A Slow Algorithm for Computing the Gabriel Graph with Double Precision

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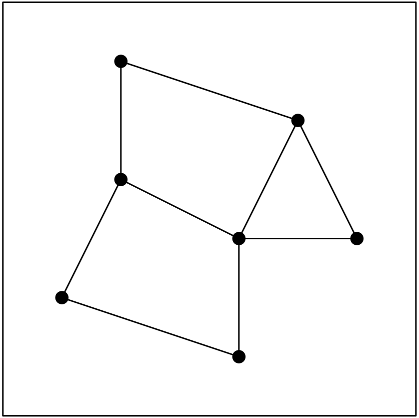
University of North Carolina at Chapel Hill

August 12, 2011

Compute the Gabriel Graph



Given

sites $S = \{s_1, \ldots, s_n\}$

Compute

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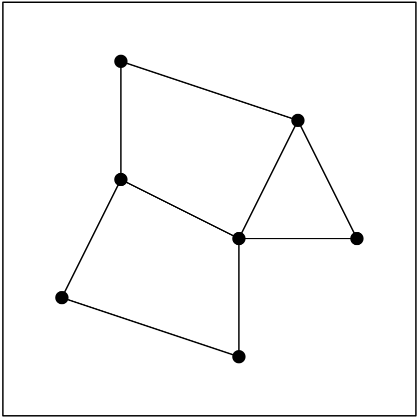
Compute the Gabriel Graph



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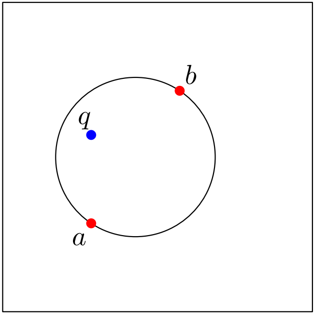
How much precision is needed to determine this?

Analyzing Precision[LPT99]

E.g., Precision of testing if a point is inside a circle



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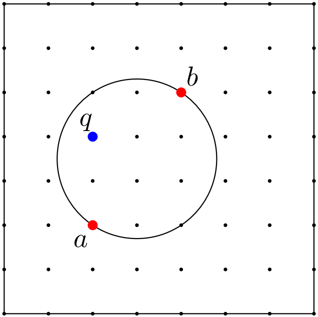

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$a, b, q \in \mathbb{U}$

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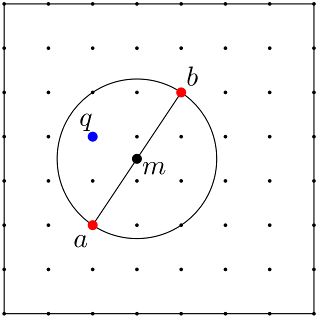

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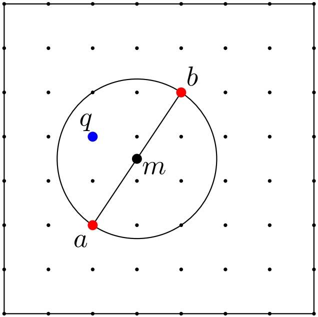

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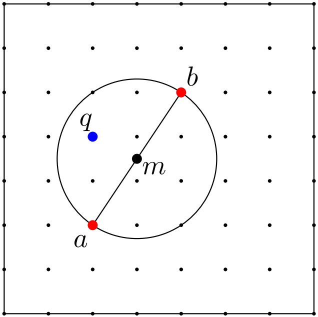

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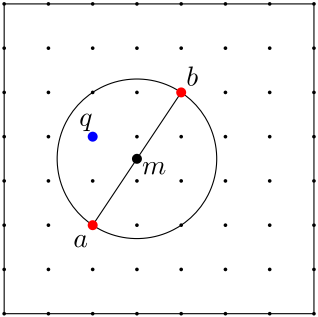

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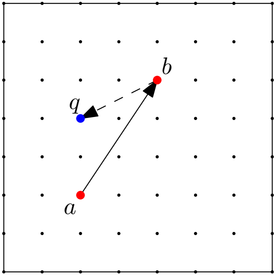
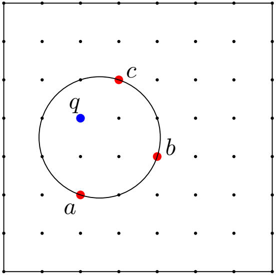
$\boxed{\text{degree 2}}$

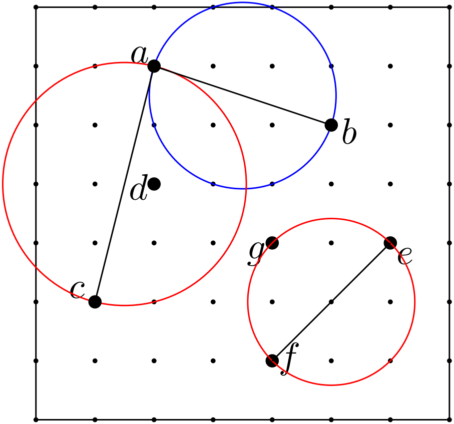
Precision of Two Well Know Predicates



`Orientation`(a, b, q)
degree ②



`InCircle`(a, b, c, q)
degree ④



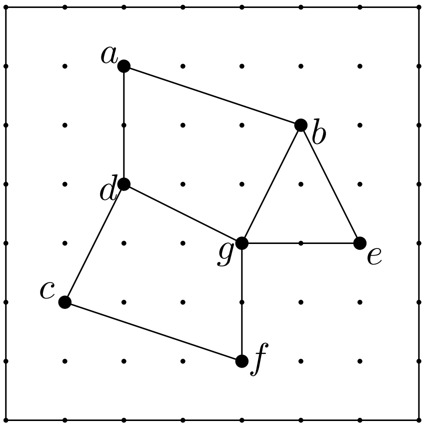
Given

sites n sites S

Definition

an edge $\overline{pq}$
is in the Gabriel graph of S
if the closed disk
centered at the midpoint of $\overline{pq}$
with diameter $|\overline{pq}|$
contains no points of $S \setminus \{p, q\}$.

Gabriel Graph



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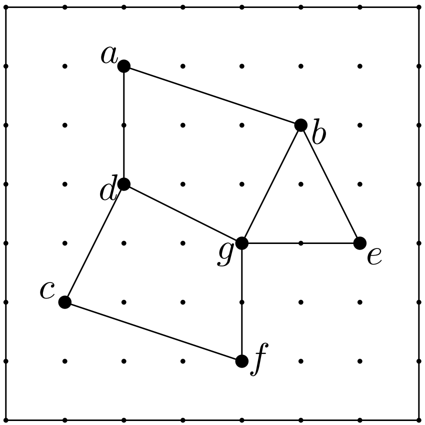

Proposed by:
Gabriel and Sokal [GS69]

Compute Gabriel from Delaunay:
[MS80] $O(n)$ time, degree 6
[L96] $O(n)$ time, degree 2

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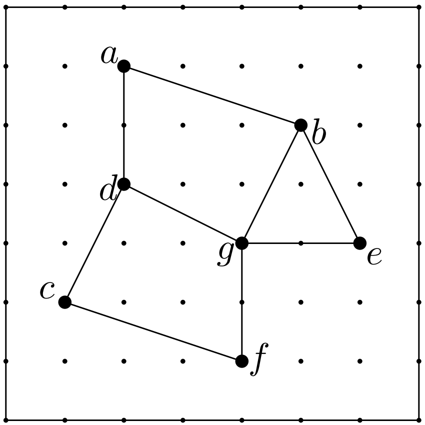

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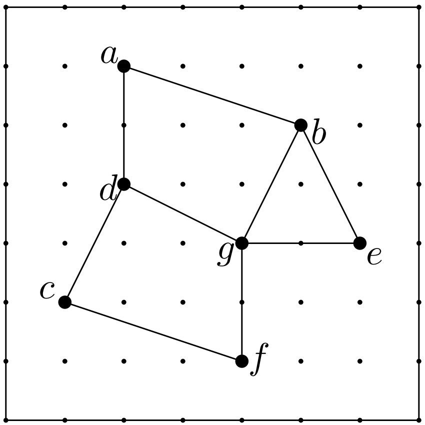

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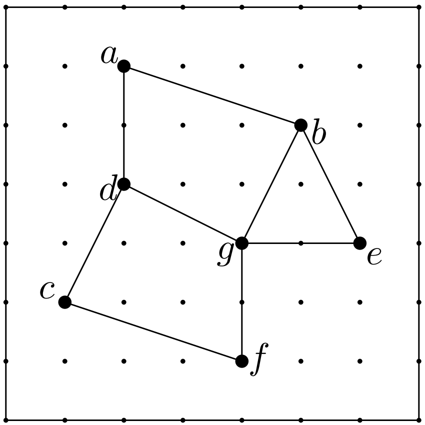
Directly compute Gabriel graph:
Brute force, $O(n^3)$ time, degree 2

Gabriel Graph



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[MV11], $O(n^2)$ time, degree 2

Robustness Approaches

Approaches for implementing geometric algorithms with finite precision computer arithmetic:

- Rely on machine precision ($+\epsilon$) [NAT90,LTH86,KMP*08]
- Topological Consistency [S99,S01,SI90,SI92,SII*00,H01]
- Exact Geometric Computation [Y97,C92,ABO*97,BEP*97]
  - Arithmetic Filters [FW93,FW96,BBP01,DP98,DP99]
  - Adaptive Predicates [P92,S97,BF09]
  - Degree-driven algorithm design [L96,LPT99,BP00,BS00,C00,MS01,MS09,CMS09,MS10]

Result

Given

sites $S = \{s_1, \ldots, s_n\}$

Arrangement of dual lines S^* and its trapezoidation

- Time: $O(n^2)$
- Space: $O(n^2)$
- Precision: degree ②

Gabriel graph

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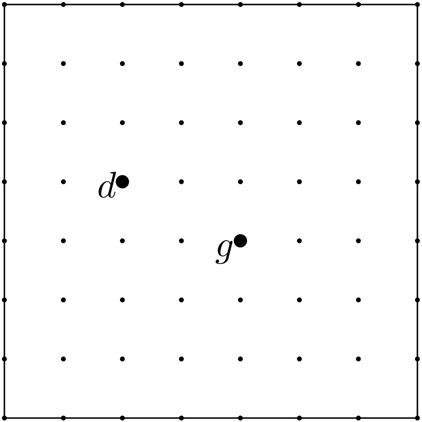
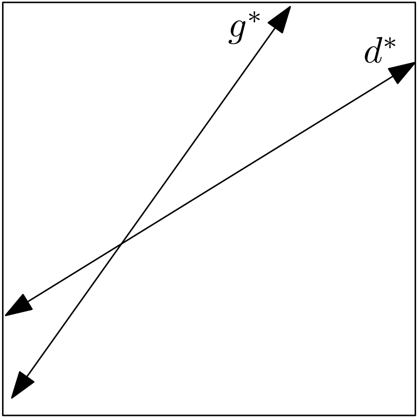
Primal

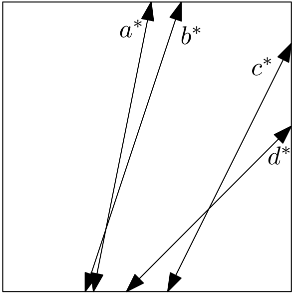
Point $p = (p_x, p_y)$
Line $\ell = (y = \ell_m x + \ell_b)$
Set of points $S = \{s_1, \ldots, s_n\}$

Dual

Line $p^* = (y = p_x x - p_y)$
Point $\ell^* = (\ell_m, -\ell_b)$
Set of lines $S^* = \{s_1^*, \ldots, s_n^*\}$



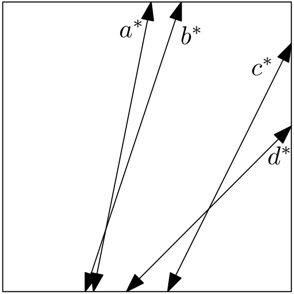




Given
dual lines a^*, b^*, c^*, and d^*

Determine
if the x-coordinate of
$a^* \cap b^*$ is left of $c^* \cap d^*$.

xIntersectonOrder(a^*, b^*, c^*, d^*)



Given

points a, b, c, and d

Determine

if the slope of $\overline{ab}$ is less than $\overline{cd}$.

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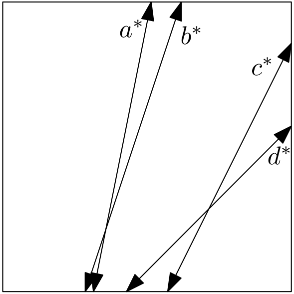

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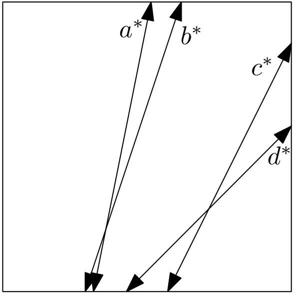
$$\texttt{xIntersectonOrder}(a^*, b^*, c^*, d^*) = \mathrm{sign}\left(\frac{a_y - b_y}{a_x - b_x} - \frac{c_y - d_y}{c_x - d_x}\right)$$



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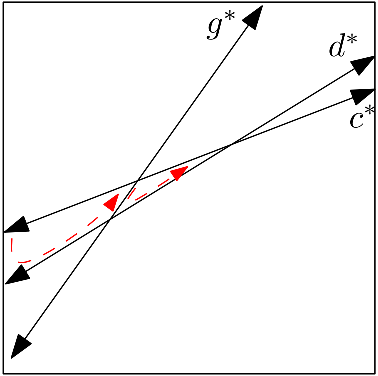
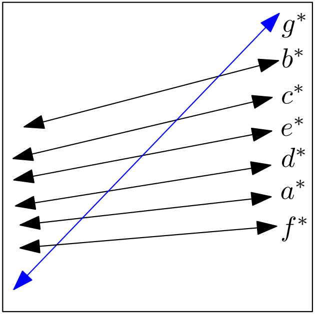
Arrangement of Dual Lines

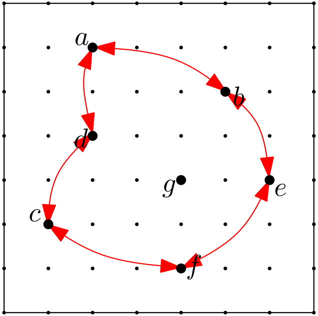


Arrangement

For n dual lines S^*, we can compute the arrangement of S^* and its trapezoidation in $O(n^2)$ time and degree ②.

Arrangement to Circular Ordering Around a Site

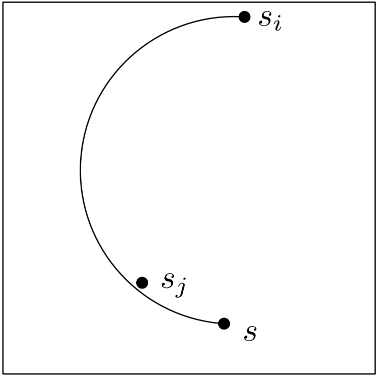




Circular Orderings

Given the arrangement of S^*, for site $s \in S$, we can compute the circular ordering of the sites in $S \setminus \{s\}$ around s in $O(n)$ time and degree ①.

Kill Edges



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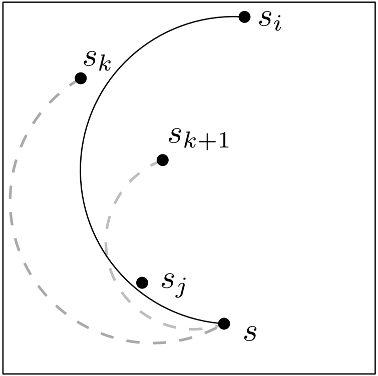
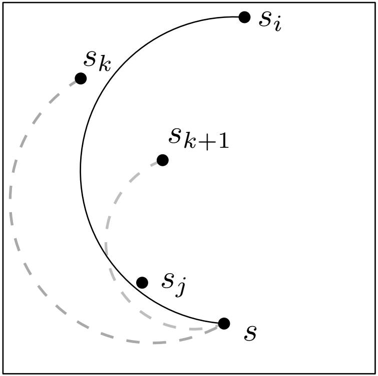


Lemma

If s_j lies in $D_l(s, s_i)$ and $\forall k \in \{i, \ldots, j-1\}$ $s_k \in D_l(s, s_i)$, then s_j also lies in $D_l(s, s_k)$, $\forall k \in \{i, \ldots, j-1\}$.

Circular Ordering Around a Site to Gabriel Edges



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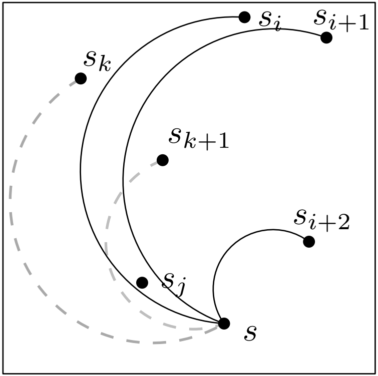

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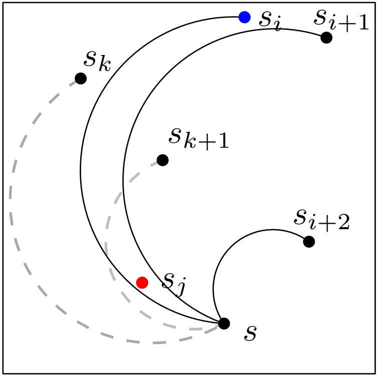

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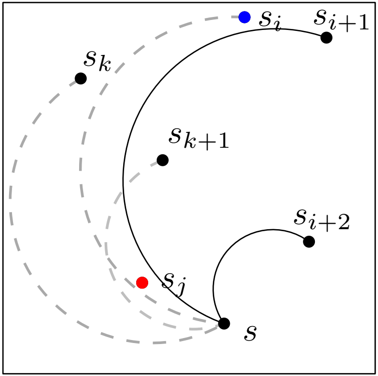

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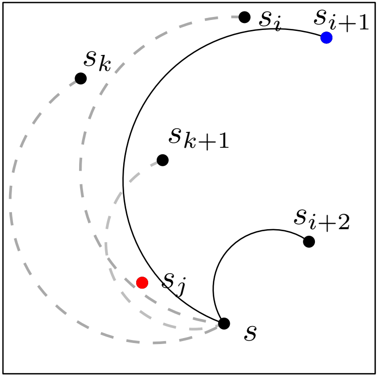

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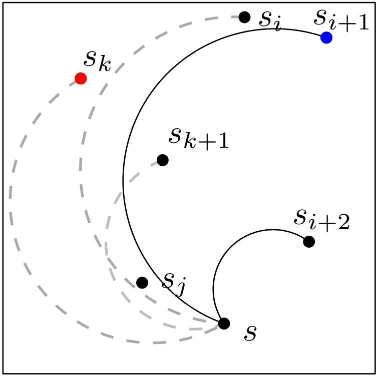

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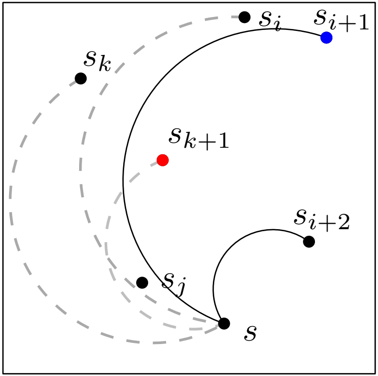

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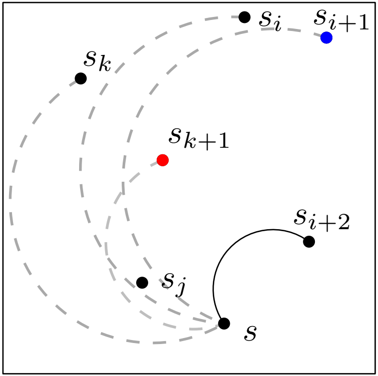

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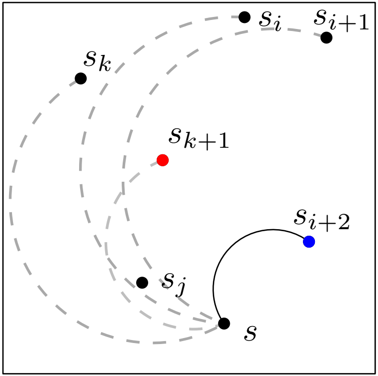

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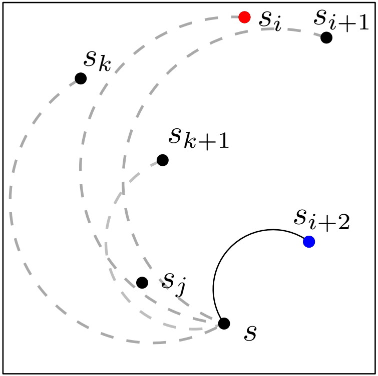

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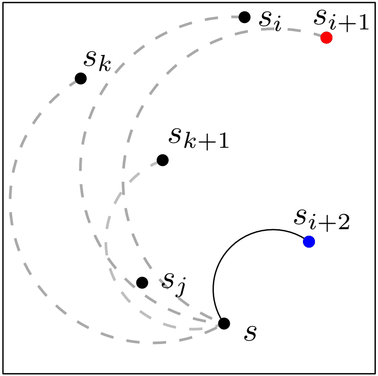

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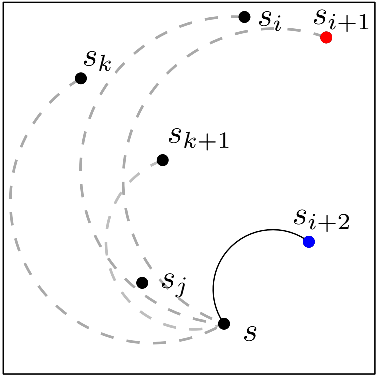

Determine Edges

Given the circular ordering of $S \setminus \{s\}$ around s, in $O(n)$ time and degree ②, we can find the Gabriel edges incident at s.

Algorithm for Computing the Gabriel Graph

Given
sites n sites S

Compute
Gabriel graph of S

1. Compute arrangement S^*
2. For each $s \in S$
   1. compute the circular ordering of $S \setminus s$ around s.
   2. determine the set of Gabriel edges in which s is incident.

Result

Given

sites $S = \{s_1, \ldots, s_n\}$

Arrangement of dual lines S^* and its trapezoidation

- Time: $O(n^2)$
- Space: $O(n^2)$
- Precision: degree 2

Gabriel graph

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Open problems

Can we...

- compute the Gabriel graph with sub-quadratic time and space in degree 2?
- compute a triangulation “close” to Delaunay?
- treat precision as a limited resource (like time and space) when solving other algorithmic problems?

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Thank you!

Contact:
Dave Millman
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http://cs.unc.edu/∼dave