Optimal PMU Placement Evaluation for Power System Dynamic State Estimation

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***Abstract*—The synchronized phasor measurement unit (PMU), developed in the 1980s, is considered to be one of the most important devices in the future of power systems. The recent development of PMU technology provides high-speed, precisely synchronized sensor data, which has been found to be useful for dynamic state estimation of the power grid. While PMU measurements currently cover fewer than 1% of the nodes in the U.S. power grid, the power industry has gained the momentum to advance the technology and install more units. However, with limited resources, the installation must be selective.**

Previous PMU placement research has focused primarily on network topology, with the goal of finding configurations that achieve full network observability with a minimum number of PMUs. Recently we introduced an approach that utilizes stochastic models of the signals and measurements, to characterize the observability and corresponding uncertainty of power system static states (bus voltage magnitudes and phase angles), for any given configuration of PMUs. Here we present a new approach to designing optimal PMU placement according to estimation uncertainties of the *dynamic* states. We hope the approach can provide planning engineers with a new tool to help in choosing between PMU placement alternatives.

***Index Terms*—Power systems, Dynamic State Estimation, Power system planning, Power system simulation, Power system state estimation.**

I. Introduction

WITH the advent of clock synchronization via the global positioning system (GPS), phasor measurement units (PMU), which measure both the magnitude and phase angle of the electrical waves in a power grid, have achieved a level of measurement precision that typically exceeds conventional measurement units. As such the PMU has been identified as one of the key enabling technologies of the *smart grid*.

Traditionally, power grid measurements have been provided by remote terminal units (RTU) at the substations. RTU measurements include real/reactive power flows, power injections, and magnitudes of bus voltages and branch currents. The most commonly used state estimation measurements are:

- Line power flow measurements: the real and reactive power flow along the transmission lines or transformers.
- Bus power injection measurements: the real and reactive power injected at the buses.
- Voltage magnitude measurements: the voltage magnitudes of the buses.

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Under certain circumstances such as state estimation of distribution systems, the line current magnitude measurements (along the transmission lines or at transformers) may be considered. The installation of PMUs makes two additional types of measurements available:

- Voltage phasor measurements: the phase angles and magnitudes of voltage phasors at system buses.
- Current phasor measurements: the phase angles and magnitudes of current phasors along transmission lines or transformers.

Recent developments of phasor measurement technologies provide high-speed sensor data (typically 30 samples/second) with precise time synchronization [1], [2]. Synchronized phasor measurements are commonly referred to as *synchrophasors*. This is in comparison with traditional Supervisory Control And Data Acquisition (SCADA) RTU measurements, which have cycle times of five seconds or longer and are not time synchronized.

PMUs are becoming increasingly attractive in various power system applications such as system monitoring, protection, control, and stability assessment. PMUs can provide real-time synchrophasor data to the SCADA system to capture the dynamic characteristics of the power system, and hence facilitate time-critical control. Compared to estimating relatively stationary state elements such as bus voltage magnitudes and phase angles, *dynamic state estimation* seeks to estimate more transient states of a power system. For example, in [3] we used PMUs to estimate generator rotor angles and speeds.

Significant previous work has also been dedicated to the selection of the best locations to install new PMUs [4]–[7]. Several algorithms have been developed, primarily with the aim of utilizing a minimum number of PMUs to ensure full network observability. However, there are two issues related to optimal PMU placement for dynamic state estimation: $1)$ It is typically assumed that all bus voltages and phase angles are measured directly by PMUs. In reality one does not have the luxury of having or installing a complete set of PMUs throughout the power grid. It would be useful to know how well a set of available and/or planned PMUs could serve the dynamic state estimator. $2)$ In the previous research, the network topology was the primary focus. In practice the networks usually have complex topologies where more than one solution for the same minimum number of PMUs will be obtained. In such cases, the planning engineers need to make a choice from among these solutions.

In this paper, we focus on bridging these gaps. Specifically our contributions include:

- We develop a stochastic model that captures dynamic state estimation uncertainties, to facilitate the assessment of PMUs installed on a subset of the buses.
- We design an optimal PMU placement evaluation algorithm by incorporating uncertainty estimates into topological considerations for the specific network.
- We present an approach to the comparison among alternative configurations via quantitative measure of expected uncertainties.

The remainder of this paper is organized as follows. Section II introduces the background. In Section III, we present an approach to determine the effects of the existing PMUs on the estimation of the power system dynamic states. We reformulate the optimal PMU placement problem in Section IV such that the solution should be not only a minimum set of PMUs that can cover the entire power system, but also the one benefits dynamic state estimation the most. Our method is tested on a multi-machine system model in Section V to demonstrate how it might help engineers make decisions regarding different candidate plans. Finally, conclusions and acknowledgement are stated in Section VI and Section VII respectively.

II. Background and Related Work

The phasor measurement unit (PMU) was first developed and utilized in [1], [2]. Considering partially observable systems (with an inadequate number of PMUs) the authors in [8] presented an estimation algorithm based on singular value decomposition (SVD), which did not require the complete network system to be observable prior to estimation.

In [3] we investigated the feasibility of applying Kalman filtering techniques to include dynamic state variables in the estimation process. The study shows a promising path forward for the implementation of Kalman filter based dynamic state estimation in conjunction with the emerging PMU measurement technologies.

Optimal PMU placement for full observability was studied in [4]. An algorithm for finding the minimum number of PMUs required for power system state estimation was developed, in which simulated annealing optimization and graph theory were utilized in formulating and solving the problem.

In [7] the authors focused on the analysis of network observability and PMU placement when using a mixed measurement set. They developed an optimal placement algorithm for PMUs using integer programming. In [5], a strategic PMU placement algorithm was developed to improve the bad data processing capability of state estimation by taking advantage of the PMU technology. Furthermore, the PMU placement problem was re-studied and a generalized integer linear programming formulation is presented in [6].

Previously we presented a stochastic approach to characterize the observability and corresponding uncertainty of the power system buses for any given configuration of PMUs, whether that configuration achieves full observability or not [9]. Here we present work on connecting the optimal PMU placement with power system dynamic state estimation: we provide a method for evaluating any candidate PMU placement design, which can be especially useful when there are multiple placement candidates.

III. PMU Placement Evaluation

Built on the previous work [9]–[11], we evaluate a PMU placement using a stochastic estimate of the *asymptotic* or *steady-state* error covariance, as a quantitative metric of the performance. As part of introducing the notion of steady-state error covariance, we first consider the relevant state space models.

A. State Space Models

The state space models are the most basic yet extensively used mathematical models in power system state estimation. An assumed *linear* system can be modeled as a pair of linear stochastic process and measurement equations

$$x_k = Ax_{k-1} + w_{k-1} \tag{1}$$

$$z_k = Hx_k + v_k \tag{2}$$

where $x \in \mathcal{R}^n$ is the state vector, $z \in \mathcal{R}^m$ is the measurement vector, A is a $n \times n$ matrix that relates the state at the previous time step $k-1$ to the state at the current step k in the absence of either a driving function or process noise[1], and H is a $m \times n$ matrix that relates the state to the measurement z_k. The process noise w_k and measurement noise v_k are assumed to be mutually independent random variables, spectrally white, and with normal probability distributions

$$p(w) \sim N(0, Q) \tag{3}$$

$$p(v) \sim N(0, R), \tag{4}$$

where the process noise covariance Q and measurement noise covariance R matrices are often assumed to be constant.

In reality, the process to be estimated and (or) the measurement relationship to the process is usually *nonlinear*. Especially when our objective is to estimate the dynamic states of the power system. A nonlinear system can be modeled using nonlinear stochastic process and measurement equations

$$x_k = a(x_{k-1}, w_{k-1}) \tag{5}$$

$$z_k = h(x_k, v_k). \tag{6}$$

One can approximate the states and measurements by

$$x_k = a(x_{k-1}) \tag{7}$$

$$z_k = h(x_k). \tag{8}$$

These nonlinear functions can then be linearized about the point of interest x in the state space. To do so one need to compute either or both of the Jacobian matrices

$$A = \frac{\partial a(x)}{\partial x}|_x \tag{9}$$

$$H = \frac{\partial h(x)}{\partial x}|_x \tag{10}$$

where A and H are the partial derivatives of a and h (respectively) with respect to x.

[1] In practice, the matrix A may change with each time step, but it is assumed to be constant here.

For the true state x_k and corresponding estimate $\hat{x}_k$ at time step k, the estimate error covariance can be defined as $P_k \equiv E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$, where E denotes statistical expectation.

B. Steady-State Performance of the Estimation Process

On-line methods such as the Kalman filter [12], [13] can be used to estimate time varying state and error covariance in a recursive predictor-corrector fashion. In these on-line scenarios the estimate error covariance P_k changes over time. However the *steady-state* error covariance

$$P^{\infty} = \lim_{k \to \infty} E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \tag{11}$$

can be computed in closed form, off line. In fact to compute the steady-state uncertainty one does not actually need to estimate x_k and $\hat{x}_k$. Instead one can estimate P^{∞} directly from state-space models of the system using stochastic models for the various noise sources.

The Discrete Algebraic Riccati Equation (DARE) represents such a closed-form solution to the steady-state covariance P^{∞} [14]. Assuming the process and measurement noise elements are uncorrelated, the DARE can be written in the form

$$\begin{aligned} P^{\infty} =& AP^{\infty}A^T + Q \qquad (12)\\ & - AP^{\infty}H^T(R + HP^{\infty}H^T)^{-1}HP^{\infty}A^T. \end{aligned}$$

We use the MacFarlane-Potter-Fath "Eigenstructure Method" [14] to calculate the DARE solution P^{∞}. Specifically, for all points of interest in the state space we perform the following steps. First we obtain or compute the model parameters A, Q, H, and R as described in the preceding section. Next we calculate the $2n \times 2n$ discrete-time Hamiltonian matrix

$$\Psi = \begin{bmatrix} A + QA^{-T}H^TR^{-1}H & QA^{-T} \\ A^{-T}H^TR^{-1}H & A^{-T} \end{bmatrix}. \tag{13}$$

We then form

$$\begin{bmatrix} B \\ C \end{bmatrix} = [e_1, e_2, ..., e_n] \tag{14}$$

from the n characteristic eigenvectors $[e_1, e_2, ..., e_n]$ of Ψ. Finally, using B and C we compute the steady-state error covariance as

$$P^{\infty} = BC^{-1}. \tag{15}$$

Note that because the Jacobians A and H are functions of the state, they will generally have to be computed at each point of interest in the state space. The noise covariance matrices R and Q might be constant, or might also vary as a function of the state.

For each point of interest in the state space, P^{∞} indicates the expected asymptotic state estimation uncertainty corresponding to the candidate design modeled by the specific A, Q, H, and R. Intuitively, given PMU placements leading to the same level of observability, the lower the overall uncertainty is, the more we prefer this design.

C. The Process Model

Without loss of generality, in a power system that consists of n generators, let us consider the generator i which is connected to the generator terminal bus i. We use a classical model for the generator composed of a voltage source $|E_i|\angle\delta_i$ with constant amplitude behind an impedance X'_{d_i}. The nonlinear differential-algebraic equations regarding the generator i can be written as

$$\begin{cases} \frac{d\delta_i}{dt} = \omega_B(\omega_i - \omega_0) \\ \frac{d\omega_i}{dt} = \frac{\omega_0}{2H_i}(P_{m_i} - \frac{|E_i||V_i|}{X'_{d_i}}\sin(\delta_i - \theta_i) - D_i(\omega_i - \omega_0)) \end{cases} \tag{16}$$

where state variables δ and ω are the generator rotor angle and speed respectively, ω_B and ω_0 are the speed base and the synchronous speed in per unit, P_{m_i} is the mechanical input, H_i is the machine inertia[2], D_i is the generator damping coefficient and $|V_i|\angle\theta_i$ is the phasor voltage at the generator terminal bus i (which is a function of $\delta_1, \delta_2, ..., \delta_n$).

For the state vector $x = [\delta_1, \omega_1, \delta_2, \omega_2, ..., \delta_n, \omega_n]^T$, the corresponding *continuous time* change in state can be modeled by the linearized equation

$$\frac{dx}{dt} = A_c x + w_c, \tag{17}$$

where w_c is an $2n \times 1$ continuous time process noise vector with $2n \times 2n$ noise covariance matrix $Q_c = E[w_c w_c^T]$, and A_c is an $2n \times 2n$ continuous time state transition Jacobian matrix, whose entries for $i \in \{1, ..., n\}$ and $j \in \{1, ..., n\}$ ($i \neq j$) are the corresponding partial derivatives

$$A_{c_{[2i-1,2i-1]}} = 0 \tag{18}$$

$$A_{c_{[2i-1,2i]}} = \omega_B \tag{19}$$

$$\begin{aligned} A_{c_{[2i,2i-1]}} =& -\frac{\omega_0|E_i|}{2H_iX'_{d_i}}\Big[\frac{\partial|V_i|}{\partial\delta_i}\sin(\delta_i - \theta_i) \qquad (20)\\ & + |V_i|\cos(\delta_i - \theta_i)(1 - \frac{\partial\theta_i}{\partial\delta_i})\Big] \end{aligned}$$

$$A_{c_{[2i,2i]}} = -\frac{\omega_0}{2H_i}D_i \tag{21}$$

$$A_{c_{[2i-1,2j-1]}} = 0 \tag{22}$$

$$A_{c_{[2i-1,2j]}} = 0 \tag{23}$$

$$\begin{aligned} A_{c_{[2i,2j-1]}} =& -\frac{\omega_0|E_i|}{2H_iX'_{d_i}}\Big[\frac{\partial|V_i|}{\partial\delta_j}\sin(\delta_i - \theta_i) \qquad (24)\\ & + |V_i|\cos(\delta_i - \theta_i)(-\frac{\partial\theta_i}{\partial\delta_j})\Big] \end{aligned}$$

$$A_{c_{[2i,2j]}} = 0. \tag{25}$$

Hence the update of the state vector x from time step $(k-1)$ to k over duration Δt has the complete corresponding *discrete-time* state transition matrix

$$A = I + A_c \cdot \Delta t. \tag{26}$$

The next issue is the discrete-time process noise covariance Q. As described by [14], if we assume the process noise

[2] The mechanical power P_{m_i} and machine inertia H_i should not be confused with the error covariance P and measurement Jacobian H from the preceding section. While potentially confusing these are the variables used by popular convention in the respective fields.

"flows" through (is shaped by) the same system of integrators represented by A, we can integrate the continuous time process equation (17) over the time interval Δt to obtain a $2n \times 2n$ discrete-time (sampled) Q matrix:

$$Q = \int_0^{\Delta t} e^{A_c t} Q_c e^{A_c^T t} dt \tag{27}$$

Now using the process model from equation (1) we have

$$x_k = A x_{k-1} + w_{k-1} \tag{28}$$
$$= (I + A_c \cdot \Delta t) x_{k-1} + w_{k-1}, \tag{29}$$

where w is the process noise with normal probability distribution $p(w) \sim N(0, Q)$.

D. The Measurement Model

In this paper, we only consider the measurements provided by PMUs. According to [5]–[7], a PMU installed at a specific bus is capable of measuring not only the bus voltage phasor, but also the current phasors along all the lines incident to the bus. So in addition to the phasor voltage at this bus, we are able to compute the phasor voltages of all of its neighboring buses, hence they all become observable.

All the observable buses can be divided into two *levels*, according to the "directness" of their observations: Level 1 contains the directly observable buses with PMUs installed on them; while Level 2 contains the more "indirect" observable buses with PMUs installed on their neighbors instead of themselves.

1) Level 1 buses: If the bus i is in Level 1 (meaning that there is a PMU placed on this bus), the measurement equation is

$$z_i = V_i + v_i, \tag{30}$$

where z_i is the measured complex voltage at bus i, V_i is the "true" complex voltage at bus i and v_i is the complex measurement noise of this PMU. Thus for all the buses in Level 1, we can write the measurement equation in the matrix form

$$z_V = I \cdot V_{L_1} + v_V, \tag{31}$$

where z_V is the complex voltage measurement subvector, I is the identity matrix, V_{L_1} is the complex state subvector ("true" complex voltages at all Level 1 buses) and v_V is the voltage measurement noise subvector.

2) Level 2 buses: If the bus i is in Level 2 (meaning that there is at least one PMU placed on an adjacent bus), then for each PMU placed at some adjacent bus j, the measurement equation will be

$$z_{ji} = \begin{pmatrix} Y_{ji} & -Y_{ji} \end{pmatrix} \begin{pmatrix} V_j \\ V_i \end{pmatrix} + v_j, \tag{32}$$

where z_{ji} is the measured complex current at bus j (towards bus i), Y_{ji} is the admittance of line (j, i), V_j and V_i are the "true" complex voltages at bus j and i respectively, and v_j is the complex measurement noise of this PMU.

So for all the buses in Level 1 and Level 2, we have the measurement equation

$$z_C = \begin{pmatrix} Y_{CL_1} & Y_{CL_2} \end{pmatrix} \begin{pmatrix} V_{L_1} \\ V_{L_2} \end{pmatrix} + v_C, \tag{33}$$

where z_C is the complex current measurement subvector, Y_{CL_1} and Y_{CL_2} are the line admittance matrices that relates Level 1 and Level 2 bus voltages to z_C respectively, V_{L_1} and V_{L_2} are the complex state subvector ("true" complex voltages at all Level 1 and Level 2 buses), and v_C is the current measurement noise subvector. In this equation, the measurement matrix $\begin{pmatrix} Y_{CL_1} & Y_{CL_2} \end{pmatrix}$ has each row sum up to zero.

If the number of PMUs installed is sufficient, or the power grid is well-connected, it is quite possible that a Level 2 bus has more than one PMU-installed neighbor buses. In such case, we allow the contributions from each adjacent PMU to be fused at this bus.

Combining the above, the measurements are functions of the phasor voltages of all observable buses:

$$\begin{pmatrix} z_V \\ z_C \end{pmatrix} = \begin{pmatrix} I & 0 \\ Y_{CL_1} & Y_{CL_2} \end{pmatrix} \begin{pmatrix} V_{L_1} \\ V_{L_2} \end{pmatrix} + \begin{pmatrix} v_V \\ v_C \end{pmatrix} \tag{34}$$

We can simply denote the equation above as

$$z' = H'' V + v' \tag{35}$$

However in this paper, our state variables are no longer bus voltages, but dynamic generator state variables, so we need to introduce the expanded system nodal equation first:

$$Y_{exp} \begin{pmatrix} E \\ V \end{pmatrix} = \begin{pmatrix} Y_{GG} & Y_{GL} \\ Y_{LG} & Y_{LL} \end{pmatrix} \begin{pmatrix} E \\ V \end{pmatrix} = \begin{pmatrix} I_G \\ 0 \end{pmatrix} \tag{36}$$

where E is the vector of internal generator complex voltages, V is the vector of bus complex voltages, I_G represents electrical currents injected by generators, Y_{exp} is called the expanded nodal matrix, which includes loads and generator internal impedances: Y_{GG}, Y_{GL}, Y_{LG} and Y_{LL} are the corresponding partitions of the expanded admittance matrix.

Therefore the relationship of the bus voltages to the generator voltages can be expressed as:

$$V = -Y_{LL}^{-1} Y_{LG} E = R_V E \tag{37}$$

where R_V is defined as the bus reconstruction matrix.

Combining equations (35) and (37), we have:

$$z' = H'' R_V E + v' = H' E + v' \tag{38}$$

Because the internal generator complex voltage vector $E = [|E_1|\angle\delta_1, |E_2|\angle\delta_2, ..., |E_n|\angle\delta_n]^T$ is a function of the state vector x, the measurements can be modeled as

$$z = h(x) + v \tag{39}$$

where $z = [|z_1'|, \arg(z_1'), |z_2'|, \arg(z_2'), ..., |z_m'|, \arg(z_m')]^T$ is the $2m \times 1$ measurement vector that consists of amplitude and phase angle of each PMU measurement.

Furthermore, the measurement model can be expressed in the form of equation (2) after linearization

$$z = Hx + v \tag{40}$$

where H is the corresponding Jacobian matrix:

$$H = \begin{pmatrix} \frac{\partial h_1}{\partial \delta_1} & 0 & \frac{\partial h_1}{\partial \delta_2} & 0 & \cdots \\ \frac{\partial h_2}{\partial \delta_1} & 0 & \frac{\partial h_2}{\partial \delta_2} & 0 & \cdots \\ \frac{\partial h_3}{\partial \delta_1} & 0 & \frac{\partial h_3}{\partial \delta_2} & 0 & \cdots \\ \frac{\partial h_4}{\partial \delta_1} & 0 & \frac{\partial h_4}{\partial \delta_2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{41}$$

and v is the measurement noise noise with normal probability distribution $p(v) \sim N(0, R)$. The measurement noise covariance R can be determined experimentally, by testing the PMUs over time.

IV. Optimal PMU Placement

Conventionally, we want the PMU placement to be optimal in the sense that it makes *all buses* in the system observable with a *minimum number of PMUs*. This is especially true for large-scale complex systems. The authors in [6], [7] presented a numerical formulation of this optimization problem:

$$\begin{aligned} \min \sum_{i}^{n(buses)} w_i \cdot X_i \\ s.t. \quad f(X) = C \cdot X \geq \hat{1} \end{aligned} \tag{42}$$

where $n(buses)$ is the number of buses in the system, w_i is the cost of the PMU installed at bus i, X is a binary decision variable vector with entries X_i defined as

$$X_i = \begin{cases} 1 & \text{if a PMU is installed at bus } i \\ 0 & \text{otherwise} \end{cases}, \tag{43}$$

$\hat{1}$ is a vector whose entries are all ones, $f(X)$ is a vector function and C is the binary connectivity matrix with entries

$$C_{[k,m]} = \begin{cases} 1 & \text{if } k = m \\ 1 & \text{if buses } k \text{ and } m \text{ are connected} \\ 0 & \text{otherwise} \end{cases}. \tag{44}$$

This formulation allows easy analysis of the network observability and concerns about the full coverage of the system. We first solve this optimization problem using integer linear programming techniques. However, the integer linear programming approach may not be sufficient for determining the optimal locations of PMUs. The reason is that very often there will be multiple optimal solutions with the same minimum number of PMUs. In such cases, one would want a means for comparing the different solutions, as some will actually offer lower estimate uncertainty and latency.

Thus the next step is to evaluate the solution(s) using steady-state uncertainty analysis as described in section III. In order to compute P^∞ for all desired points in the state space, we need to identify the parameter matrices A, Q, H and R. Notice that in these matrices, only the generator rotor angles $\{\delta_1, \delta_2, ..., \delta_n\}$ are variables, so one only need to decide at which *interest* points $\{\bar{x}_1, \bar{x}_2, ...\bar{x}_p\}$ where $\bar{x}_i \in [0, 2\pi]^n$, to evaluate P^∞. Finally, our approach can be described by algorithm 1.

Algorithm 1: Optimal PMU placement evaluation

Build the optimization problem as shown in (42) based on the network topology
Obtain all solutions to this problem
for *each solution set of PMUs* **do**
  for *each n-dimensional point $\bar{x}_i \in \{\bar{x}_1, \bar{x}_2, ...\bar{x}_p\}$* **do**
    Determine Δt for the PMUs
    Evaluate A with Δt using (26)
    Evaluate Q with Δt using (27)
    Evaluate H using (41)
    Evaluate R
    Compute Ψ using (13)
    Compute the n eigenvectors of Ψ using (14)
    Compute P_i^∞ using (15)

V. Simulation Results

In this section, we first test the uncertainty analysis on an ideal multi-machine system with PMU installed on every bus. We then apply our approach to the evaluation of four different optimal PMU placement alternatives, to find the best one.

A. Estimation Uncertainty of a Fully PMU-installed System

We carried out our steady-state uncertainty analysis on a small test system consisting of three machines in a looped network of nine buses as shown in Fig. 1(a). The dynamic state vector is $x = \{\delta_1, \omega_1, \delta_2, \omega_2, \delta_3, \omega_3\}$. To better illustrate the results, we represent the entire set of interest points $\hat{x} = \{\delta_1, \delta_2, \delta_3\} \in [0, 2\pi]^3$ by a 3D grid. For each point $\hat{x}_i$, we obtain a 6×6 steady-state error covariance matrix P_i^∞, and then aggregate the information in P_i^∞ to the following value for visualization

$$f(\hat{x}_i) = \frac{\sum\limits_{j=1,3,5} \sqrt{P^\infty_{i_{[j,j]}}} + w_B \Delta t \sum\limits_{j=2,4,6} \sqrt{P^\infty_{i_{[j,j]}}}}{3}, \tag{45}$$

where we use Δt to convert speeds to changes in angle over the inter-measurement period. Because

$$\sqrt{P^\infty_{i_{[2k-1,2k-1]}}} + w_B \Delta t \sqrt{P^\infty_{i_{[2k,2k]}}} \cdot \Delta t \tag{46}$$

represents the uncertainty in estimating the rotor angle of generator k over time Δt, the value $f(\hat{x}_i)$ is the average uncertainty of the three generators over Δt at this particular point $\hat{x}_i$.

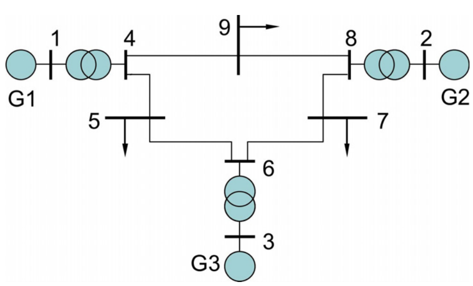
To begin with we consider an ideal scenario, where PMUs are installed on all the nine buses. Fig. 1(b) illustrates the $f(\hat{x})$ values throughout the $[0, 2\pi]^3$ space. One can clearly tell a periodic pattern in the plot, because the Jacobian parameter matrices involve trigonometric functions of $\hat{x}$. The darker areas reflect lower uncertainty, hence imply better expected estimation. This ideal case indicates the best estimation uncertainty level we can reach with a PMU placed on every bus.

B. Comparison Among Multiple Optimal Solutions

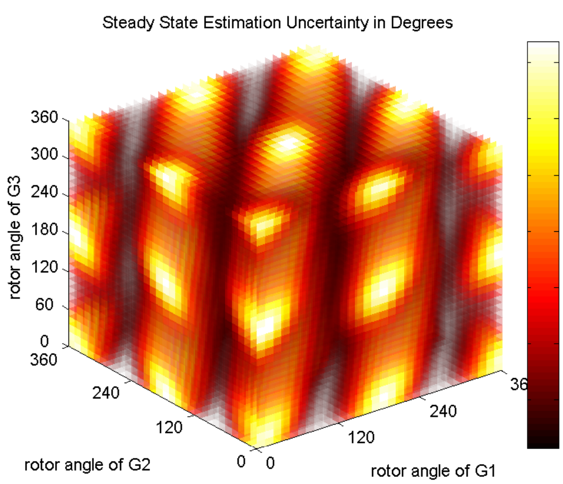
In this subsection, we used the same test system in Fig. 1(a) as an example of finding the optimal PMU placement solution.

After solving the optimization problem (42) with the linear integer programming method, we obtained four solutions, with PMUs installed at buses $\{1,6,8\}$, $\{2,4,6\}$, $\{3,4,8\}$ and $\{4,6,8\}$ respectively. According to previous research they can all be called "optimal" in the sense that they each make the entire system observable by using a minimum number of PMUs. To visualize the differences between some different PMU placement strategies via our approach, we provide two of the uncertainty plots in Fig. 2 and Fig. 3, with the same colormap and plot range. We use the same visualization method as in Fig. 1(b), where darker color depicts lower uncertainty value.

One can choose different criteria depending on the circumstances. For instance, one could use the maximum, minimum, or mean of these aggregated uncertainties as depicted in Fig. 4, or even some weighted function of them. In our example, the scheme with PMUs placed at buses $\{4,6,8\}$ is more likely to be chosen by the designer, because it has smaller max, min and mean uncertainties than the alternative scenarios. In practice, many factors may affect the final decision, *e.g.*, existing infrastructure, cost, *etc.*. Our quantitative approach helps guide decision-making by providing concrete information about tradeoffs, which can not be offered by pure network topology analysis.



(a) A 3-machine 9-bus system. (Figure from [3].)



(b) Steady-state estimation uncertainty versus rotor angles for the ideal case, i.e. when PMUs are installed on *all* buses.

Fig. 1. The test system (a) and results for the ideal case (b).

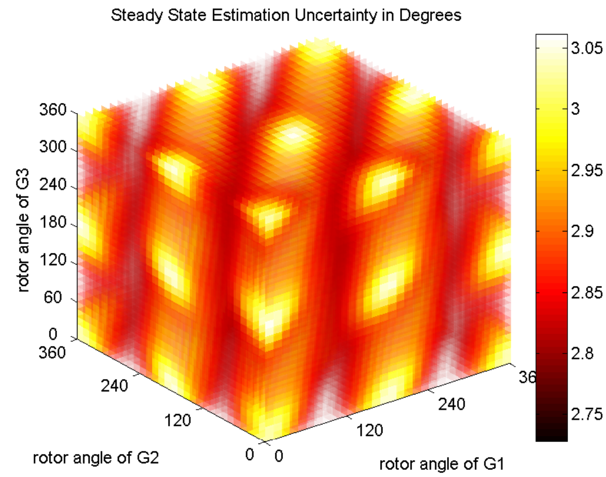


Fig. 2. Steady-state estimation uncertainty versus rotor angles for PMUs installed at buses 1, 6 and 8.

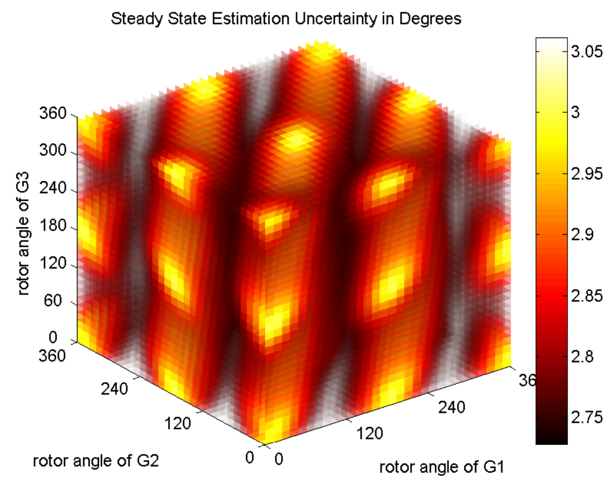


Fig. 3. Steady-state estimation uncertainty versus rotor angles for PMUs installed at buses 4, 6 and 8. Note the darker colors corresponding to the better performance compared to Fig. 2. See also Fig. 4.

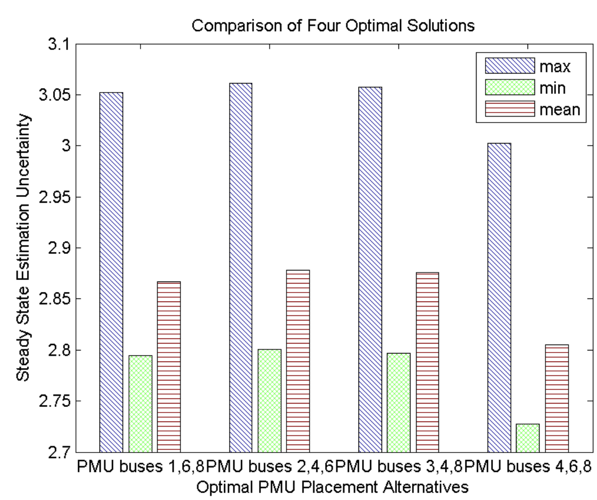


Fig. 4. Comparison of steady-state uncertainties for four "optimal" solutions, based upon three criteria: maximum (max), minimum (min) and average (mean).

VI. Conclusions

We have presented a stochastic framework to quantify the steady-state performance of any candidate PMU placement design, and reformulated the optimal PMU placement problem to tie it in with dynamic state estimation. This method can be readily used without running the actual estimation procedure. We applied the method to a test system, and visualized the results for several candidate PMU placements.

While we initially chose to work with PMU measurements only, we plan to extend the measurement set to include conventional RTU measurements. We also plan to employ our steady-state approach in a system-wide sensor placement optimization framework.

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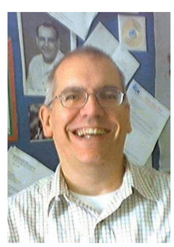

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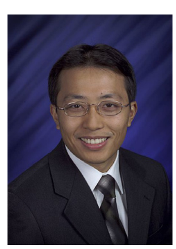

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