Statistical Computing on Riemannian manifolds

From Riemannian Geometry to Computational Anatomy

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Computational Anatomy

Modeling and Analysis of the Human Anatomy

- Estimate representative / average organ anatomies
- Model organ development across time
- Establish normal variability
- Detection and classification of pathologies from structural deviations
- From generic (atlas-based) to patients-specific models

⇨ **Statistical analysis**

Statistical computing on manifolds

The geometric framework

- (Geodesically complete) Riemannian manifolds

The statistical tools

- Mean, Covariance, Parametric distributions / tests
- Interpolation, filtering, diffusion PDEs

The application examples

- Rigid body transformations (evaluation of registration performances)
- Tensors: Diffusion tensor imaging, Variability of brain sulci
- Statistics of deformations for non-linear registration

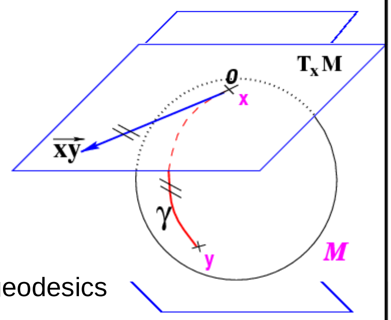
Overview

- ➔ **Statistics on point-wise geometric features**
  - ⇨ **The Riemannian framework and first statistical tools**
  - Example on rigid registration performances evaluation
- **Fields of geometric features: tensor computing**
- **Statistics of deformations for non-linear registration**
- **Conclusion**

Riemannian Manifolds: geometrical tools

Riemannian metric :

- Dot product on tangent space
- Speed, length of a curve
- Distance and geodesics
  (angle, great circles)



Exponential chart (Normal coord. syst.) :

- Development in tangent space along geodesics
- Geodesics = straight lines
- Distance = Euclidean
- Star shape domain limited by the cut-locus
- Covers all the manifold if **geodesically complete**

Reinterpretation of Basic Operations

Operation	Euclidean space	Riemannian manifold
Subtraction	$\overrightarrow{xy} = y - x$	$\overrightarrow{xy} = \log_x(y)$
Addition	$y = x + \overrightarrow{xy}$	$y = \exp_x(\overrightarrow{xy})$
Distance	$dist(x, y) = \lVert y - x \rVert$	$dist(x, y) = \lVert \overrightarrow{xy} \rVert_x$
Gradient descent	$\Sigma_{t+\varepsilon} = \Sigma_t - \varepsilon \nabla C(\Sigma_t)$	$\Sigma_{t+\varepsilon} = \exp_{\Sigma_t}(-\varepsilon \nabla C(\Sigma_t))$

Metric choice on Transformation (Lie) Group

Metric choice: left invariant $\text{dist}(g, h) = \text{dist}(f \circ g, f \circ h)$

- The principal chart (exp. chart at the origin) can be translated at any point : only one chart.

$$\text{dist}(g, h) = \left\| \overrightarrow{f^{(-1)} \circ g} \right\|$$

Practical computations $\overrightarrow{fg} = g - f \quad \Leftrightarrow \quad \overrightarrow{fg} = f^{(-1)} \circ g$

$$f + \overrightarrow{\delta f} \quad \Leftrightarrow \quad \exp_{\vec{f}}\left(\overrightarrow{\delta f}\right) = f \circ \overrightarrow{\delta f}$$

- Atomic operations $\left[\vec{f} \circ \vec{g}\right]$, $\left[\overrightarrow{f^{(-1)}}\right]$ and their Jacobian

Metric choice on Homogeneous manifolds

Metric choice: invariant $\text{dist}(x, y) = \text{dist}(g * x, g * y)$

- Isotropy group of the origin: $H = \{h * o = o\}$
- Existence condition: $\text{dist}(x, o) = \text{dist}(h * x, o)$
- Placement function: $f_x * o = x$

Practical computations $\overrightarrow{xy} = y - x \quad \Leftrightarrow \quad \overrightarrow{xy} = f_x^{(-1)} * \vec{y}$

$$x + \overrightarrow{\delta x} \quad \Leftrightarrow \quad \exp_x\left(\overrightarrow{\delta x}\right) = f_x * \overrightarrow{\delta x}$$

- Atomic operations $\left[f * \vec{x}\right]$, $\left[f_x\right]$ and their Jacobian

Statistical tools on Riemannian manifolds

Metric -> Volume form (measure) $d\mathrm{M}(\mathrm{x})$

Probability density functions $\forall X, P(x \in X) = \int_X p_{\mathbf{x}}(y).d\,\mathrm{M}(y)$

Expectation of a function ϕ from M into R :

- Definition : $\mathrm{E}[\phi(x)] = \int_{\mathrm{M}} \phi(y).p_{\mathbf{x}}(y).d\mathrm{M}(y)$
- Variance : $\sigma_{\mathbf{x}}^2(y) = \mathrm{E}\left[\mathrm{dist}(y,\underline{\mathrm{x}})^2\right] = \int_{\mathrm{M}} \mathrm{dist}(y,z)^2.p_{\mathbf{x}}(z).d\mathrm{M}(z)$
- Information (neg. entropy): $\mathrm{I}[\mathbf{x}] = \mathrm{E}[\log(p_{\mathbf{x}}(\mathbf{x}))]$

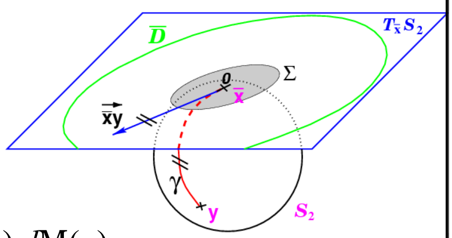
Statistical tools: Moments

Frechet / Karcher mean minimize the variance

$$\mathrm{E}[\mathbf{x}] = \operatorname*{argmin}_{y \in \mathrm{M}} \left(\mathrm{E}\left[\mathrm{dist}(y,\mathbf{x})^2\right]\right) \Rightarrow \mathrm{E}\left[\overrightarrow{\bar{\mathrm{x}}\mathbf{x}}\right] = \int_{\mathrm{M}} \overrightarrow{\bar{\mathrm{x}}\mathbf{x}}.p_{\mathbf{x}}(z).d\mathrm{M}(z) = 0 \quad [P(C) = 0]$$

Geodesic marching

$$\bar{\mathrm{x}}_{t+1} = \exp_{\bar{\mathrm{x}}_t}(v) \quad \text{with} \quad v = \mathrm{E}\left[\overrightarrow{\mathrm{y}\mathbf{x}}\right]$$



Covariance et higher moments

$$\Sigma_{\mathbf{xx}} = \mathrm{E}\left[\left(\overrightarrow{\bar{\mathrm{x}}\mathbf{x}}\right)\left(\overrightarrow{\bar{\mathrm{x}}\mathbf{x}}\right)^{\mathrm{T}}\right] = \int_{\mathrm{M}} \left(\overrightarrow{\bar{\mathrm{x}}z}\right)\left(\overrightarrow{\bar{\mathrm{x}}z}\right)^{\mathrm{T}}.p_{\mathbf{x}}(z).d\mathrm{M}(z)$$

[Pennec, INRIA Research Report RR-5093, NSIP'99]

Distributions for parametric tests

Uniform density:

- maximal entropy knowing X $\quad p_{\mathbf{x}}(z) = \mathrm{Ind}_X(z)/\mathrm{Vol}(X)$

Generalization of the Gaussian density:

- Stochastic heat kernel p(x,y,t) [complex time dependency]
- Wrapped Gaussian [Infinite series difficult to compute]
- Maximal entropy knowing the mean and the covariance

$$N(y) = k.\exp\left(\left(\overrightarrow{\bar{\mathbf{x}}\mathbf{x}}\right)^{\mathrm{T}}.\mathbf{\Gamma}.\left(\overrightarrow{\bar{\mathbf{x}}\mathbf{x}}\right)/2\right)$$

$$\mathbf{\Gamma} = \mathbf{\Sigma}^{(-1)} - \tfrac{1}{3}\mathrm{Ric} + O(\sigma) + \varepsilon(\sigma/r)$$

$$k = (2\pi)^{-n/2}.\det(\mathbf{\Sigma})^{-1/2}.\left(1 + O(\sigma^3) + \varepsilon(\sigma/r)\right)$$

Mahalanobis D2 distance / test: $\quad \mu_{\mathbf{x}}^2(\mathrm{y}) = \overrightarrow{\bar{\mathrm{x}}\mathrm{y}}^{t}.\Sigma_{\mathbf{xx}}^{(-1)}.\overrightarrow{\bar{\mathrm{x}}\mathrm{y}}$

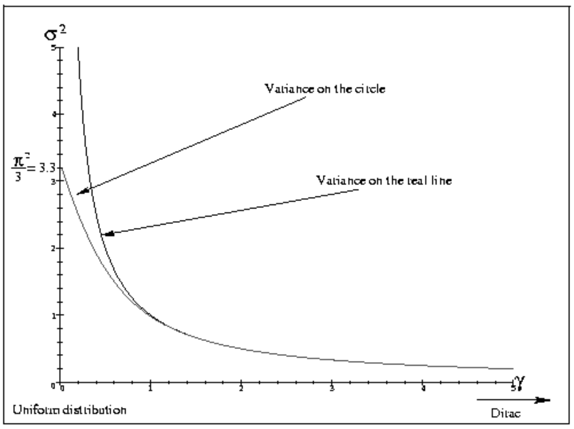
- Any distribution: $\quad \mathrm{E}\left[\mu_{\mathbf{x}}^2(\mathbf{x})\right] = n$
- Gaussian: $\quad \mu_{\mathbf{x}}^2(\mathbf{x}) \propto \chi_n^2 + O(\sigma^3) + \varepsilon(\sigma/r)$

[Pennec, INRIA Research Report RR-5093, NSIP'99]

Gaussian on the circle

Exponential chart: $\quad x = r\theta \in \left]-\pi.r\,;\pi.r\right[$

Gaussian: truncated standard Gaussian



$r \to \infty$: standard Gaussian (Ricci curvature → 0)

$\gamma \to 0$: uniform pdf with $\sigma^2 = (\pi.r)^2/3$ (compact manifolds)

$\gamma \to \infty$: Dirac

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  - ✓ **The Riemannian framework and first statistical tools**
  - ⇨ **Example on registration performances evaluation**
- **Fields of geometric features: tensor computing**
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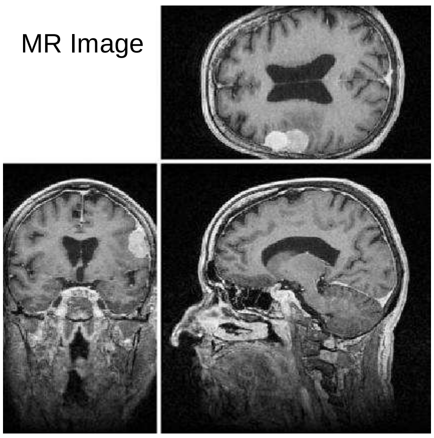
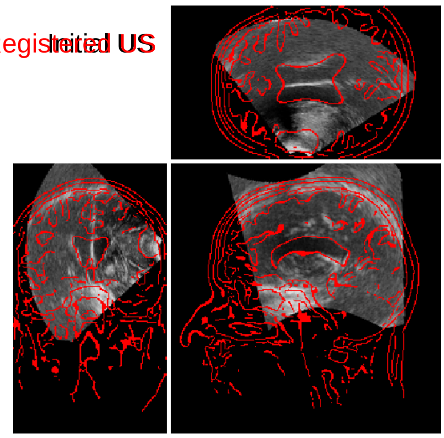
Per-operative registration of MR/US images

MR Image



Registered US / Initial US

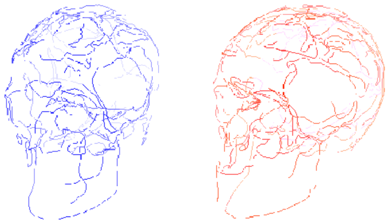


Performance Evaluation?

Uncertainty of feature-based registration

Matches estimation (landmarks)

- Alignment
- Geometric hashing
- ICP



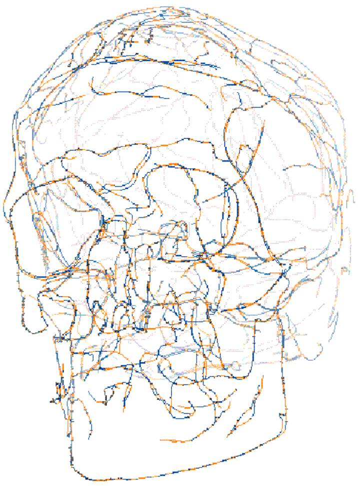
Least square registration

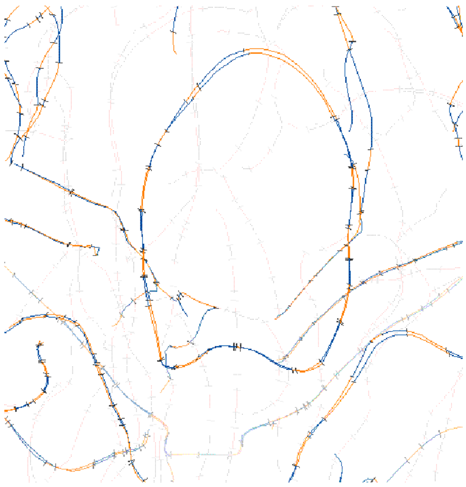
$$C(T,\chi) = \sum_i \left\| y_i - T * x_i \right\|^2$$

- Propagation of the errors from the data to the optimal transformation at the first order (implicit function theorem):

$$\Sigma_{\chi\chi} = \sigma^2.Id \quad \Rightarrow \quad \boxed{\Sigma_{TT} = \sigma^2.H^{-1}} \quad \text{with} \quad H = \frac{\partial^2 C(T,\chi)}{\partial T^2}$$

Registration of CT images of a dry skull



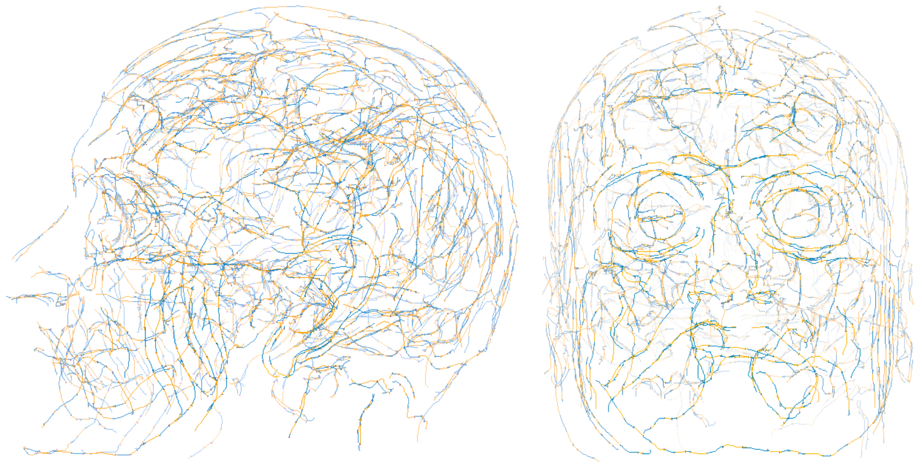


550 matched frames among 2000

Typical object accuracy: 0.04 mm

Typical corner accuracy: 0.10 mm

Registration of MR T1 images of the head



Typical object accuracy: 0.06 mm

860 matched frames among 3600

Typical corner accuracy: 0.125 mm

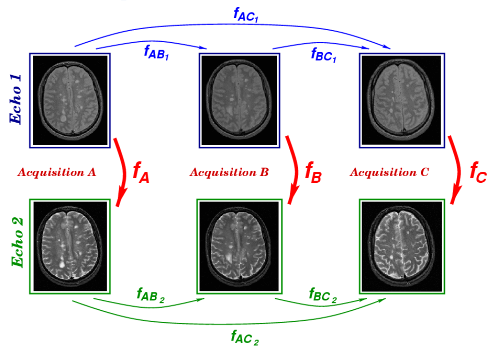
Validation of the error prediction

Comparing two transformations and their Covariance matrix :

$$\mu^2(T_1, T_2) \approx \chi_6^2$$

Mean: 6, Var: 12

KS test



Brigham and Women's Multiple sclerosis database

- 24 3D acquisitions over one year per patient
- T2 weighted MR, 2 different echo times, voxels 1x1x3 mm
- Predicted object accuracy: 0.06 mm.

[X. Pennec et al., Int. J. Comp. Vis. 25(3) 1997, MICCAI 1998]

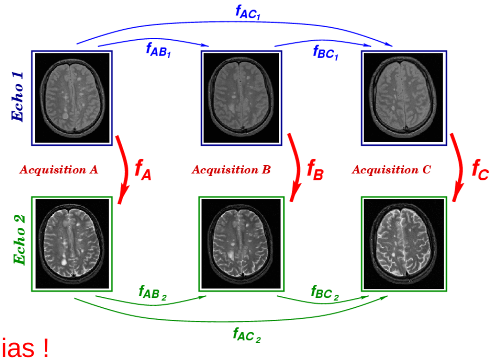
Validation of the error prediction

Comparing two transformations and their Covariance matrix :

$$\mu^2(T_1, T_2) \approx \chi_6^2$$

Mean: 6, Var: 12

KS test



Intra-echo: $\mu^2 \approx 6$, KS test OK

Inter-echo: $\mu^2 > 50$, KS test failed, Bias !

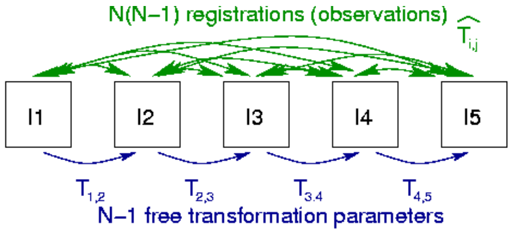
Bias estimation: (chemical shift, susceptibility effects)

- $\sigma_{rot} = 0.06$ deg (not significantly different from the identity)
- $\sigma_{trans} = 0.2$ mm (significantly different from the identity)

Inter-echo with bias corrected: $\mu^2 \approx 6$, KS test OK

[X. Pennec et al., Int. J. Comp. Vis. 25(3) 1997, MICCAI 1998]

Bronze Standard: Multiple registration



Best explanation of the observations (ML) : $C = \sum_{ij} d^2(T_{ij}, \hat{T}_{ij})$

- LSQ criterion
- Robust Fréchet mean $d^2(T_1, T_2) = \min\left(\mu^2(T_1, T_2), \chi^2\right)$
- Robust initialization and Newton gradient descent

Result $T_{i,j}, \sigma_{rot}, \sigma_{trans}$

Results on per-operative patient images

Data (per-operative US)

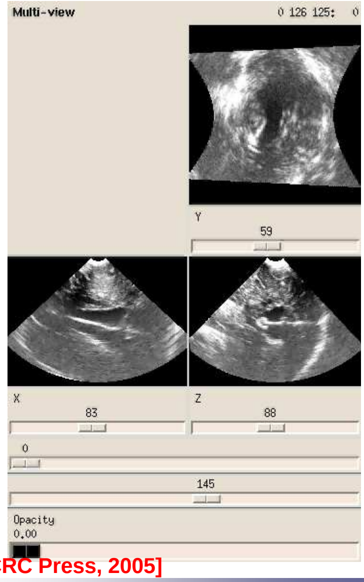
- 2 pre-op MR (0.9 x 0.9 x 1.1 mm)
- 3 per-op US (0.63 and 0.95 mm)
- 3 loops

Robustness and precision

	Success	var rot (deg)	var trans (mm)
MI	29%	0.53	0.25
CR	90%	0.45	0.17
BCR	**85%**	**0.39**	**0.11**

Consistency of BCR

	var rot (deg)	var trans (mm)	var test (mm)
Multiple MR	0.06	0.06	0.10
Loop	2.22	0.82	2.33
MR/US	**1.57**	**0.58**	**1.65**

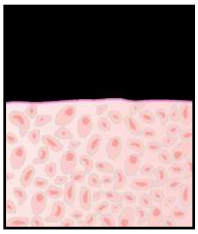


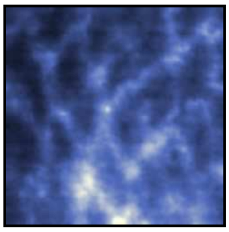
[Roche et al, TMI 20(10), 2001]

[Pennec et al, Multi-Sensor Image Fusion, Chap. 4, CRC Press, 2005]

Mosaicing of Confocal Microscopic in Vivo Video Sequences.

Cellvizio: Fibered confocal fluorescence imaging

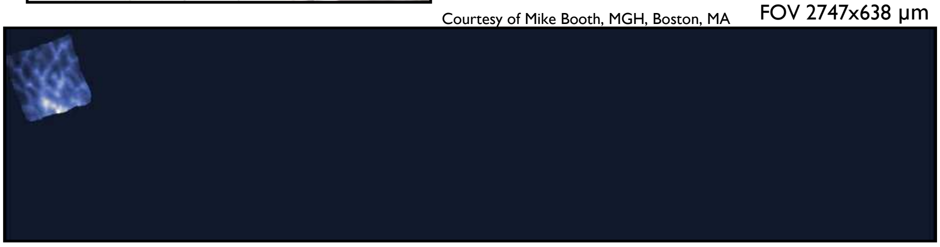




FOV 200x200 µm

Courtesy of Mike Booth, MGH, Boston, MA

FOV 2747x638 µm



[T. Vercauteren et al., MICCAI 2005, T.1, p.753-760, Talk on Friday]

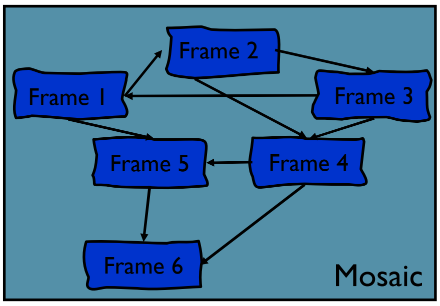
Mosaicing of Confocal Microscopic in Vivo Video Sequences.

Common coordinate system

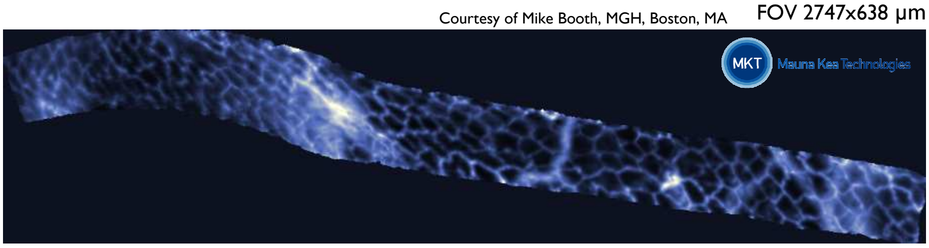
- **Multiple rigid** registration
- Refine with **non rigid**

Mosaic image creation

- Interpolation / approximation with **irregular sampling**



Courtesy of Mike Booth, MGH, Boston, MA FOV 2747x638 µm



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Overview

✓ **Statistics on point-wise geometric features**

⇨ **Fields of geometric features: Tensor computing**
- ⇨ **Interpolation, filtering, diffusion**
- ❍ Morphometry of sulcal lines on the brain

❍ **Statistics of deformations for non-linear registration**

❍ **Conclusion**

Diffusion tensor imaging

Very noisy data

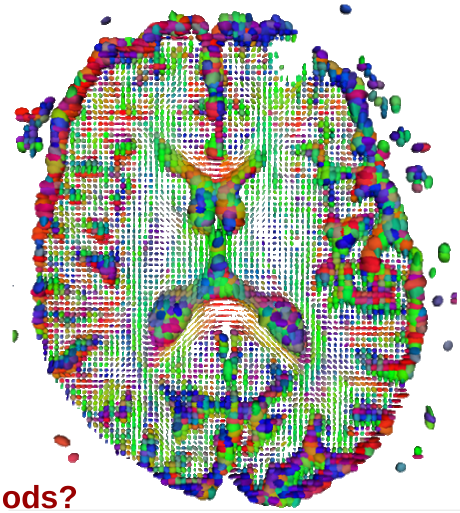
Preprocessing steps

- Filtering
- Regularization
- Robust estimation

Processing steps

- Interpolation / extrapolation
- Statistical comparisons

Can we generalize scalar methods?



DTI Tensor field (slice of a 3D volume)

Tensor computing

Tensors = space of positive definite matrices

- Linear convex combinations are stable (mean, interpolation)
- More complex methods are not (null or negative eigenvalues) (gradient descent, anisotropic filtering and diffusion)

Current methods for DTI regularization

- Principle direction + eigenvalues [Poupon MICCAI 98, Coulon Media 04]
- Iso-spectral + eigenvalues [Tschumperlé PhD 02, Chef d'Hotel JMIV04]
- Choleski decomposition [Wang&Vemuri IPMI03, TMI04]
- Still an active field…

Riemannian geometric approaches

- Statistics [Pennec PhD96, JMIV98, NSIP99, IJCV04, Fletcher CVMIA04]
- Space of Gaussian laws [Skovgaard84, Forstner99,Lenglet04]
- Geometric means [Moakher SIAM JMAP04, Batchelor MRM05]

Affine Invariant Metric on Tensors

Action of the Linear Group GL_n $A * \Sigma = A.\Sigma.A^T$

Invariant distance $dist(A * \Sigma_1, A * \Sigma_2) = dist(\Sigma_1, \Sigma_2)$

Invariant metric $\langle W_1 | W_2 \rangle_\Sigma \stackrel{def}{=} \langle \Sigma^{-1/2} * W_1, \Sigma^{-1/2} * W_2 \rangle_{Id}$

- Usual scalar product at identity $\langle W_1 | W_2 \rangle_{Id} \stackrel{def}{=} Tr\left(W_1^T W_2\right)$
- Geodesics $\exp_\Sigma(\overrightarrow{\Sigma\Psi}) = \Sigma^{1/2} \exp(\Sigma^{-1/2}.\overrightarrow{\Sigma\Psi}.\Sigma^{-1/2})\Sigma^{1/2}$
- Distance $dist(\Sigma, \Psi)^2 = \langle \overrightarrow{\Sigma\Psi} | \overrightarrow{\Sigma\Psi} \rangle_\Sigma = \left\| \log(\Sigma^{-1/2}.\Psi.\Sigma^{-1/2}) \right\|_{L_2}^2$

[X Pennec, P.Fillard, N.Ayache, IJCV 65(1), Oct. 2005 and RR-5255, INRIA, 2004]

Log Euclidean Metric on Tensors

Exp/Log: global diffeomorphism Tensors/sym. matrices

- Vector space structure carried from the tangent space to the manifold
  - Log. product $\Sigma_1 \otimes \Sigma_2 \equiv \exp\left(\log(\Sigma_1) + \log(\Sigma_2)\right)$
  - Log scalar product $\alpha \bullet \Sigma \equiv \exp(\alpha \log(\Sigma)) = \Sigma^\alpha$
  - Bi-invariant metric $dist(\Sigma_1, \Sigma_2)^2 \equiv \left\| \log(\Sigma_1) - \log(\Sigma_2) \right\|^2$

Properties

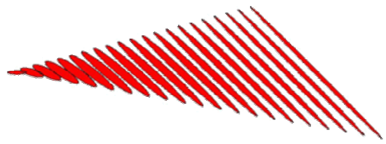
- Invariance by the action of similarity transformations only
- Very simple algorithmic framework
- Affine and Log-Euclidean means are geometric
  - Log Euclidean slightly more anisotropic
  - Speedup ratio: 7 (aniso. filtering) to >50 (interp.)

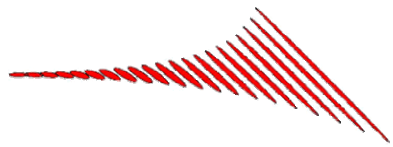
[Arsigny, Fillard, Pennec, Ayache, MICCAI 2005, T1, p.115-122, talk on Thursday]

Tensor interpolation

Geodesic walking in 1D

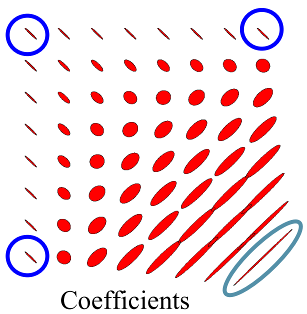
$$\Sigma(t) = \exp_{\Sigma_1}(t\overrightarrow{\Sigma_1\Sigma_2})$$



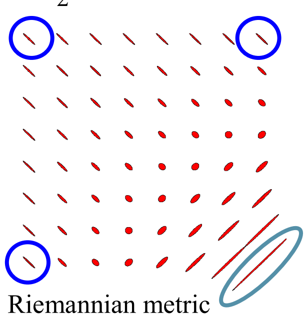


Weighted mean in general

$$\Sigma(x) = \min_{\Sigma} \sum w_i(x)\ dist(\Sigma, \Sigma_i)^2$$



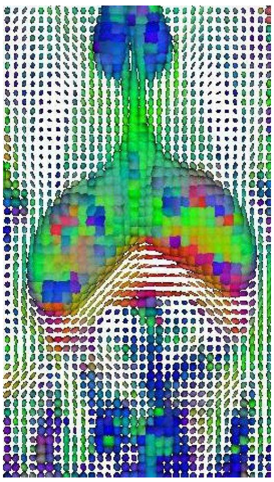
Coefficients



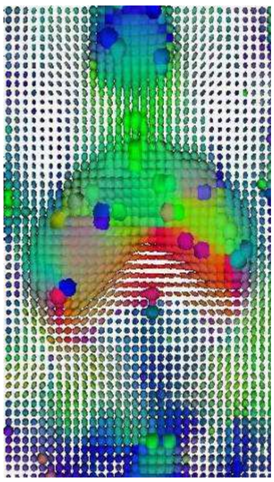
Riemannian metric

Gaussian filtering: Gaussian weighted mean

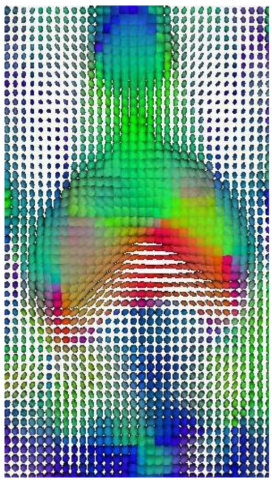
$$\Sigma(x) = \min \sum_{i=1}^{n} G_\sigma(x - x_i)\ dist\ (\Sigma, \Sigma_i)^2$$



Raw



Coefficients σ=2



Riemann σ=2

PDE for filtering and diffusion

Harmonic regularization

$$C(\Sigma) = \int_{\Omega} \left\| \nabla \Sigma(x) \right\|_{\Sigma(x)}^{2} dx$$

- Gradient = Laplace Beltrami operator $\nabla C(x) = -2 \Delta \Sigma(x)$

$$\Delta \Sigma(x) = \sum_i \partial_i^2 \Sigma - \sum_i (\partial_i \Sigma) \Sigma^{(-1)} (\partial_i \Sigma) = \sum_u \frac{\overrightarrow{\Sigma(x)\Sigma(x+u)}}{\|u\|^2} + O\left(\|u\|^2\right)$$

- Integration scheme = geodesic marching

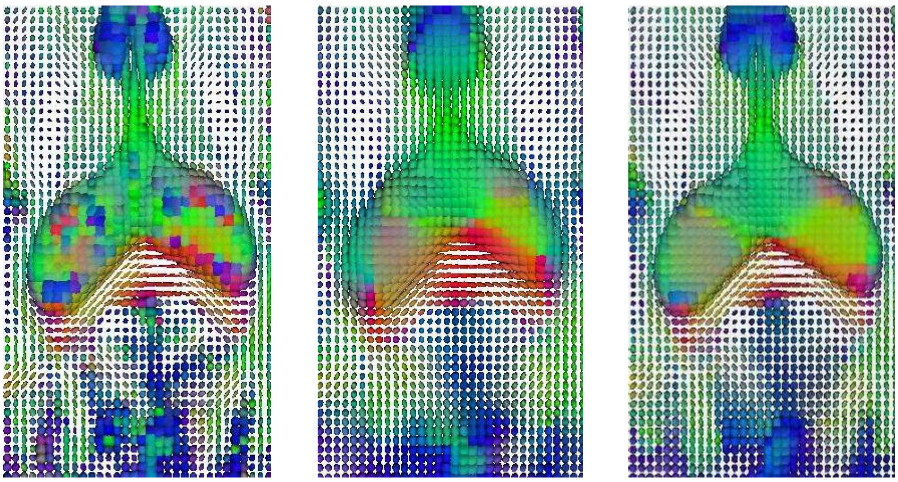
$$\Sigma_{t+1}(x) = \exp_{\Sigma_t(x)}\left(-\varepsilon \nabla C(\Sigma)(x)\right)$$

Anisotropic regularization

- Perona-Malik 90 / Gerig 92
- Phi functions formalism

Anisotropic filtering

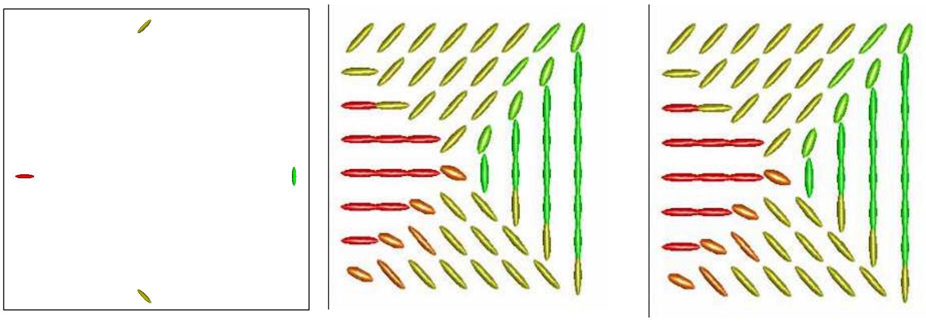
$$\Delta_w \Sigma(x) = \sum_u w(\partial_u \Sigma(x)) \; \Delta_u \Sigma(x) \quad \text{with} \quad w(t) = \exp(-t^2 / \kappa^2)$$



Raw Riemann Gaussian Riemann anisotropic

Extrapolation by Diffusion

$$C(\Sigma) = \frac{1}{2}\int_{\Omega}\sum_{i=1}^{n} G_{\sigma}(x - x_i)\, dist(\Sigma(x), \Sigma_i)^2\, dx + \frac{\lambda}{2}\int_{\Omega}\left\|\nabla\Sigma(x)\right\|^2_{\Sigma(x)}$$



Original Tensor Data | Diffusion without data attachment | Diffusion with data attachment

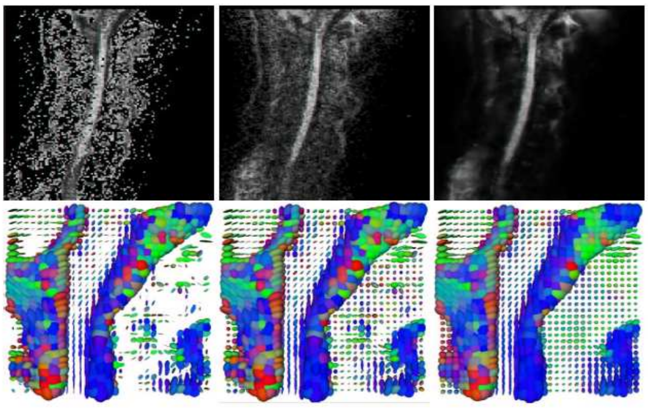
Joint Estimation and regularization from DWI

$$C(\Sigma) = \int \sum_i \left(S_i - S_0 \exp\left(-b\, \mathbf{g}_i^T \Sigma(x)\, \mathbf{g}_i\right)\right)^2 + \Phi\left(\left\|\nabla\Sigma(x)\right\|^2_{\Sigma(x)}\right)$$

Clinical DTI of the spinal cord

FA

Estimated tensors

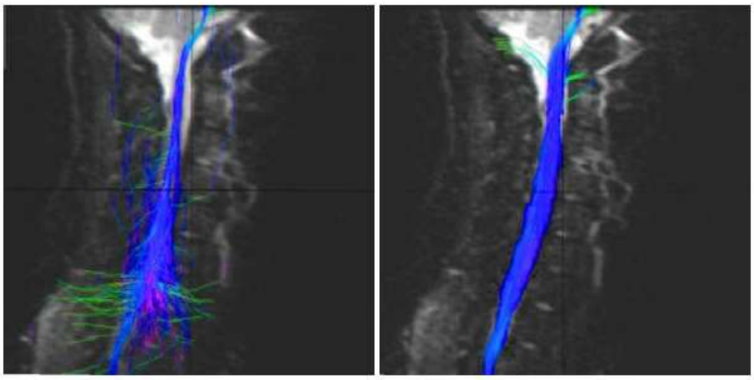


Standard | LSQ | LSQ + Φ-regul

[Fillard, Arsigny, Pennec, Ayache, RR-5607, June 2005]

Joint Estimation and regularization from DWI

Clinical DTI of the spinal cord: fiber tracking



Standard | LSQ + Phi-regul

Overview

- ✓ **Statistics on point-wise geometric features**
- ⇨ **Fields of geometric features: Tensor computing**
  - ✓ **Interpolation, filtering, diffusion**
  - ⇨ **Morphometry of sulcal lines on the brain**
- ❍ **Statistics of deformations for non-linear registration**
- ❍ **Conclusion**

Morphometry of Sucal Lines

Goal:

- Learn local brain variability from sulci
- Better constrain inter-subject registration
- Correlate this variability with age, pathologies

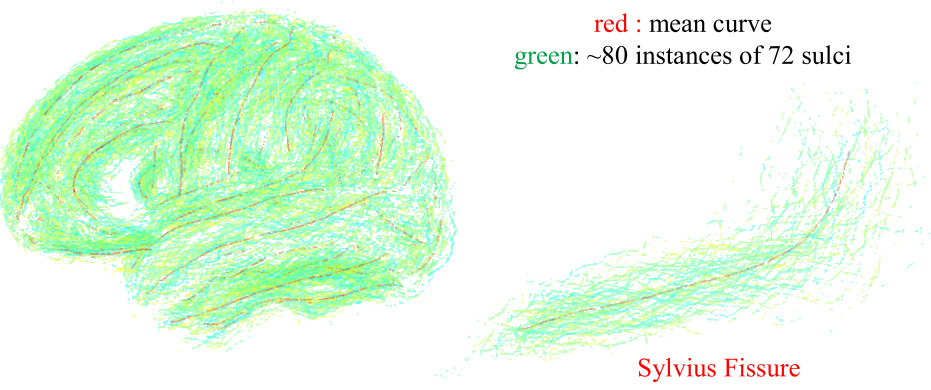
Collaborative work between Epidaure (INRIA) and LONI (UCLA)
V. Arsigny, N. Ayache, P. Fillard, X. Pennec and P. Thompson

[Fillard, Arsigny, Pennec, Ayache, Thompson, IPMI'05]

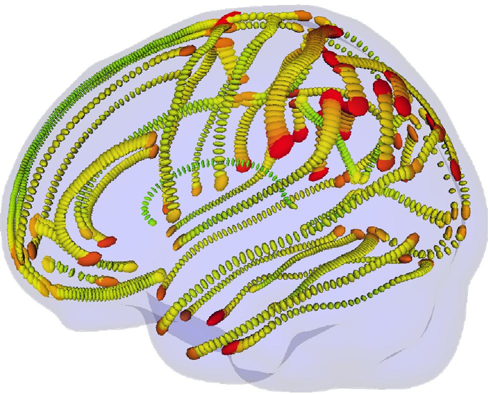
Computation of Average Sulci

Alternate minimization of global variance

- Dynamic programming to match the mean to instances
- Gradient descent to compute the mean curve position



Extraction of Covariance Tensors

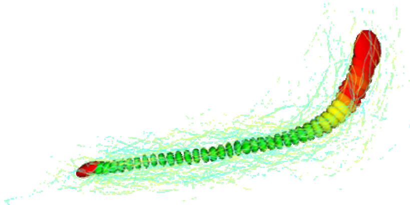


Color codes Trace

Currently:

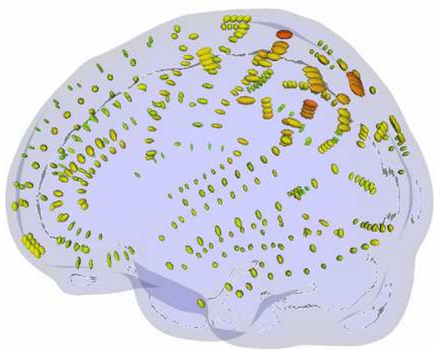
80 instances of 72 sulci

About 1250 tensors

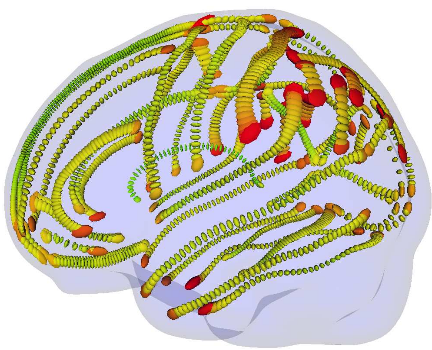


Covariance Tensors
along Sylvius Fissure

Compressed Tensor Representation

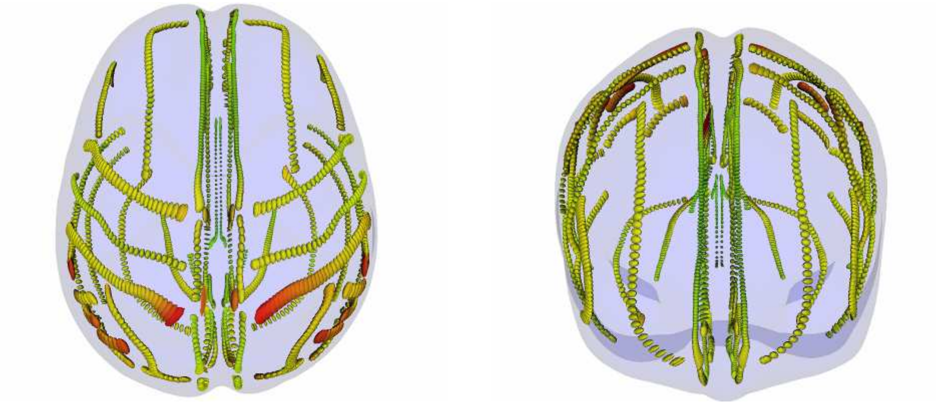


Representative Tensors (250)



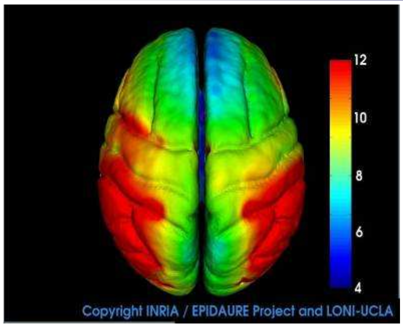
Reconstructed Tensors (1250)
Original Tensors (1250)
(Riemannian Interpolation)

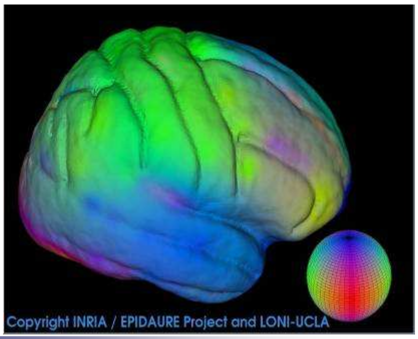
Variability Tensors

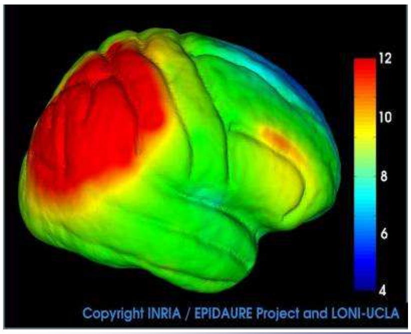


Color codes tensor trace

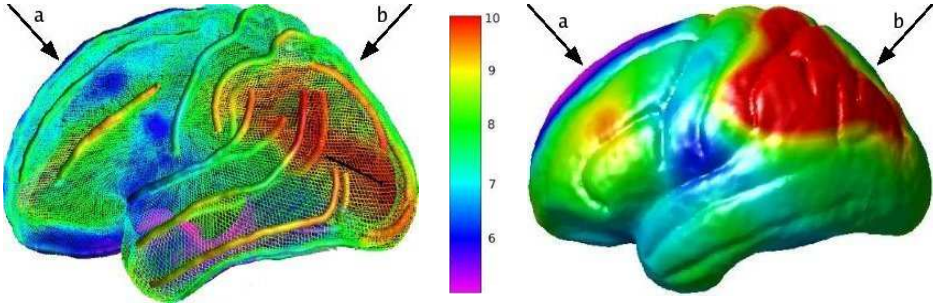
Full Brain extrapolation of the variability







Comparison with cortical surface variability



P. Thompson at al, HMIP, 2000

Average of 15 normal controls by non-linear registration of surfaces

P. Fillard et al, IPMI 05

Extrapolation of our model estimated from 98 subjects with 72 sulci.

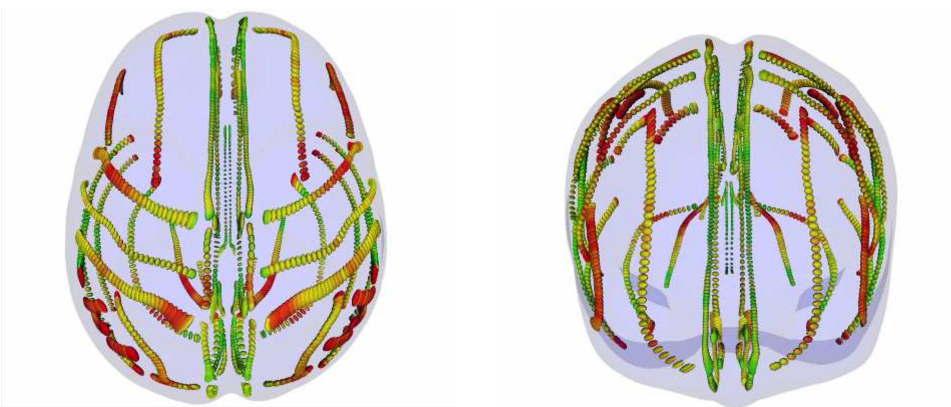
Consistent low variability in phylogenetical older areas

- (a) superior frontal gyrus

Consistent high variability in highly specialized and lateralized areas

- (b) temporo-parietal cortex

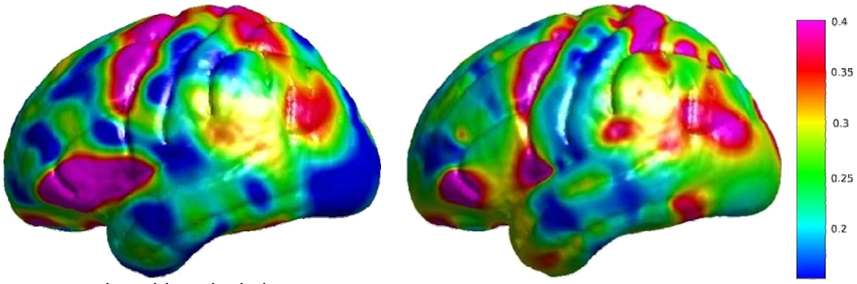
Quantitative comparison: Asymmetry Measure



Color Codes Distance between tensors at "symmetric" positions

$$dist(\Sigma,\Sigma')^2 = \left\langle \overrightarrow{\Sigma\Sigma'} \mid \overrightarrow{\Sigma\Sigma'} \right\rangle_\Sigma = \left\| \log(\Sigma^{-1/2}.\Sigma'.\Sigma^{-1/2}) \right\|_{L_2}^2$$

Asymmetry Measures



w.r.t the mid-sagittal plane. w.r.t opposite (left-right) sulci

Greatest asymmetry	Lowest asymmetry
Broca's speech area and Wernicke's language comprehension area	Primary sensorimotor areas

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⇨ **Statistics on deformations for non linear registration**

❍ **Conclusion**

Non-linear elastic regularization

Gradient descent

$$C(\Phi) = \text{Sim}(\text{Images}, \Phi) + \text{Reg}(\Phi)$$

$$\Phi_{t+1} = \Phi_t - \kappa \nabla C(\Phi_t)$$

Regularization

- Local deformation measure: Cauchy Green strain tensor
  - Id for local rotations
  - Small for local contractions
  - Large for local expansions

$$\Sigma = \nabla\Phi^t . \nabla\Phi$$

- St Venant Kirchoff elastic energy

$$\text{Reg}(\Phi) = \int \mu \,\text{Tr}\left((\Sigma - I)^2\right) + \frac{\lambda}{2} \text{Tr}(\Sigma - I)^2$$

[Pennec, et al, MICCAI 2005, T2, p.943-950, poster #S47 Saturday]

Statistical Riemannian elasticity

Problems

- Elasticity is not symmetric $\quad d(\Sigma, 0) = d(\Sigma, 2\Sigma)$
- Statistics are not easy to include

Idea:

$$\text{Tr}\left((\Sigma - I)^2\right) \rightarrow dist_{LE}(\Sigma, I)^2 = \|\log(\Sigma)\|^2$$

- Replace the Euclidean by the Log-Euclidean metric

Statistics on strain tensors

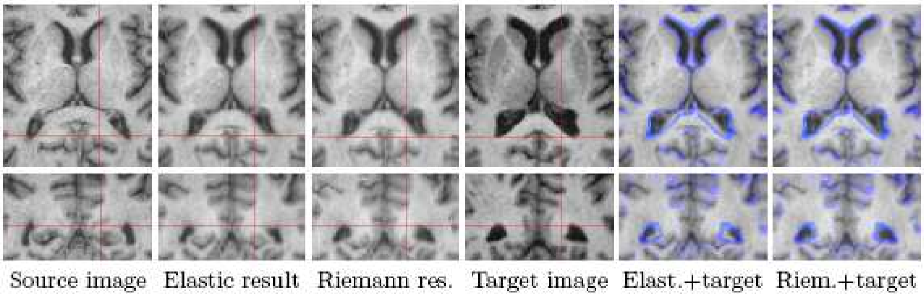
- Mean, covariance, Mahalanobis computed in Log-space

$$\text{Re}\,g(\Phi) = \int \text{Vect}\left(\log(\Sigma) - \overline{W}\right)^T . \text{Cov}^{-1} . \text{Vect}\left(\log(\Sigma) - \overline{W}\right)$$

- Isotropic Riemannian Elasticity

$$\text{Re}\,g_{\text{iso}}(\Phi) = \int \mu \,\text{Tr}\left(\log(\Sigma)^2\right) + \frac{\lambda}{2} \text{Tr}(\log(\Sigma))^2$$

Isotropic Riemannian Elasticity Results



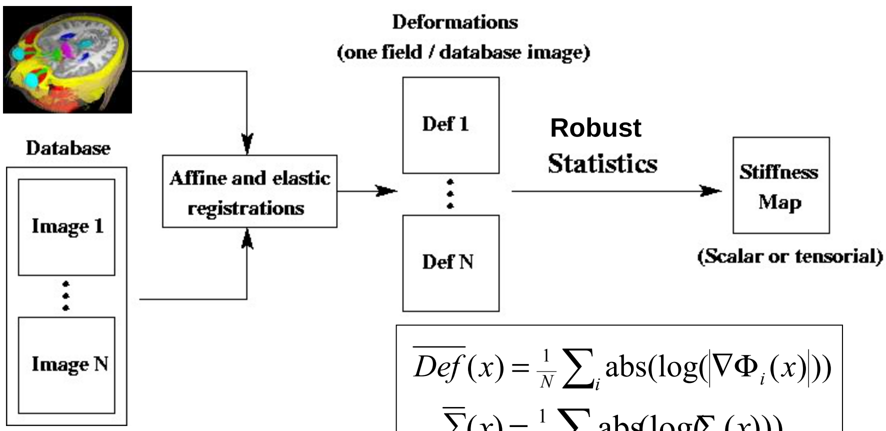
Roi 186x124x216 voxels, λ=μ=0.2, 12 PC 2Gh.

- Larger computation times 3h vs 1h
- Slightly larger and better deformation of the right ventricle
  without any statistical information yet…

[Pennec, et al, MICCAI 2005, T2, p.943-950, poster #S47 Saturday]

Statistics on the deformation field

- Objective: planning of conformal brain radiotherapy
- 30 patients, 2 to 5 time points (P-Y Bondiau, MD, CAL, Nice)



$$\overline{Def}(x) = \frac{1}{N}\sum_i \mathrm{abs}(\log(|\nabla\Phi_i(x)|))$$

$$\overline{\Sigma}(x) = \frac{1}{N}\sum_i \mathrm{abs}(\log(\Sigma_i(x)))$$

[Commowick, et al, MICCAI 2005, T2, p. 927-931, poster #S45, Saterday]

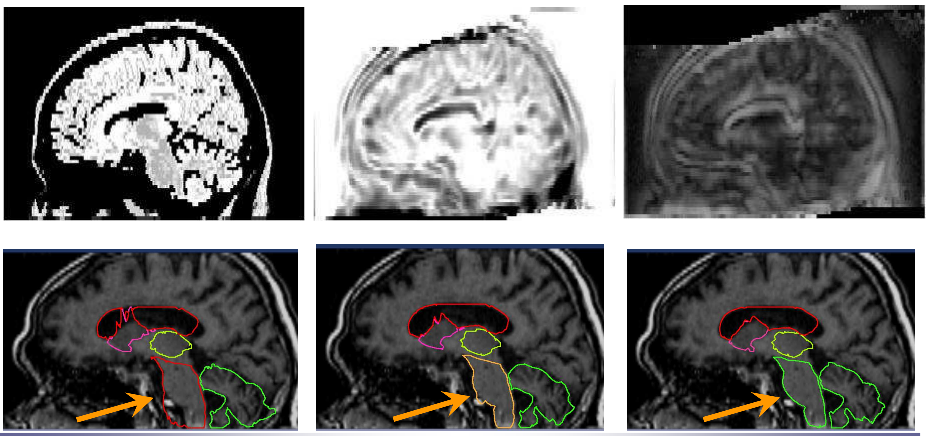
Introducing deformation statistics into RUNA

RUNA [R. Stefanescu et al, Med. Image Analysis 8(3), 2004]

- non linear-registration with non-stationary regularization
- Scalar or tensor stiffness map

$$D(x) = (Id + \lambda \overline{\Sigma}(x))^{-1}$$

Heuristic RUNA stiffness **Scalar statistical stiffness** **Tensor stat. stiffness (FA)**



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Conclusion : geometry and statistics

A Statistical computing framework on “simple” manifolds

- Mean, Covariance, statistical tests…
- Interpolation, diffusion, filtering…
- Which metric for which problem?

Extend to more complex groups and manifolds

- Deformations (Trouvé, Younes, Miller)
- Shapes (Kendall, Olsen)

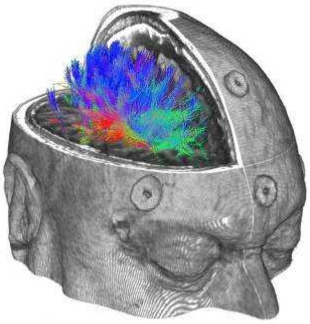
Spatially extended features (curves, surfaces, volumes…)

- Homology assumption (mixtures ?)
- Spatial correlation between neighbors… and distant points
- Probability density for curves and surfaces

Applications of Riemannian Computing

Registration

- Performance evaluation
- Introducing a-priori distributions
- Statistical deformations



Diffusion tensor imaging

- Regularization for fiber tracts estimation
- Registration (atlases)

Variability of the brain

- Learn Variability from Large Group Studies
- Statistical Comparisons between Groups
- Improve Inter-Subject Registration

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