Now Playing:

My Mathematical Mind
Spoon
From *Gimme Fiction*
Released May 10, 2005

PetShop (BYU Students, SIGGRAPH 2006)

Geometric Objects in Computer Graphics

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COMP 575
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Last Time

- Overview of the second half of the semester
- Talked about real cameras and light transport
- Talked about how to turn those ideas into a ray-tracer
  - Generate rays
  - Intersect rays with objects
  - Determine pixel color

Time for some math

- Today we're going to review some of the basic mathematical constructs used in computer graphics
  - Scalars
  - Points
  - Vectors
  - Matrices
  - Other stuff (rays, planes, etc.)

Scalars

- A scalar is a quantity that does not depend on direction
  - In other words, it's just a regular number
    - *i.e.* 7 is a scalar
    - so is 13.5
    - or -4

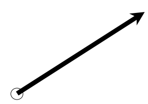
Points

- A point is a list of n numbers referring to a location in n-D
- The individual components of a point are often referred to as coordinates
  - *i.e.* (2, 3, 4) is a point in 3-D space
    - This point's x-coordinate is 2, it's y-coordinate is 3, and it's z-coordinate is 4

Vectors

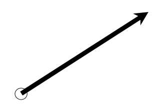
- A vector is a list of n numbers referring to a direction (and magnitude) in n-D
  - *i.e.*
- N.B. - From a data structures perspective, a vector looks exactly the same as a point
  - This will be important later

Rays



- A ray is just a vector with a starting point
  - Ray = (Point, Vector)

Rays



- Let a ray be defined by point **p** and vector **d**
- The parametric form of a ray expresses it as a function as some scalar t, giving the set of all points the ray passes through:
  - $r(t) = \mathbf{p} + t\mathbf{d}, 0 \leq t \leq \infty$

Vectors

- We said that a vector encodes a direction and a magnitude in n-D
  - How does it do this?
- Here are two ways to denote a vector in 2-D:

$$\mathbf{V} = \langle V_x, V_y \rangle$$

$$\mathbf{V} = \begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

Vector Magnitude

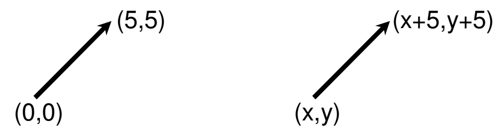
- Geometrically, the magnitude of a vector is the Euclidean distance between its start and end points, or more simply, it's length
- Vector magnitude in n-D: $||\mathbf{V}|| = \sqrt{\sum_{i=1}^{n} V_i^2}$
- Vector magnitude in 2-D: $||\mathbf{V}|| = \sqrt{V_x^2 + V_y^2}$

Normalized Vectors

- Most of the time, we want to deal with normalized, or unit, vectors
- This means that the magnitude of the vector is 1: $||\mathbf{V}|| = 1$
- We can normalize a vector by dividing the vector by its magnitude:
  - N.B. The '^' denotes a normalized vector $\hat{V} = \frac{V}{||\mathbf{V}||}$

Question

- Are these two vectors the same?
  - (x,y) != (0,0)

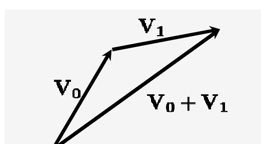


- A: Yes and no
- They are the same displacement vectors, which is what we will normally care about

Vector Addition

- Vectors are closed under addition
  - Vector + Vector = Vector

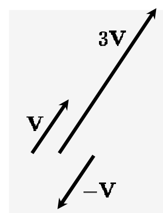
Vector Addition



$$\mathbf{V}_0 + \mathbf{V}_1 = \begin{bmatrix} V_{0,x} + V_{1,x} \\ V_{0,y} + V_{1,y} \end{bmatrix}$$

Vector Scaling

- Vectors are closed under multiplication with a scalar
  - Scalar * Vector = Vector



Vector Scaling

$$a\mathbf{V} = \begin{bmatrix} aV_x \\ aV_y \end{bmatrix}$$

Properties of Vector Addition & Scaling

Addition is Commutative	$\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}$
Addition is Associative	$(\mathbf{P} + \mathbf{Q}) + \mathbf{R} = \mathbf{P} + (\mathbf{Q} + \mathbf{R})$
Scaling is Commutative and Associative	$(ab)\mathbf{P} = a(b\mathbf{P})$
Scaling and Addition are Distributive	$a(\mathbf{P} + \mathbf{Q}) = a\mathbf{P} + a\mathbf{Q}$
	$(a + b)\mathbf{P} = a\mathbf{P} + b\mathbf{P}$

Points and Vectors

- Can define a vector by 2 points
  - Point - Point = Vector
- Can define a new point by a point and a vector
  - Point + Vector = Point

Linear Interpolation

- Can define a point in terms of 2 other points and a scalar
  - Given points **P**, **R**, **Q** and a scalar a
    - **P** = a**R** + (1 - a)**Q**
    - How does this work?
      - It's really **P** = **Q** + a**V**
        - **V** = **R** - **Q**
      - Point + Vector = Point

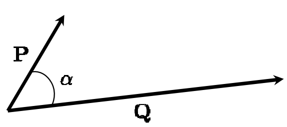
Vector Multiplication?

- What does it mean to multiply two vectors?
  - Not uniquely defined
- Two product operations are commonly used:
  - Dot (scalar, inner) product
    - Result is a scalar
  - Cross (vector, outer) product
    - Result is a new vector

Dot Product

$$\mathbf{P}\cdot\mathbf{Q}=\sum_{i=1}^{n}P_iQ_i=\begin{bmatrix}P_1 & P_2 & \dots & P_n\end{bmatrix}\begin{bmatrix}Q_1\\Q_2\\\dots\\Q_n\end{bmatrix}$$

$$\mathbf{P}\cdot\mathbf{Q}=||\mathbf{P}||\;||\mathbf{Q}||\;\cos\alpha$$

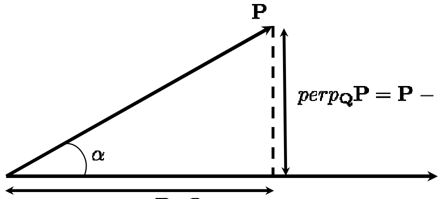


$$\alpha=cos^{-1}\left(\frac{\mathbf{P}\cdot\mathbf{Q}}{||\mathbf{P}||\;||\mathbf{Q}||}\right)$$

Properties of Vector Dot Products

Commutative	$\mathbf{P}\cdot\mathbf{Q}=\mathbf{Q}\cdot\mathbf{P}$
Associative with Scaling	$(a\mathbf{P})\cdot\mathbf{Q}=a(\mathbf{P}\cdot\mathbf{Q})$
Distributive with Addition	$\mathbf{P}\cdot(\mathbf{Q}+\mathbf{R})=\mathbf{P}\cdot\mathbf{Q}+\mathbf{P}\cdot\mathbf{R}$
	$\mathbf{P}\cdot\mathbf{P}=\|\|\mathbf{P}\|\|^2$
	$\|\mathbf{P}\cdot\mathbf{Q}\|\leq\|\|\mathbf{P}\|\|\;\|\|\mathbf{Q}\|\|$

Perpendiculars and Projections



$$perp_{\mathbf{Q}}\mathbf{P}=\mathbf{P}-\frac{\mathbf{P}\cdot\mathbf{Q}}{||\mathbf{Q}||^2}\mathbf{Q}$$

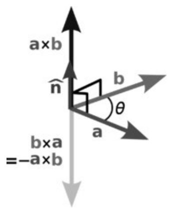
$$proj_{\mathbf{Q}}\mathbf{P}=\frac{\mathbf{P}\cdot\mathbf{Q}}{||\mathbf{Q}||^2}\mathbf{Q}$$

$$||proj_{\mathbf{Q}}\mathbf{P}||=||\mathbf{P}||\cos\alpha$$

Dot Product Application: Lighting

- P • Q = ||P|| ||Q|| cos a
- So what does this mean if P and Q are normalized?
  - Can get cos a for just 3 multiplies and 2 adds
    - Very useful in lighting and shading calculations
    - Example: Lambert's cosine law

Cross Product



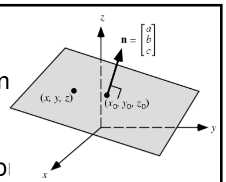
$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

$\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$

Cross Product Application: Normals

- A <u>normal</u> (or <u>surface normal</u>) is a vector that is perpendicular to a surface at a given point
  - This is often used in lighting calculations
- The cross product of 2 orthogonal vectors on the surface is a vector perpendicular to the surface
  - Can use the cross product to compute the normal

Planes



- How can we define a plan
  - 3 non-linear points
    - Use linear interpolatio
  - A perpendicular vector and an incident point
    - $\mathbf{n} \cdot (\mathbf{x}-\mathbf{x}_0) = 0$
      - $ax + by + cz + d = 0$
    - <u>Hessian normal form</u>: Normalize **n** first
      - $\mathbf{n} \cdot \mathbf{x} = -p$

Columns and Rows

- In this class, we will generally assume that a list forms a column vector:

$$(a, b, c, d) \Longrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

- The reason for this will become clear when we talk about matrices

Matrices

- Reminder: A matrix is a rectangular array of numbers
  - An m x n matrix has m rows and n columns
- M_{ij} denotes the entry in the i-th row and j-th column of matrix M
  - These are generally thought of as1-indexed (instead of 0-indexed)

Matrices

- Here, M is a 2x5 matrix:

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \end{bmatrix}$$

Matrix Transposes

- The transpose of an m x n matrix is an n x m matrix
  - Denoted M^T
  - $M^T_{ij} = M_{ji}$

$$\mathbf{M}^T = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \end{bmatrix}^T = \begin{bmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \\ M_{13} & M_{23} \\ M_{14} & M_{24} \\ M_{15} & M_{25} \end{bmatrix}$$

Matrix Addition

- Only well defined if the dimensions of the 2 matrices are the same
  - That is, $m_1 = m_2$ and $n_1 = n_2$
  - Here, M and G are both 2 x 5

$$(\mathbf{M}+\mathbf{G})_{ij} = M_{ij} + G_{ij}$$

$$\mathbf{M}+\mathbf{G} = \begin{bmatrix} M_{11}+G_{11} & M_{12}+G_{12} & M_{13}+G_{13} & M_{14}+G_{14} & M_{15}+G_{15} \\ M_{21}+G_{21} & M_{22}+G_{22} & M_{23}+G_{23} & M_{24}+G_{24} & M_{25}+G_{25} \end{bmatrix}$$

Matrix Scaling

- Just like vector scaling
  - Matrix * Scalar = Matrix

$$(a\mathbf{M})_{ij} = aM_{ij}$$

$$a\mathbf{M} = \begin{bmatrix} aM_{11} & aM_{12} & aM_{13} & aM_{14} & aM_{15} \\ aM_{21} & aM_{22} & aM_{23} & aM_{24} & aM_{25} \end{bmatrix}$$

Properties of Matrix Addition and Scaling

Addition is Commutative	$\mathbf{F}+\mathbf{G} = \mathbf{G}+\mathbf{F}$
Addition is Associative	$(\mathbf{F}+\mathbf{G})+\mathbf{H} = \mathbf{F}+(\mathbf{G}+\mathbf{H})$
Scaling is Associative	$a(b\mathbf{F}) = (ab)\mathbf{F}$
Scaling and Addition are Distributive	$a(\mathbf{F}+\mathbf{G}) = a\mathbf{F}+a\mathbf{G}$ $(a+b)\mathbf{F} = a\mathbf{F}+b\mathbf{F}$

Matrix Multiplication

- Only well defined if the number of columns of the first matrix and the number of rows of the second matrix are the same
  - Matrix * Matrix = Matrix
  - *i.e.* if F is m x n, and G is n x p, then FG if m x p
- Let's do an example

$$(\mathbf{FG})_{ij} = \sum_{k=1}^{m} F_{ik} G_{kj}$$

The Identity Matrix

- Defined such that the product of any matrix M and the identity matrix I is M
  - **IM** = **MI** = **M**
- Let's derive it
- The identity matrix is a square matrix with ones on the diagonal and zeros elsewhere

The Identity Matrix

- Defined such that the product of any matrix M and the identity matrix I is M
  - **IM** = **MI** = **M**
- Let's derive it
- The identity matrix is a square matrix with ones on the diagonal and zeros elsewhere

$$(\mathbf{I}_n)_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Linear Systems, Matrix Inverses, etc.

- I'm not planning to cover this material in this course
- If there is any interest in going over this, let me know and I'll cover it on Tuesday

Next Time

- Going to cover coordinate systems and transforms, focusing on 2D