Statistical Analysis on Riemannian Shape Spaces

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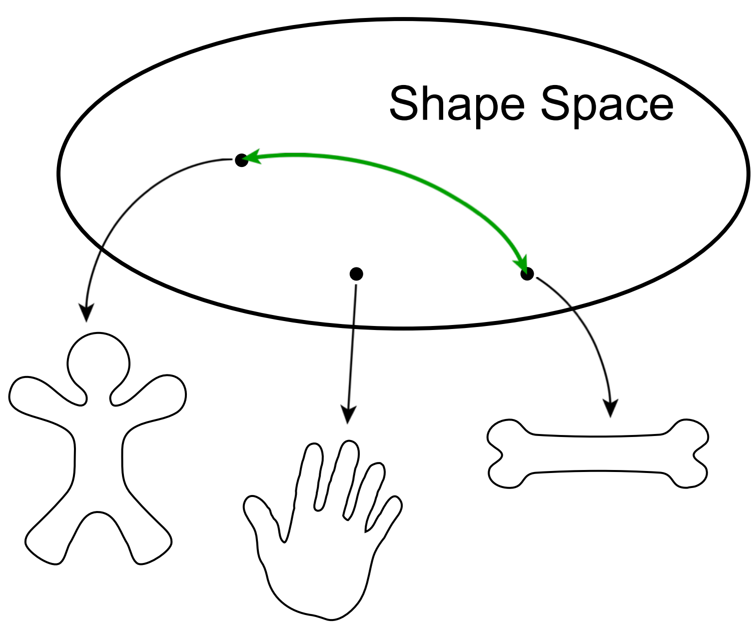
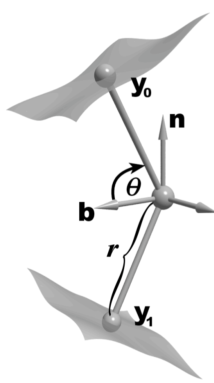
Shape Analysis

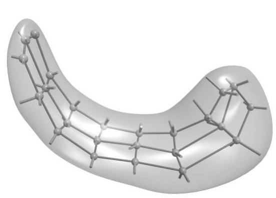


Application Areas

- Medial representations of shape (m-reps).
- Diffusion tensor MRI.
- Continuous models of solid shape.

The M-rep Shape Space





Medial Atom:

$$\mathbf{m} = \{\mathbf{x}, r, \mathbf{n}_0, \mathbf{n}_1\} \in \mathcal{M}(1)$$

$$\mathcal{M}(1) = \mathbb{R}^3 \times \mathbb{R}^+ \times S^2 \times S^2$$

M-rep Model with n atoms:

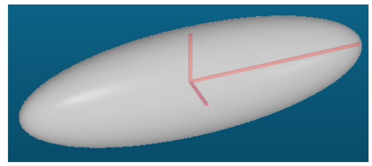
$$\mathbf{M} \in \mathcal{M}(n) = \mathcal{M}(1)^n$$

Shape change in terms of local translation, bending, & widening.

Diffusion Tensors

- DT-MRI produces a 3×3 symmetric, positive-definite matrix at each voxel.

$$D = D^T,$$
$$x^T D x > 0 \quad \text{for} \quad x \neq 0.$$

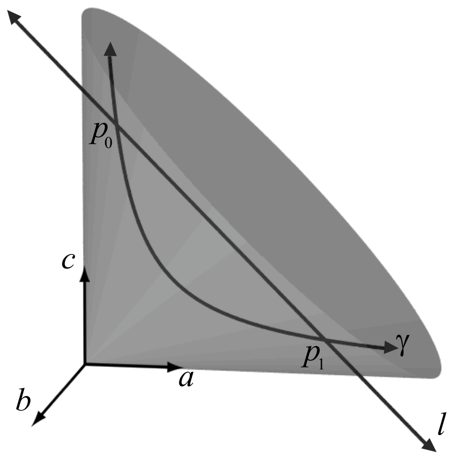


- Represents covariance in Brownian motion model of water diffusion - fiber tracts in major axis direction.
- What about statistical studies of DTI across subjects?

Geometry of the Diffusion Tensor Space

- Let $PD(n)$ denote the space of all $n \times n$ symmetric, positive-definite real matrices.
- $PD(n)$ is not a vector space (doesn't contain 0, not closed under negation).
- $PD(n)$ is a curved manifold, a Riemannian symmetric space.

Example: $PD(2)$



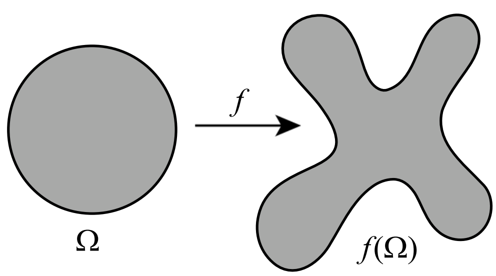
$A \in PD(2)$ is of the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

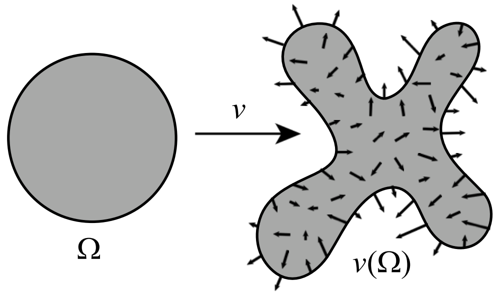
$$ac - b^2 > 0, \quad a > 0.$$

Similar situation for $PD(3)$ (6-dimensional).

Models of Continuous Solid Shape

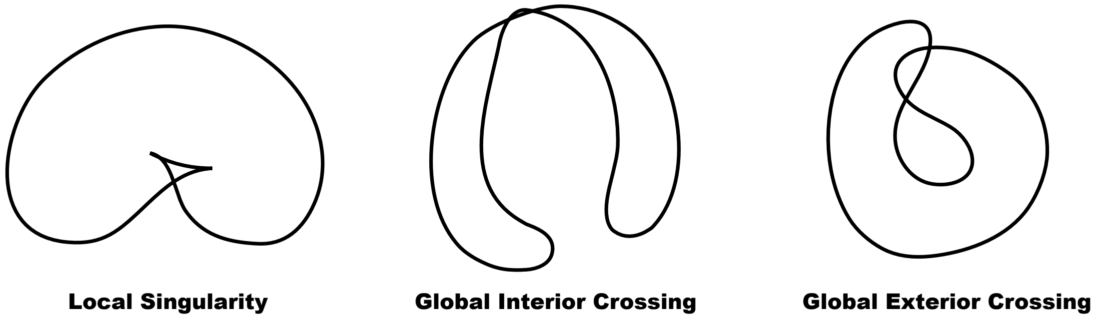


$Emb(\Omega, \mathbb{R}^n)$



$C^\infty(\Omega, \mathbb{R}^n)$

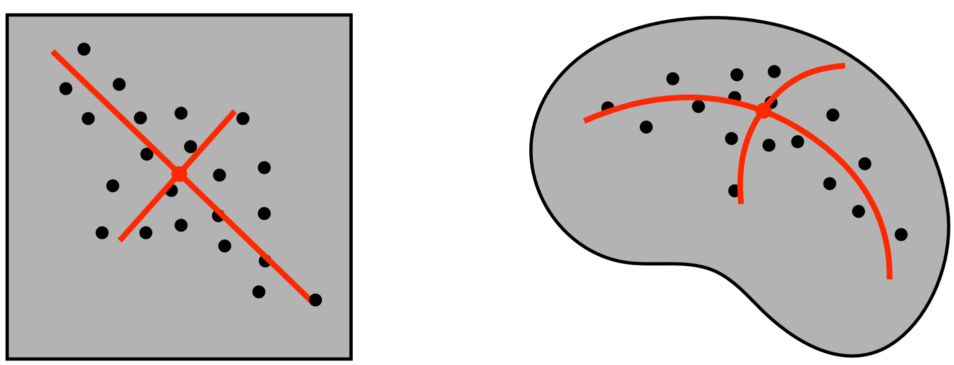
Prevent Shape Self-Intersections



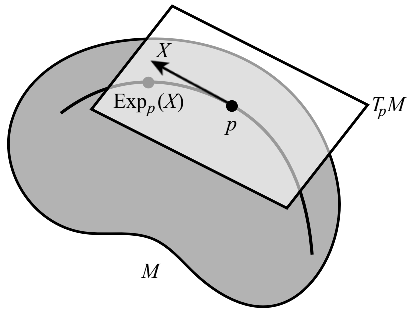
Principal Geodesic Analysis

Linear Statistics (PCA)

Curved Statistics (PGA)



The Exponential Map



- Maps tangent vectors to points along geodesics.
- Inverse is the log map – gives distance between points: $d(p, q) = \| \mathrm{Log}_p(q) \|$.

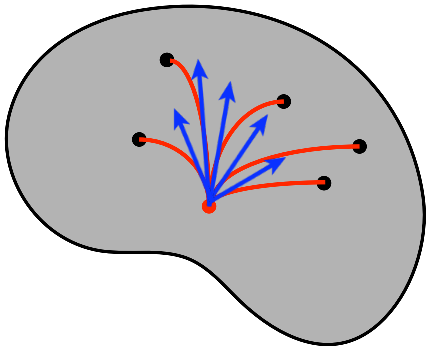
Intrinsic Means (Fréchet)

The *intrinsic mean* of a collection of points $x_1, \ldots, x_N$ on a Riemannian manifold M is

$$\mu = \arg\min_{x \in M} \sum_{i=1}^{N} d(x, x_i)^2,$$

where $d(\cdot, \cdot)$ denotes Riemannian distance on M.

Computing Means



Gradient Descent Algorithm:

Input: $\mathbf{x}_1, \ldots, \mathbf{x}_N \in M$

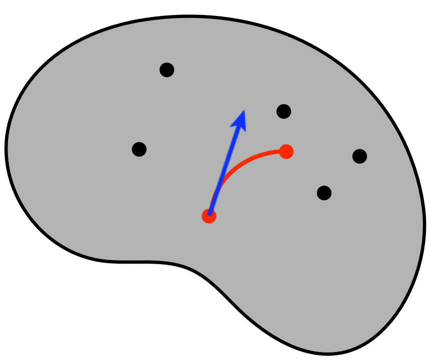
$\mu_0 = \mathbf{x}_1$

Repeat:

$\Delta\mu = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Log}_{\mu_k}(\mathbf{x}_i)$

$\mu_{k+1} = \mathrm{Exp}_{\mu_k}(\Delta\mu)$

Computing Means



Gradient Descent Algorithm:

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Covariance

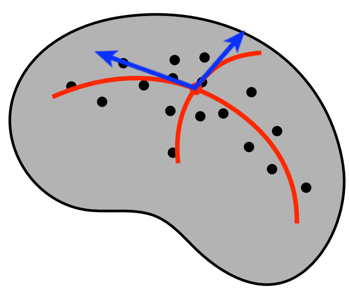
Sample covariance in the tangent space:

$$S = \frac{1}{N-1} \sum_{i=1}^{N} \operatorname{Log}_\mu(x_i) \operatorname{Log}_\mu(x_i)^T$$

Gives a "Gaussian" probability model:

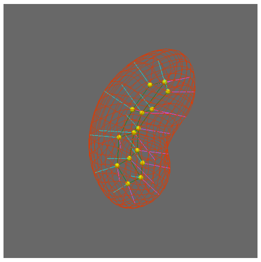
$$p(x) = k \exp\left(-\frac{1}{2} \operatorname{Log}_\mu(x)^T S^{-1} \operatorname{Log}_\mu(x) \right)$$

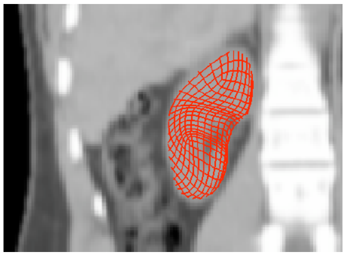
Principal Geodesic Analysis



- Find nested linear subspaces $V_k \subset T_pM$ such that $\operatorname{Exp}_\mu(V_k)$ maximizes variance of projected data.
- First-order approximation: PCA in tangent space of sample covariance matrix S.

M-rep Shape Statistics in Segmentation



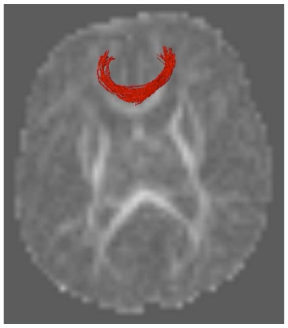


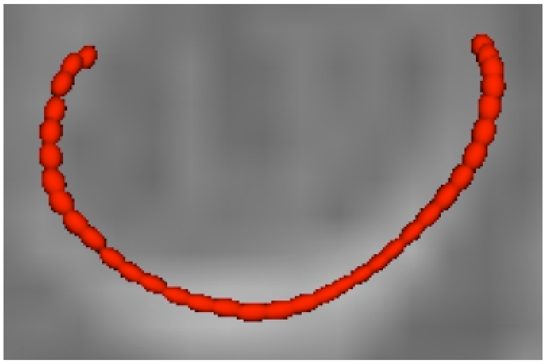
Optimize shape parameters $\{\alpha_1, \ldots, \alpha_d\}$, generating m-rep models:

$$\mathbf{M} = \mathrm{Exp}_\mu \left(\sum_{k=1}^{d} \alpha_k v_k \right).$$

Maximize log-posterior in Bayesian framework.

Tract-Oriented Diffusion Tensor Statistics



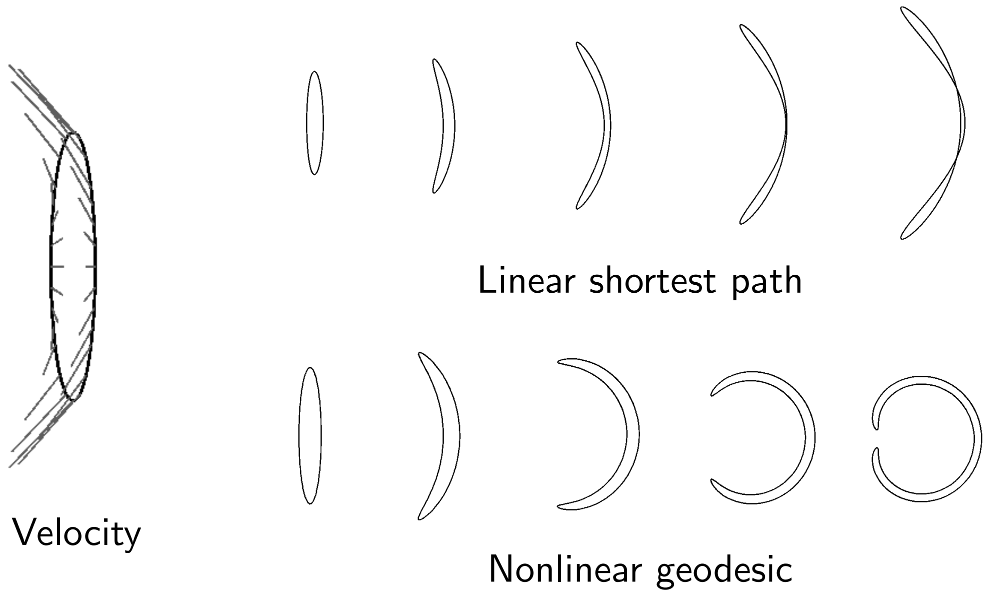


Average tensors along a fiber tract.

Corouge et al., Fiber Tract-Oriented Statistics for Quantitative Diffusion Tensor MRI Analysis, MICCAI 2005.

Part of the NA-MIC project: http://www.na-mic.org

Continuous Solid Shape Geodesics



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Papers available at:
http://www.cs.unc.edu/~fletcher/research.html

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