UNC-Chapel Hill, COMP 145

Team 18: The Kalman Filter Learning Tool
Dynamic and Measurement Models

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1. DYNAMIC (PROCESS) MODELS

We allow the dynamic model of the water level, also known as the *process model*, to be modled as the sum of three possible components: a constant component, a steadily increasing or decreasing component, and a sinusoidal component. The simplest model would consist of the constant component alone; the most complicated would consist of the combination (sum) of all three.

In any case, the primary element of the *state* of the system is the water level L as shown in Figure 5. We represent the *true* state of the system as $\bar{x}$ and the estimated state as $\hat{x}$. The estimated process covariance is P, and the process noise (covariance) matrix is Q. The state and the covariance matrices are all the same dimension, with the dimensionality depending on the specific combination of dynamic components.

1.1 Actual Dynamics (Truth)

Here we look at the three components of the *actual* water level dynamics in the most fundamental forms. The actual water level would be modeled as summed combinations of these components. The valid combinations are the same as those in Section 1.2, where I cover the modeled (Kalman filter) formulations.

1.1.1 Constant

In this case, the water level L does not change. In other words,

$$L(t) = c \tag{1}$$

for some constant c.

1.1.2 Filling (Steady Increase)

In this case the water level is increasing or decreasing at a constant rate r. In other words,

$$\frac{d}{dt}L(t) = r \tag{2}$$

for some $r > 0$.

1.1.3 Sloshing (Sinusoidal)

In this case the water level is changing as sinusoidal function of time. In other words,

$$L(t) = k_s \sin(\omega t + \phi) \tag{3}$$

and

$$\frac{d}{dt}L(t) = k_s \omega \cos(\omega t + \phi).$$

1.2 The Models (Kalman Filter)

Here I describe the finite set of possible KF (and EKF) models that we will use. Each combination would have a corresponding *truth* signal that could be generated using the models in Section 1.1, however of course the user will be able to select combinations of *actual* and *modeled* dynamics that do not match.

1.2.1 Constant

In this case the estimated state would have only one element, i.e. it would be the scalar water level, simply $\hat{x}_L \equiv L$. The continuous time process model is depicted in Figure 1.

Because the level is modeled as constant, the *continuous time state transition matrix* would be simply $A_c = 0$ and the continuous time process noise matrix would be $Q_c = q_c$. The

$$\nu_c \longrightarrow \int \xrightarrow{\hat{x}_L} \qquad \nu_c \sim \mathrm{N}(0, q_c)$$

Figure 1: Constant level process model.

corresponding *discrete time* state transition matrix is $A(\delta t) = 1$, and from (17) the discrete time process noise matrix would be $Q(\delta t) = q_c \delta t$, where $0 < \sqrt{q_c} \ll F$ for the full water level F. The *time update* equations are simply

$$\hat{x}^{+}(t + \delta t) = \hat{x}(t)$$

and

$$\begin{aligned} P^{+}(t + \delta t) &= A(\delta t) P(t) A^{T}(\delta t) + Q(\delta t) \\ &= P(t) + q \delta t \end{aligned}$$

1.2.2 Filling

In this case the estimated state would have *two* elements, the current water level $\hat{x}_L \equiv L$ and the water fill rate,

$$\hat{x}_r \equiv \frac{d\hat{x}_L}{dt}.$$

as in (2). The continuous time process model is depicted in Figure 2.

$$\nu_f \longrightarrow \int \xrightarrow{\hat{x}_r} \int \xrightarrow{\hat{x}_L} \qquad \nu_f \sim \mathrm{N}(0, q_f)$$

Figure 2: Filling process model.

The overall state is then

$$\hat{x} = \begin{bmatrix} \hat{x}_L \\ \hat{x}_r \end{bmatrix}.$$

The *continuous time* state transition matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the *continuous time* process noise matrix is

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & q_f \end{bmatrix}.$$

The *discrete time* state transition matrix is

$$A(\delta t) = \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix} \tag{4}$$

and from (17) the *discrete time* process noise matrix is

$$Q(\delta t) = \begin{bmatrix} \frac{q_f \delta t^3}{3} & \frac{q_f \delta t^2}{2} \\ \frac{q_f \delta t^2}{2} & q_f \delta t \end{bmatrix} \tag{5}$$

where $0 < \sqrt{q_f} \ll F$, for the full water level F. The filter update equations are the usual linear Kalman filter equations:

$$\hat{x}^+(t + \delta t) = A(\delta t)\hat{x}(t)$$

and

$$P^+(t + \delta t) = A(\delta t)P(t)A^T(\delta t) + Q(\delta t)$$

1.2.3 Sloshing

In this case the estimated state would have *two* elements, the current water level $\hat{x}_L \equiv L$ and the magnitude of the sinusoidal component $\hat{x}_s \equiv k_s$ for some constant k_s. The continuous time process model is depicted in Figure 3.

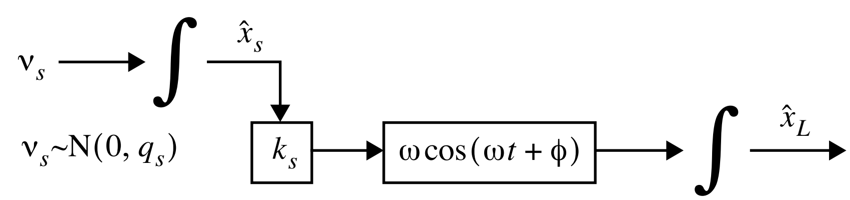

Figure 3: Sloshing process model.

In state form we have,

$$\hat{x} = \begin{bmatrix} \hat{x}_L \\ \hat{x}_s \end{bmatrix}.$$

The present water level would be modeled as function of both the previous level and the changing sinusoidal components. For the sake of simplicity we will assume that ω is known and $\phi = 0$.

We model the new state $\hat{x}$ at time $t + \delta t$ as

$$\begin{aligned} \hat{x}_L(t + \delta t) &= \hat{x}_L(t) + \hat{x}_s(t)\omega\cos(\omega t) \\ \hat{x}_s(t + \delta t) &= \hat{x}_s(t) \end{aligned}.$$

The *continuous time* state transition matrix is

$$A = \begin{bmatrix} 0 & \omega\cos(\omega t) \\ 0 & 0 \end{bmatrix}$$

and the *continuous time* process noise matrix is

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & q_s \end{bmatrix}.$$

The *discrete time* state transition matrix is

$$A(t) = \begin{bmatrix} 1 & \omega\cos(\omega t) \\ 0 & 1 \end{bmatrix} \tag{6}$$

and from (17) the *discrete time* process noise matrix is

$$Q(t, \delta t) = \begin{bmatrix} \frac{q_s\gamma^2\delta t(3t^2 + 3t\delta t + \delta t^2)}{3} & \frac{q_s\gamma\delta t\beta}{2} \\ \frac{q_s\gamma\delta t\beta}{2} & q_s\delta t \end{bmatrix} \tag{7}$$

where $\gamma = \omega\cos(\omega t)$, $\beta = 2t + \delta t$, and $0 < \sqrt{q_s} \ll F$, for the full water level F. The filter update equations are the usual linear Kalman filter equations:

$$\hat{x}^+(t + \delta t) = A(\delta t)\hat{x}(t)$$

and

$$P^+(t + \delta t) = A(\delta t)P(t)A^T(\delta t) + Q(\delta t)$$

1.2.4 Filling + Sloshing

In this case the estimated state would have *three* elements, the present water level $\hat{x}_L \equiv L$, the magnitude of the sinusoidal component $\hat{x}_s \equiv k_s$, and the water fill rate,

$$\hat{x}_r \equiv \frac{d\hat{x}_L}{dt}.$$

The continuous time process model is depicted in Figure 4.

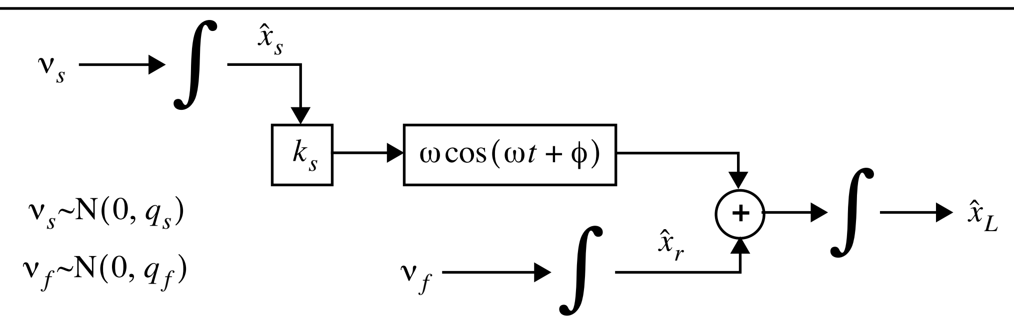

Figure 4: Filling + sloshing process model.

In state form we have,

$$\hat{x} = \begin{bmatrix} \hat{x}_L \\ \hat{x}_r \\ \hat{x}_s \end{bmatrix}.$$

The present water level would be modeled as function of the previous level, the fill rate, and the changing sinusoidal components. Again for the sake of simplicity we will assume that ω is known and $\phi = 0$. We model the new state $\hat{x}$ at time $t + \delta t$ as

$$\begin{aligned} \hat{x}_L(t+\delta t) &= \hat{x}_L(t) + \delta t \hat{x}_r(t) + \hat{x}_s(t)\omega\cos(\omega t) \\ \hat{x}_r(t+\delta t) &= \hat{x}_r(t) \\ \hat{x}_s(t+\delta t) &= \hat{x}_s(t) \end{aligned}.$$

The *continuous time* state transition matrix is

$$A = \begin{bmatrix} 0 & 1 & \omega\cos(\omega t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the *continuous time* process noise matrix is

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & q_f & 0 \\ 0 & 0 & q_s \end{bmatrix}.$$

So the *discrete time* state transition matrix is

$$A(\delta t) = \begin{bmatrix} 1 & \delta t & \omega\cos(\omega t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

and from (17) the *discrete time* process noise matrix is

$$Q(t, \delta t) = \begin{bmatrix} \dfrac{\delta t(3t^2 + 3t\delta t + \delta t^2)(q_f + q_s\gamma^2)}{3} & \dfrac{q_f \delta t \beta}{2} & \dfrac{q_s \delta t \gamma \beta}{2} \\ \dfrac{q_f \delta t \beta}{2} & q_f \delta t & 0 \\ \dfrac{q_s \delta t \gamma \beta}{2} & 0 & q_s \delta t \end{bmatrix} \tag{9}$$

where $\gamma = \omega\cos(\omega t)$, $\beta = 2t + \delta t$, and $0 < \sqrt{q_f}, \sqrt{q_s} \ll F$, for the full water level F. The filter update equations are the usual linear Kalman filter equations:

$$\hat{x}^{+}(t+\delta t) = A(\delta t)\hat{x}(t)$$

and

$$P^{+}(t+\delta t) = A(\delta t)P(t)A^{T}(\delta t) + Q(\delta t)$$

2. MEASUREMENT MODELS

To determine the measurement models we need to determine (specify in this case) the mechanical and electrical characteristics of the actual system. The following schematic diagram will be used to identify variables in the measurement equations.

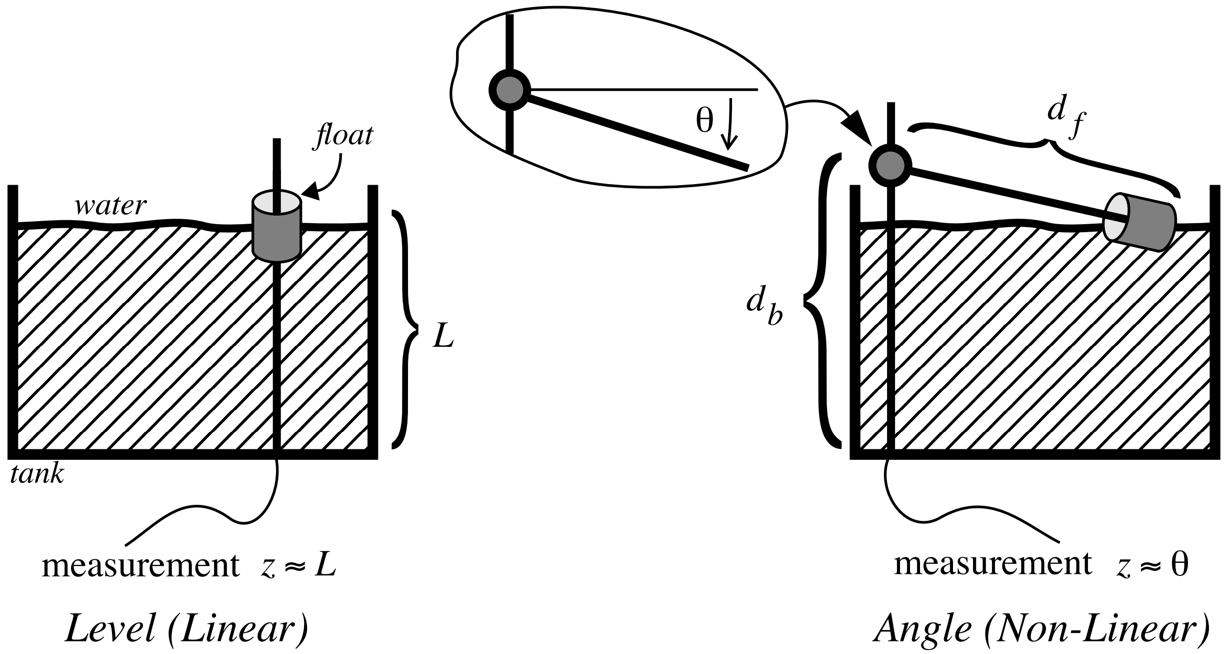

Figure 5: The two possible measurement methods. *Left:* the system returns a noisy voltage z representing the *height* of the float. *Right:* system returns a noisy voltage z representing the *angle* θ between the fixed base segment d_b and the pivoting float segment d_f.

2.1 Level (Linear)

In the simplest case, the system returns a noisy voltage z representing the *height* of the float, which is proportional to (i.e. a linear function of) the water level L. In other words,

$$L = \frac{z}{k_l}$$

where k_l is some *a priori* known constant scale factor. By solving this linear expression for z we obtain a linear model of the measurement:

$$z = k_l L .$$

In more general notation, we model the measurement z as a *linear* function of the system state $\bar{x}$,

$$\hat{z} = H\hat{x}$$

where the "hats" on $\hat{z}$ and $\hat{x}$ reflect the notion that they are *estimates* of the actual measurement and state.

If we use a one-dimensional state $\hat{x}$ to estimate the water level L, the measurement matrix is simply

$$H = k_l. \tag{10}$$

If we use a *two*-dimensional state $\hat{x}$ to estimate the water level L, and the estimate of the water level is in the first position/element of the state, the measurement matrix is

$$H = \begin{bmatrix} k_l & 0 \end{bmatrix}. \tag{11}$$

Finally if we use a *three*-dimensional state to estimate the water level L, and the estimate of the water level is in the first position/element of the state, the measurement matrix is

$$H = \begin{bmatrix} k_l & 0 & 0 \end{bmatrix}. \tag{12}$$

In the actual filter, either (10), (11), or (12) would be used to both for measurement prediction and in the Kalman gain equation. Which of (10), (11), and (12) is used depends on the number of state elements.

2.2 Angle (Non-Linear)

In this case the system returns a noisy voltage z representing the *angle* θ between the fixed base segment d_b and the pivoting float segment d_f. The water level L is a non-linear function of this angle as follows.

$$L = d_b - d_f \sin\left(\frac{z}{k_a}\right)$$

where k_a is some *a priori* known constant scale factor. Bysolving this non-linear expression for z we obtain a non-linear model of the measurement:

$$z = k_a \operatorname{asin}\frac{d_b - L}{d_f}.$$

Again in more general notation, we model the measurement z as a *non-linear* function of the system state $\bar{x}$,

$$\hat{z} = h(\hat{x})$$

where

$$h(\hat{x}) = k_a \operatorname{asin}\frac{d_b - \hat{x}_L}{d_f}, \tag{13}$$

$\hat{x}_L$ is the element of the state vector that represents the water level, and again the “hats” on $\hat{z}$ and $\hat{x}$ reflect the notion that they are *estimates* of the actual measurement and state.

Because we have a non-linear measurement model, we will have to use an *extended Kalman filter.* For the EKF we need the Jacobian of the measurement function—the derivative of the measurement function (13) with respect to the state:

$$H = \frac{\partial}{\partial \hat{x}} h(\hat{x}).$$

In our situation, the measurement model (13) is only a function of $\hat{x}_L$—the element of the state representing the water level, so the Jacobian elements corresponding to any other state elements will always be zero. Specifically if we use a one-dimensional state $\hat{x}$ to estimate the water level L (Section 1.2.1), the measurement matrix is simply

$$H = \frac{d}{d\hat{x}_L}h(\hat{x}) \tag{14}$$

where

$$\frac{d}{d\hat{x}_L}h(\hat{x}) = \frac{-k_a}{d_f\sqrt{1-\left(\frac{d_b-\hat{x}_L}{d_f}\right)^2}}.$$

If we use a *two*-dimensional state $\hat{x}$ to estimate the water level L (Section 1.2.2 or Section 1.2.3), and the estimate of the water level is in the first position/element of the state, the measurement matrix is

$$H = \left[\frac{d}{d\hat{x}_L}h(\hat{x}) \; 0\right]. \tag{15}$$

Finally if we use a *three*-dimensional state to estimate the water level L (Section 1.2.4), and the estimate of the water level is in the first position/element of the state, the measurement matrix is

$$H = \left[\frac{d}{d\hat{x}_L}h(\hat{x}) \; 0 \; 0\right]. \tag{16}$$

In the actual filter, (13) would be used to predict the measurement, and the Jacobian (14), (15), or (16) would be used in the Kalman gain equation.

APPENDIX A: DISCRETE TIME PROCESS NOISE

Given continuous time state transition and process noise matrices A and Q, one can compute the discrete time (sampled) process noise matrix as follows:

$$Q(\delta t) = \int_0^{\delta t} e^{A\tau} Q e^{A^T\tau} d\tau . \tag{17}$$

For example, given

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & q_c \end{bmatrix},$$

where q_c is is the autocorrelation of the continuous process noise, the discrete time time process noise matrix would be

$$Q(\delta t) = \begin{bmatrix} \frac{\delta t^3 q_c}{3} & \frac{\delta t^2 q_c}{2} \\ \frac{\delta t^2 q_c}{2} & \delta t q_c \end{bmatrix}.$$