Review of Probability (Chapter 5)

COMP122: Algorithm and Analysis
Lecture for Tuesday/Thursday, September 13-15, 2005

- **Sample Space:** a set whose elements are *elementary events*.
  For example: flipping 2 coins, $S = \{HH, HT, TH, TT\}$.
- **Events:** a subset A of the sample space S, i.e. $A \subseteq S$.
- **Certain event:** S, **null event:** $\emptyset$.
- Two events A and B are **mutually exclusive**, if $A \cap B = \emptyset$.
- **Axioms of Probability:** A *probability distribution* $Pr\{\}$ on a sample space S is a mapping from events of S to real numbers such that the following are satisfied:
  1. $Pr\{A\} \geq 0$ for any event A.
  2. $Pr\{S\} = 1$.
  3. $Pr\{A \cup B\} = Pr\{A\} + Pr\{B\}$ for any two mutually exclusive events A and B.

  $Pr\{A\}$ is the probability of event A. Axiom 2 is a normalization requirement. Axiom 3 can be generalized to the following:
- $$Pr\{\bigcup_i A_i\} = \sum_i Pr\{A_i\}$$
- $Pr\{\emptyset\} = 0$
- If $A \subseteq B$, then $Pr\{A\} \leq Pr\{B\}$.
- $Pr\{\bar{A}\} = Pr\{S - A\} = 1 - Pr\{A\}$.
- For any two events A and B,
  $Pr\{A \cup B\} = Pr\{A\} + Pr\{B\} - Pr\{A \cap B\} \leq Pr\{A\} + Pr\{B\}$

- **Conditional Probability** of an event A given another event B is:

$$Pr\{A|B\} = \frac{Pr\{A \cap B\}}{Pr\{B\}}, \quad provided\ Pr\{B\} \neq 0 \tag{1}$$

  For example, $A = \{TT\}$ and $B = \{HT, TH, TT\}$. $Pr\{A|B\} = \frac{1/4}{3/4} = 1/3$

- **Independent Events:** Events A and B are independent if
  $Pr\{A \cap B\} = Pr\{A\} * Pr\{B\}$, provided $Pr\{B\} \neq 0$
  Equivalently, $Pr\{A|B\} = Pr\{A\}$

- **Bayes's Theorem:**

  $Pr\{A \cap B\} = Pr\{B\} * Pr\{A|B\} = Pr\{A\} * Pr\{B|A\}$
  Therefore,

$$Pr\{A|B\} = \frac{Pr\{A\} * Pr\{B|A\}}{Pr\{B\}} \tag{2}$$

$$\begin{aligned} Pr\{B\} &= Pr\{B \cap A\} + Pr\{B \cap \bar{A}\} \\ &= Pr\{A\} * Pr\{B|A\} + Pr\{\bar{A}\} * Pr\{B|\bar{A}\} \end{aligned}$$

  Therefore,

$$Pr\{A|B\} = \frac{Pr\{A\} * Pr\{B|A\}}{Pr\{A\} * Pr\{B|A\} + Pr\{\bar{A}\} * Pr\{B|\bar{A}\}} \tag{3}$$

  For example, given a fair coin and a biased coin that always come up heads. We choose one of the two and flip the coin twice. The chosen coin comes up with heads both times. What is the probability that it is biased?

  Let A be the event that the biased coin is chosen and B be the event that the coins comes up heads both times. $Pr\{A\} = 1/2$, $Pr\{\bar{A}\} = 1/2$, $Pr\{B|A\} = 1$, $Pr\{B|\bar{A}\} = (1/2) * (1/2) = 1/4$

$$Pr\{A|B\} = \frac{(1/2) * 1}{(1/2) * 1 + (1/2) * (1/4)} = 4/5$$

- A **discrete random variable** X is a function from a finite or countable infinite sample space S to R.

  We define the event $X = x$ to be $\{s \in S : X(s) = x\}$.

$$f(x) = Pr\{X = x\} = \sum_{s \in S : X(s) = x} Pr\{s\}$$

  is the *probability density function* of the random variable X.

- **Expected Value** (or **expectation** or **mean**) of a discrete random variable is

$$E[X] = \sum_x x Pr\{X = x\} = \mu_x$$

  1. $E[aX + Y] = aE[X] + E[Y]$
  2. When events $X_1, X_2, ...X_n$ are mutually independent, then $E[X_1 X_2 ... X_n] = E[X_1]E[X_2]...E[X_n]$.

- **Variance** of a random variable X with mean $E[X]$ is

$$\begin{aligned}
Var[X] &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + E^2[X]] \\
&= E[X^2] - 2E[XE[X]] + E^2[X] \\
&= E[X^2] - 2E^2[X] + E^2[X] \\
&= E[X^2] - E^2[X].
\end{aligned}$$

- **Standard deviation** of random variable X is $SD(X) = \sqrt{Var(X)} = \sigma_x$

- **Chebychev's inequality:**

$$Pr\{|X - \mu_x| \geq \alpha\} \leq \frac{Var(X)}{\alpha^2} = \frac{\sigma_x^2}{\alpha^2}, \ \forall \alpha > 0$$

  For example, let $\alpha = c\sigma_x$, then $Pr\{|X - \mu_x| \geq c * \sigma_x\} \leq \frac{1}{c^2}$

  The probability that a random variable differs from its expected value by more than c times standard deviation is at most $1/c^2$.

An Example: Given an array A[1...n] of n distinct numbers randomly ordered with each random permutation equally likely. Where can you expect to find the maximum element in the array?

Let I_m be the index of the maximum element.

$$\begin{aligned} E[I_m] &= \sum_{i=1}^{n} i Pr\{I_m = i\} \\ &= \sum_{i=1}^{n} i * \frac{1}{n} \\ &= \frac{1}{n} * \frac{n(n+1)}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

$$\begin{aligned} E[I_m^2] &= \sum_{i=1}^{n} i^2 Pr\{I_m = i\} \\ &= \sum_{i=1}^{n} i^2 * \frac{1}{n} \\ &= \frac{1}{n} * \frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n+1)}{6} \end{aligned}$$

$$\begin{aligned} Var[I_m] &= E[I_m^2] - E^2[I_m] \\ &= \frac{(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 \\ &= \frac{n^2-1}{12} \end{aligned}$$

Therefore,

$$SD(I_m) = \sqrt{\frac{(n^2-1)}{12}}$$