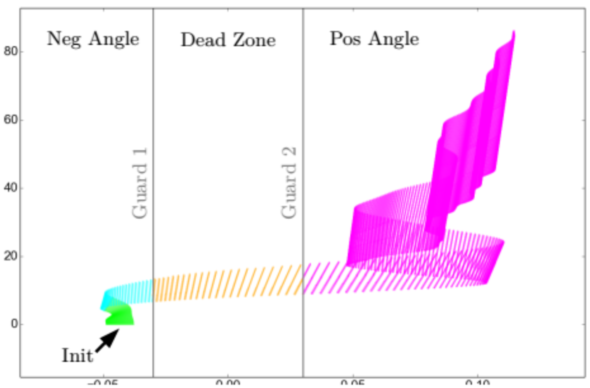
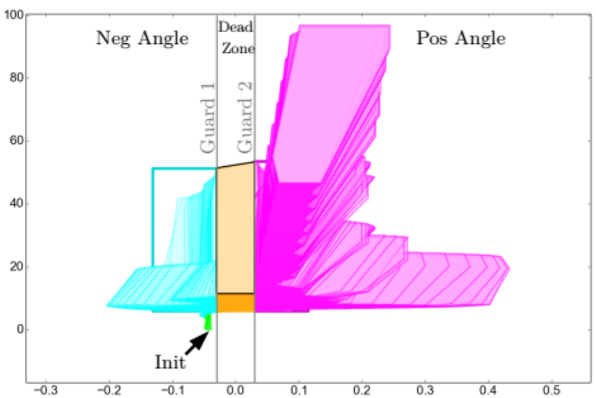
HyLAA
Aggregation and Deaggregation



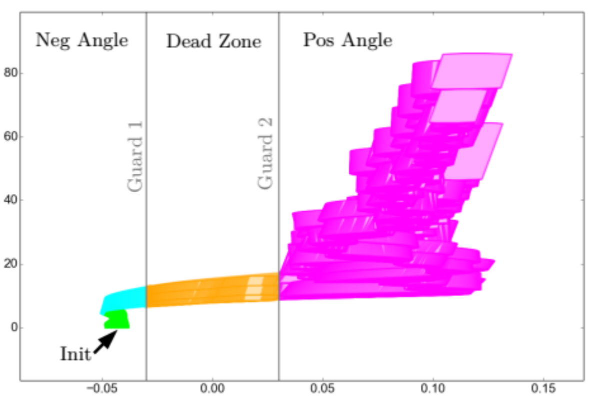
(a) Simulations



(b) Unaggregated



(c) Aggregated (incomplete)



(d) Deaggregated

- Without aggregation is very expensive
- Completely aggregated introduces new transitions and doesn't terminate.
- Dynamic deaggregation has 1.2x – 5x speedup based on the system.

http://stanleybak.com/hylaa/

HyLAA
Aggregation and Deaggregation

# Dims	10	12	14	16	18	20	24	30	42
Deaggregated	25.70	44.94	24.71	131.82	47.72	267.71	450.42	331.57	516.21
Unaggregated	112.94	79.24	98.63	145.87	214.80	409.55	561.47	384.55	672.60

- Automotive drivetrain system with additional masses ($8 + 2\theta$).
- In lower dimensions, the synchronous behavior of masses gives a better performance for aggregation.
- In higher dimensions, the benefits of aggregation are low because deaggregation is performed more often.

http://stanleybak.com/hylaa/

Conclusion

- Notion of simulation equivalent reachable set and safety verification.
- New invariant constraint propagation methods for handling invariants.
- Dynamic aggregation and deaggregation for handling discrete transitions.
- Implemented these in a tool called HyLAA and demonstrated the benefits of these techniques.

Future work

- Giving guarantees over *dense-time* semantics.
- Templates for aggregation and deaggregation.

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