A Slow Algorithm for Computing the Gabriel Graph with Double Precision

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Given sites $S = \{s_1, \ldots, s_n\}$

Compute the Gabriel graph of $S$. 
Given
sites $S = \{s_1, \ldots, s_n\}$

Compute
the Gabriel graph of $S$.

How much precision is needed to determine this?
E.g., Precision of testing if a point is inside a circle
Analyzing Precision[LPT99]

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\[ U = \{1, \ldots, U\}^2 \]
\[ a, b, q \in U \]
Analyzing Precision \cite{LPT99}

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\[ \mathbb{U} = \{1, \ldots, U\}^2 \]
\[ a, b, q \in \mathbb{U} \]
\[ a = (a_x, a_y) \]
\[ b = (b_x, b_y) \]
\[ q = (q_x, q_y) \]
\[ m = \left( \frac{a_x + b_x}{2}, \frac{a_y + b_y}{2} \right) \]
Analyzing Precision [LPT99]

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$$\mathbb{U} = \{1, \ldots, U\}^2$$

$$a, b, q \in \mathbb{U}$$

$$a = (a_x, a_y)$$

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$$\text{IsKiller}(a, b, q) = \text{sign}(\|m - p\|^2 - \|m - r\|^2)$$

$$= \text{sign}(p_x r_x + q_x r_x - p_x q_x - r_x^2)$$

$$p_y r_y + q_y r_y - p_y q_y - r_y^2)$$
Analyzing Precision [LPT99]

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\[ \text{IsKiller}(a, b, q) = \text{sign} \left( \|m - p\|^2 - \|m - r\|^2 \right) \]
\[ = \text{sign} \left( p_x r_x + q_x r_x - p_x q_x - r_x^2 \right. \]
\[ \left. + p_y r_y + q_y r_y - p_y q_y - r_y^2 \right) \]

\[ \text{degree \ 2} \]
Precision of Two Well Known Predicates

Orientation\((a, b, q)\)
degree \(2\)

InCircle\((a, b, c, q)\)
degree \(4\)
Given
sites $n$ sites $S$

Definition
an edge $\overline{pq}$
is in the Gabriel graph of $S$
if the closed disk
centered at the midpoint of $\overline{pq}$
with diameter $|\overline{pq}|$
contains no points of $S \setminus \{p, q\}$.
Given sites \( n \) sites \( S \)

**Definition**

an edge \( \overline{pq} \) is in the Gabriel graph of \( S \) if the closed disk centered at the midpoint of \( \overline{pq} \) with diameter \( |\overline{pq}| \) contains no points of \( S \setminus \{p, q\} \).
Proposed by: Gabriel and Sokal [GS69]

Compute Gabriel from Delaunay:
[MS80] $O(n)$ time, degree 6
[L96] $O(n)$ time, degree 2

Directly compute Gabriel graph:
Brute force, $O(n^3)$ time, degree 2
[MV11], $O(n^2)$ time, degree 2
Gabriel Graph

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Robustness Approaches

Approaches for implementing geometric algorithms with finite precision computer arithmetic:

- Rely on machine precision ($+\epsilon$) [NAT90,LTH86,KMP*08]
- Topological Consistency [S99,S01,SI90,SI92,SII*00,H01]
- Exact Geometric Computation [Y97,C92,ABO*97,BEP*97]
  - Arithmetic Filters [FW93,FW96,BBP01,DP98,DP99]
  - Adaptive Predicates [P92,S97,BF09]
  - Degree-driven algorithm design [L96,LPT99,BP00,BS00,C00,MS01,MS09,CMS09,MS10]
Given
sites \( S = \{s_1, \ldots, s_n\} \)

Arrangement of dual lines \( S^* \) and its trapezoidation
- Time: \( O(n^2) \)
- Space: \( O(n^2) \)
- Precision: degree 2

Gabriel graph
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Point/Line Duality [dBCvKO08]

**Primal**
Point \( p = (p_x, p_y) \)
Line \( \ell = (y = \ell_m x + \ell_b) \)
Set of points \( S = \{ s_1, \ldots, s_n \} \)

**Dual**
Point \( p^\ast = (y = p_x x - p_y) \)
Line \( \ell^\ast = (\ell_m, -\ell_b) \)
Set of lines \( S^\ast = \{ s_1^\ast, \ldots, s_n^\ast \} \)
Given
dual lines $a^*, b^*, c^*,$ and $d^*$

Determine
if the $x$-coordinate of
$a^* \cap b^*$ is left of $c^* \cap d^*$. 

$x\text{IntersectonOrder}(a^*, b^*, c^*, d^*)$
**xIntersectonOrder\((a^*, b^*, c^*, d^*)\)**

**Given**
points \(a, b, c,\) and \(d\)

**Determine**
if the slope of \(ab\) is less than \(cd\).

\[\text{xIntersectonOrder}(a^*, b^*, c^*, d^*)\]
xIntersectonOrder($a^*, b^*, c^*, d^*$)

Given

points $a$, $b$, $c$, and $d$

Determine

if the slope of $\overline{ab}$ is less than $\overline{cd}$.

\[
xIntersectonOrder(a^*, b^*, c^*, d^*) = \text{sign} \left( \frac{a_y - b_y}{a_x - b_x} - \frac{c_y - d_y}{c_x - d_x} \right)
\]
Given points \( a, b, c, \) and \( d \)

Determine if the slope of \( \overline{ab} \) is less than \( \overline{cd} \).

\[
x\text{IntersectonOrder}(a^*, b^*, c^*, d^*) = \text{sign}\left(\frac{a_y-b_y}{a_x-b_x} - \frac{c_y-d_y}{c_x-d_x}\right)
\]

\( \text{degree } 2 \)
Arrangement

For $n$ dual lines $S^*$, we can compute the arrangement of $S^*$ and its trapezoidation in $O(n^2)$ time and degree 2. 
Circular Orderings

Given the arrangement of $S^*$, for site $s \in S$, we can compute the circular ordering of the sites in $S \setminus \{s\}$ around $s$ in $O(n)$ time and degree $\circledast$. 

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Gabriel Graph with Double Precision
Lemma

If \( s_j \) lies in \( D(s, s_i) \) and \( \forall k \in \{i, \ldots, j-1\} \) \( s_k \in D(s, s_i) \), then \( s_j \) also lies in \( D(s, s_k) \), \( \forall k \in \{i, \ldots, j-1\} \).
Lemma

If $s_j$ lies in $D_l(s, s_i)$ and $\forall k \in \{i, \ldots, j - 1\}$ $s_k \in D_l(s, s_i)$, then $s_j$ also lies in $D_l(s, s_k)$, $\forall k \in \{i, \ldots, j - 1\}$. 
Determine Edges

Given the circular ordering of $S\{s\}$ around $s$, in $O(n)$ time and degree 2, we can find the Gabriel edges incident at $s$. 
Determine Edges
Given the circular ordering of \( S = \{ s \} \) around \( s \), in \( O(n) \) time and degree 2, we can find the Gabriel edges incident at \( s \).
Circular Ordering Around a Site to Gabriel Edges

Given the circular ordering of $S\{s\}$ around $s$, in $O(n)$ time and degree 2⃝, we can find the Gabriel edges incident at $s$. 

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Gabriel Graph with Double Precision
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Determine Edges

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![Diagram showing Gabriel edges](image-url)
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Gabriel Graph with Double Precision
Determine Edges

Given the circular ordering of $S \setminus \{s\}$ around $s$, in $O(n)$ time and degree 2, we can find the Gabriel edges incident at $s$. 

Circular Ordering Around a Site to Gabriel Edges
Algorithm for Computing the Gabriel Graph

Given
sites \( n \) sites \( S \)

Compute
Gabriel graph of \( S \)

1. Compute arrangement \( S^* \)

2. For each \( s \in S \)
   1. compute the circular ordering of \( S \setminus s \) around \( s \).
   2. determine the set of Gabriel edges in which \( s \) is incident.
Given
sites \( S = \{s_1, \ldots, s_n\} \)

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Gabriel graph
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Open problems

Can we...

- compute the Gabriel graph with sub-quadratic time and space in degree 2?
- compute a triangulation “close” to Delaunay?
- treat precision as a limited resource (like time and space) when solving other algorithmic problems?
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Thank you!

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