Quaternions and Rotations in 3-Space: The Algebra and its Geometric Interpretation

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Summary

Think of a quaternion Q as a vector augmented by a real number to make a four element entity. It has a *real* part $Q\rfloor_{re}$ and a *vector* part $Q\rfloor_{ve}$. If $Q\rfloor_{re}$ is zero, Q represents an ordinary vector; if $Q\rfloor_{ve}$ is zero, it represents an ordinary real number. In any case, the ratio between the real part and the *magnitude* of the vector part $|Q\rfloor_{ve}|$ plays an important role in rotations, and is conveniently represented by the parameter $\phi = tan^{-1}(|Q\rfloor_{ve}|/Q\rfloor_{re})$. A unit magnitude quaternion U has a Pythagorean sum of 1 over its four elements, and its product with any vector S_v gives another vector having the same magnitude as S_v but rotated in space. If $S_v \perp U\rfloor_{ve}$, the rotation is by an angle ϕ about the vector $U\rfloor_{ve}$ (or simply about U). If S_v is arbitrary, however, certain cross-terms of the product spoil this convenient relationship. Even in this general case however, these cross-terms cancel in the triple product $R_v = US_vU^{-1}$, where $U^{-1} \equiv 1/U$. The rotations of the two successive products are in the same direction, so R_v represents a rotation of S_v about $U\rfloor_{ve}$ by an angle 2ϕ, which depends only on U. Thus, the operation US_vU^{-1} performs a rotation of S_v which is entirely characterized by the unit quaternion U. The rotation occurs about an axis parallel to U by an amount $2\,tan^{-1}(|U\rfloor_{ve}|/U\rfloor_{re})$. Quaternion notation conveniently handles composition of any number of successive rotations into one equivalent rotation: $U = U_1U_2\cdots U_n$ where each unit quaternion U_i represents one of the succession of rotations. Other operations useful in inertial navigation problems are also presented.

1 Historical background

Quaternions were devised by Sir William Hamilton in his extensions of vector algebras to satisfy the properties of division rings (roughly, quotients exist in the same domain as the operands). In [1], Art.112, Hamilton notes, "...that *for the complete determination,* of what we have called the *geometrical* QUOTIENT *of two Co-initial Vectors,* a *System of Four Elements,* admitting each separately of numerical expression, *is generally required.* ... we have already a *motive* for saying, that 'the Quotient of two Vectors is generally a Quaternion.' "

Quaternions can also be considered to be an extension of classical algebra into the hypercomplex number domain D, satisfying a property that $|p|^2 \cdot |q|^2 = |p \cdot q|^2$ for $(p, q) \in D$ [2]. This domain consists of symbolic expressions of n terms with real coefficients where n may be 1 (real numbers), 2 (complex numbers), 4 (quaternions), 8 (Cayley numbers), but no other possible values (proved by Hurwitz in 1898). Thus, quaternions also share many properties with complex numbers.

While Hamilton provides geometrical interpretations of various proved properties throughout [1], the development itself is fundamentally algebraic, that is, based on the properties of a particular axiomatic set of symbolic operations. The geometric properties of quaternions are nevertheless sweeping, the composition of successive rotations through successive multiplications being just one, albeit an important one.

2 Axiomatic properties of quaternions

Quaternions are defined as sums of 4 terms of the form $Q = 1 \cdot q_1 + i \cdot q_2 + j \cdot q_3 + k \cdot q_4$ where q_1, q_2, q_3, q_4 are reals, 1 is the multiplicative identity element, and i, j, k are symbolic elements having the properties:

$$
\begin{gathered}
i^2 = -1,\ j^2 = -1,\ k^2 = -1, \\
ij = k,\ ji = -k, \\
jk = i,\ kj = -i, \\
ki = j,\ ik = -j.
\end{gathered}
$$

Customarily, the extension of an algebra should attempt to preserve the properties of the operators defined in the original algebra. Generalizing from the classical algebra of real and complex numbers to quaternions motivates the following operator rules.

2.1 Addition of quaternions

The addition rule for quaternions is component-wise addition:

$$P+Q = (p_1+ip_2+jp_3+kp_4)+(q_1+iq_2+jq_3+kq_4) = (p_1+q_1)+i(p_2+q_2)+j(p_3+q_3)+k(p_4+q_4).$$

This rule preserves the associativity and commutativity properties of addition, and provides a consistent behavior for the subset of quaternions corresponding to real numbers, i.e.,

$$P_r + Q_r = (p + 0i + 0j + 0k) + (q + 0i + 0j + 0k) = p + q.$$

2.2 Multiplication of quaternions

The multiplication rule for quaternions is the same as for polynomials, extended by the multiplicative properties of the elements i, j, k given above. Written out for close inspection, we have:

$$\begin{aligned} PQ &= (p_1 + ip_2 + jp_3 + kp_4)(q_1 + iq_2 + jq_3 + kq_4) \\ &= (p_1q_1 - p_2q_2 - p_3q_3 - p_4q_4) + i(p_1q_2 + p_2q_1 + p_3q_4 - p_4q_3) \\ &+ j(p_1q_3 + p_3q_1 + p_4q_2 - p_2q_4) + k(p_1q_4 + p_4q_1 + p_2q_3 - p_3q_2). \end{aligned}$$

A term-wise inspection reveals that commutativity is not preserved. Associativity and distributivity over addition are preserved, however, the proof being left to the reader. And as desired for the subset of reals, $P_rQ_r = pq$.

2.3 Conjugates of quaternions

Consistent with complex numbers, let us define the *conjugate* operation on a given quaternion Q to be,

$$\overline{Q} = \overline{(q_1 + iq_2 + jq_3 + kq_4)} \equiv (q_1 - iq_2 - jq_3 - kq_4).$$

As with complex numbers, note that both $(Q + \overline{Q})$ and $(Q\overline{Q})$ are real. Moreover, if we define the absolute value or *norm* of Q to be,

$$|Q| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2},$$

then apparently $Q\overline{Q} = \overline{Q}Q = |Q|^2$. The conjugate operation is distributive over addition, that is, $\overline{P + Q} = \overline{P} + \overline{Q}$. With respect to multiplication however, $\overline{PQ} = \overline{Q}\,\overline{P}$, the proof of which is left as an exercise to the reader.

3 Other properties of quaternions

The axioms in the previous section completely *define* quaternions in terms of the desired properties under three basic operations. Many other properties may be proved.

3.1 General properties

Mathematically, the most important property is that the quaternions form a division ring (i.e., quaternion quotients exist).

3.1.1 Division of quaternions

Since multiplication is not commutative, let us derive both a *left quotient* Q_L^{-1} and a *right quotient* Q_R^{-1} by defining the symbolic expression P/Q to be solutions of the following two identities,

$$QQ_L^{-1} = P, \qquad Q_R^{-1}Q = P.$$

Multiplying both sides of these identities respectively on the left and right by $\overline{Q}/|Q|^2$ we have immediately,

$$Q_L^{-1} = \frac{\overline{Q}P}{|Q|^2}, \qquad Q_R^{-1} = \frac{P\overline{Q}}{|Q|^2}.$$

Thus in general two distinct quotients will occur, however in the special case where $P = 1$, we have by definition the multiplicative *inverse* of a quaternion,

$$Q_L^{-1} = Q_R^{-1} = Q^{-1} = \frac{\overline{Q}}{|Q|^2}$$

3.1.2 Quaternion multiplication is distributive over addition

A term-wise expansion of $P(Q+S) = PQ + PS$ proves this property and is left as an exercise for the reader.

3.1.3 Unit quaternions

The subspace U of *unit quaternions* which satisfy the condition $|U| = 1$ have some important properties. A trivially apparent one is,

$$U^{-1} = \overline{U}.$$

A less obvious, but very useful one is,

$$U = U_r cos\,\phi + U_v sin\,\phi = cos\,\phi + U_v sin\,\phi,$$

where $U_r = (1, 0, 0, 0)$ is a real unit quaternion, $U_v = (0, iu_2, ju_3, ku_4)$ is a vector unit quaternion parallel to the vector part of U, and ϕ is a real number. The proof is straightforward:

$$\begin{aligned}
|U|^2 &= U\overline{U} = (U_r cos\,\phi + U_v sin\,\phi)\overline{(U_r cos\,\phi + U_v sin\,\phi)} \\
&= U_r\overline{U}_r cos^2\phi + (U_r\overline{U}_v + U_v\overline{U}_r) sin\,\phi\, cos\,\phi + U_v\overline{U}_v sin^2\phi \\
&= cos^2\phi + sin^2\phi = 1.
\end{aligned}$$

At this time, let's interpret ϕ as simply quantifying the ratio of the real part to the magnitude of the vector part of a quaternion. Its geometrical representation as specifying an angle of rotation will be presented later.

3.2 Vector properties of quaternions

The quaternion $Q = (q_1 + iq_2 + jq_3 + kq_4)$ can be interpreted as having a real part q_1, and a vector part $(iq_2 + jq_3 + kq_4)$, where the elements $\{i, j, k\}$ are given an added *geometric* interpretation as unit vectors along the x, y, z axes, respectively. Accordingly, the subspace $Q_r = (q_1 + 0i + 0j + 0k)$ of *real quaternions* may be regarded as being equivalent to the real numbers, $Q_r = q$. Similarly, the subspace $Q_v = (0 + iq_2 + jq_3 + kq_4)$ of *vector quaternions* may be regarded as being equivalent to the ordinary vectors, $Q_v = \boldsymbol{q} \equiv (iq_x + jq_y + kq_z)$.

3.2.1 Products of real quaternions

The product of real quaternions is real, and the operation is commutative:

$$P_r Q_r = pq = qp = Q_r P_r.$$

Moreover, the operation is associative:

$$(P_r Q_r) S_r = (pq)s = p(qs) = P_r (Q_r S_r).$$

3.2.2 Product of a real quaternion with a vector quaternion

The product of a real and a vector quaternion is a vector, and the operation is commutative:

$$P_r Q_v = (0 + p_1 q_2 i + p_1 q_3 j + p_1 q_4 k) = (0 + q_2 p_1 i + q_3 p_1 j + q_4 p_1 k) = Q_v P_r.$$

3.2.3 Products of vector quaternions

The product of two vector quaternions has the remarkable property,

$$\begin{aligned} P_v Q_v &= -(p_2 q_2 + p_3 q_3 + p_4 q_4) + (p_3 q_4 - p_4 q_3) i + (p_4 q_2 - p_2 q_4) j + (p_2 q_3 - p_3 q_2) k \\ &= -\boldsymbol{p} \cdot \boldsymbol{q} + \boldsymbol{p} \times \boldsymbol{q}, \end{aligned}$$

where the "$\cdot$" and "$\times$" operators are respectively the "dot" and "cross" products of classical vector algebra. This is clearly a general quaternion except in two special cases: if $P_v \parallel Q_v$ the product is a real quaternion equal to $-\boldsymbol{p} \cdot \boldsymbol{q}$ and if $P_v \perp Q_v$ the product is a vector quaternion equal to $\boldsymbol{p} \times \boldsymbol{q}$.

3.2.4 Parallel and perpendicular quaternions

We call quaternions P and Q *parallel* ($P \parallel Q$) if their *vector parts* $P\rfloor_{ve} = (P - \overline{P})/2$ and $Q\rfloor_{ve} = (Q - \overline{Q})/2$ are parallel; i.e., if $(S - \overline{S}\,) = 0$, where $S = P\rfloor_{ve} Q\rfloor_{ve}$. Similarly, we call them *perpendicular* ($P \perp Q$) if $P\rfloor_{ve}$ and $Q\rfloor_{ve}$ are perpendicular; i.e. if $(S + \overline{S}\,) = 0$.

3.2.5 Product of a unit quaternion and a perpendicular vector quaternion

Properties of this curiously specialized case are useful in understanding how quaternions can be used to rotate vectors in 3-space. Let S_v be a vector quaternion, U be a unit quaternion, and $S_v \perp U$. Then according to section 3.1.3, we can write,

$$T = U S_v = (cos\, \phi + sin\, \phi\, U_v) S_v = cos\, \phi\, S_v + sin\, \phi\, U_v S_v,$$

where $U_v \parallel U$. The first term is a vector $T_{v(1)} \parallel S_v$. Since $S_v \perp U_v$, the second term must also be a vector $T_{v(2)}$; moreover $T_{v(2)} \perp S_v$ and $T_{v(2)} \perp U \parallel U_v$. Since the product T is a sum of vectors it must also be a vector, i.e., $T = T_v$. Both $T_{v(1)}$ and $T_{v(2)}$ lie in a plane perpendicular to U. Thus $T_v = T_{v(1)} + T_{v(2)}$ can be geometrically interpreted as a rotation of S_v by an angle ϕ in this plane, i.e., about an axis parallel to U.

Now consider the product,

$$R_v = T_v U^{-1} = T_v \overline{U} = cos\,\phi\, T_v + sin\,\phi\, T_v \overline{U}_v = cos\,\phi\, T_v - sin\,\phi\, T_v U_v.$$

The vector identity $T_v U_v = -U_v T_v$ can be used to rewrite this as,

$$R_v = cos\,\phi\, T_v + sin\,\phi\, U_v T_v,$$

which is another rotation of angle ϕ about U. The rotation ϕ is in the same sense for these two products, so the operation

$$R_v = U S_v U^{-1}$$

performs a rotation of S_v about U by an angle 2ϕ.

3.3 General rotations in 3-space; Reference frames

In section 3.2.5 we saw how the operation $U S_v U^{-1}$ rotated a *perpendicular* vector S_v about a unit quaternion U. Now let's consider how this operation behaves with an *arbitrary* vector V_v. We can decompose $V_v = W_v + S_v$ where $W_v \parallel U$ and $S_v \perp U$. Then,

$$U V_v U^{-1} = U(W_v + S_v)U^{-1} = U W_v U^{-1} + U S_v U^{-1} = U W_v U^{-1} + R_v,$$

where R_v is S_v rotated about U by an angle 2ϕ. To evaluate the first term, note that since $W_v \parallel U$ we can write $W_v = zU_v$, where z is a real number and unit vector $U_v \parallel U$. Thus,

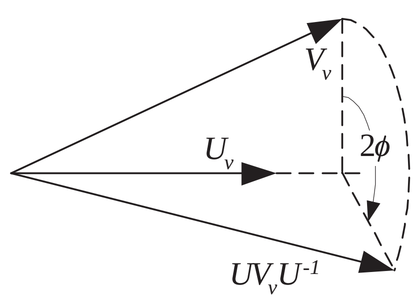
$$U W_v U^{-1} = U z U_v U^{-1} = z U U_v U^{-1} = z U_v U U^{-1} = z U_v = W_v.$$

That $U U_v = U_v U$ is left as an exercise to the reader. Finally then, we have:

$$U V_v U^{-1} = W_v + R_v.$$

Geometrically, we interpret this as a rotation of V_v about U by an angle of 2ϕ.

Figure 1:
Arbitrary vector V_v is rotated by unit quaternion U about a unit vector $U_v \parallel U$, through angle 2ϕ.



This operation performs the same rotation on *all* vectors including the unit vectors of a coordinate system. Therefore, it can be used to rigidly transform the coordinates of any *reference frame* into a new frame of different orientation. This is a very useful property.

3.4 Composition of successive rotations

Let Q_1 and Q_2 be two unit quaternions representing arbitrary rotations in 3-space as described in section 3.3. Applying them in succession to a vector V_v,

$$Q_2(Q_1 V_v Q_1^{-1})Q_2^{-1} = (Q_2 Q_1)V_v(Q_1^{-1}Q_2^{-1}) = (Q_2 Q_1)V_v(Q_2 Q_1)^{-1} = Q_i V_v Q_i^{-1},$$

where the unit quaternion $Q_i = Q_2 Q_1$ is the successive composition of two rotations. This property generalizes to the composition of any number of rotations. In this reverse order composition, each successive rotation is relative to the *initial* reference frame as is illustrated in Figure 2a.

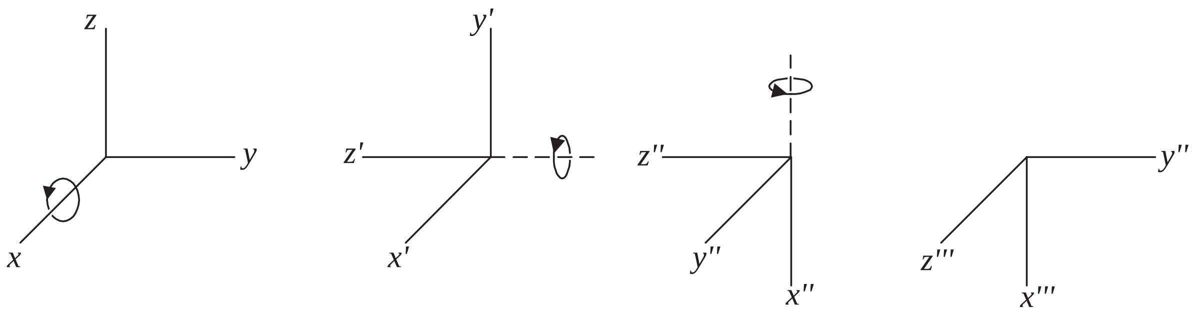


Figure 2a: 90° rotations of a reference frame about the initial x, y, z axes, respectively

Composing a rotation in the forward order, $Q_c = Q_1 Q_2 \ldots$, has the effect of performing each successive rotation relative to its *current* reference frame, illustrated in Figure 2b.

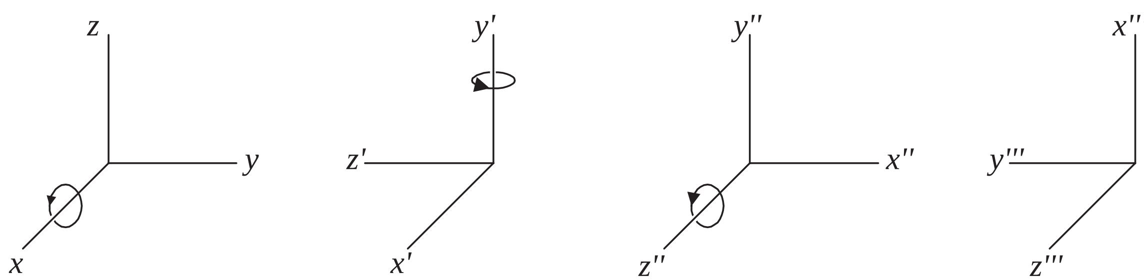


Figure 2b: 90° rotations of a reference frame about its current x, y, z axes, respectively.

4 Strapdown inertial navigation system (INS) applications

Usage of quaternions by this branch of engineering is common, but the notation often differs in some respects from the above, and a more detailed annotation is provided to relate variables to reference frames. Specifically in this section, I'll follow the notation used in Titterton and Weston [3]. I will introduce this notation, then derive expressions for some of the commonly used operations for INS engineering.

4.1 Frames and coordinates

It is often convenient to represent the same physical situation in a number of different frames of reference which may differ by displacement, rotation, and system of coordinates. Each frame comprises a complete definition of these parameters. A privileged, *inertial* family of frames are those in which physical objects experience no inertial forces.

Cartesian coordinate systems, while not necessary, are generally used as coordinate systems of the frames discussed in [3]. The non-scalar data types used are vectors, matrices, and quaternions. Distinct from the data types, are the kinds of variables treated, i.e., positions, linear velocities, and angular rates.

4.2 Superscripts and subscripts

Superscripts and subscripts are used to associate certain attributes of a variable with coordinate frames. On a gross level, the notation is consistent, but there are fine nuances, depending on the kind of the variable but not its type.

Superscripts are used consistently for all kinds of variables. S^i indicates that the variable S is expressed in the coordinates of the i^{th} frame.

4.2.1 The position variable X^i_j

X^i_j represents the position of a point relative to the origin of the j^{th} frame, expressed in the coordinates of the i^{th} frame. In most cases $i = j$, and it is common to use implicit notations. X_j and X^j both represent X^j_j, where the choice of super- or subscript depends on what is being emphasized.

4.2.2 The velocity variable V^i_j

The variable V^i_j represents a velocity taken relative to the j^{th} frame, expressed in coordinates of the i^{th} frame. The velocity in any frame is not dependent on the location of the origin of the frame; rather it may be taken relative to the velocity of *any fixed point* in that frame. Just as for position variables, $V_j = V^j_j$ is implied.

4.2.3 The angular rate variable Ω^i_{jk}

The variable Ω^i_{jk} represents an angular rate of rotation of the k^{th} entity relative to the j^{th} frame, expressed in coordinates of the i^{th} frame. Just as for velocities, the location of origin of reference frame j is not relevant; rather the angular rate is taken relative to the angular rate of any fixed point in the j^{th} frame. Often, the k^{th} entity is another frame, so this notation conveniently expresses the angular rate of rotation of the k^{th} frame relative to the j^{th}.

4.3 A pure vector representation of a rotation

It is also possible to completely represent a 3D rotation with a pure vector. The geometric properties of algebraic operations on this representation are naturally quite different than for unit quaternions. For some purposes these properties are particularly useful.

Let vector $\boldsymbol{a} = a\hat{\boldsymbol{a}}$ represent a rotation where its unit vector $\hat{\boldsymbol{a}}$ specifies the axis of rotation and its magnitude a specifies the angular amount of rotation. From this we can uniquely construct a unit quaternion, $A = cos(a/2) + sin(a/2)\hat{\boldsymbol{a}}$, such that $AS_v\overline{A}$ performs a rotation of S_v about $\hat{\boldsymbol{a}}$ by an angle equal to a.

Let us define a transform $\mathcal{Q}$ of the vector representation $\boldsymbol{a}$ to the unit quaternion representation A of a 3D rotation:

$$A = \mathcal{Q}(\boldsymbol{a}) = \mathcal{Q}(a\hat{\boldsymbol{a}}) = cos(a/2) + sin(a/2)\hat{\boldsymbol{a}}.$$

Likewise, let us define the inverse transform,

$$\boldsymbol{a} = \mathcal{Q}^{-1}(A) = \mathcal{Q}^{-1}(A_r + A_v\hat{\boldsymbol{a}}) = 2tan^{-1}(A_v/A_r)\hat{\boldsymbol{a}}.$$

4.4 Time derivative of a rotation quaternion

Assume a b-frame that is rotating with respect to a reference n-frame. At any instant, let the unit quaternion U represent a rotation of an arbitrary constant vector C^b in the b-frame into a vector $C^n = UC^b\overline{U}$ in the n-frame. Since this rotation progresses continuously in time, $U = U(t)$ has a time derivative $\dot{U}$ which we now derive.

Applying the derivative of products rule to C^n, we have, (since $\dot{C}^b = 0$),

$$\dot{C}^n = \dot{U}\,C^b\,\overline{U} + U\,C^b\,\dot{\overline{U}} = \dot{U}\,C^b\,\overline{U} + \overline{\dot{U}\,\overline{C}^b\,\overline{U}} = \dot{U}\,C^b\,\overline{U} - \overline{\dot{U}\,C^b\,\overline{U}}.$$

In the vector formulation of classical mechanics [4], a vector $\boldsymbol{p}$ is used to represent an instantaneous rate of rotation, $\dot{\boldsymbol{c}} = \boldsymbol{p} \times \boldsymbol{c}$, where $\boldsymbol{c}$ is an arbitrary vector, and $\dot{\boldsymbol{c}}$ is its variation with time. In the n-frame, a quaternion formulation of this equation is,

$$\dot{C}^n = (P^nC^n - \overline{P^nC^n})/2.$$

Since $\boldsymbol{c}$ is arbitrary, this equation can be applied to an entire coordinate system, and we can represent the rate of rotation of the b-frame in the n-frame as $P^n = P^n_{nb}$.

Equating the expressions for $\dot{C}^n$, we have, $\dot{U}C^b\overline{U} = P^n_{nb}C^n/2 = P^n_{nb}(UC^b\overline{U})/2$, or $\dot{U} = P^n_{nb}U/2$. It is often the case that the rotational rate is measured in the rotating b-frame, so we can substitute the identity $P^n_{nb} = UP^b_{nb}\overline{U}$, to obtain

$$\dot{U} = UP^b_{nb}/2.$$

4.5 Interpolation between rotations

Given two arbitrary rotations U_{10}, U_{20} from the 0-frame to the 1 and 2-frames respectively, geometric intuition would suggest an interpolation between them would be along the single rotation U_{21} taking the 1-frame into the 2-frame. In fact, this can be visualized as a great circle on a unit 4-sphere which connects the images of U_{10} and U_{20}. This great circle lies in a plane normal to $U_{21}\rfloor_{ve}$. The locus of points lying between U_{10} and U_{20} on the great circle corresponds to a rotational angle of between 0 and $cos^{-1}(U_{21}\rfloor_{re})$.

Now $U_{20} = U_{10}U_{21} \;\Rightarrow\; U_{21} = \overline{U_{10}}U_{20}$. Let $U_{21} = cos(\phi_{21}) + \hat{\boldsymbol{u}}_{21}sin(\phi_{21})$, whence we can calculate $\phi_{21} = cos^{-1}(U_{21}\rfloor_{re})$ and $\hat{\boldsymbol{u}}_{21} = U_{21}\rfloor_{ve}/sin(\phi_{21})$.

Given $\phi_{x1} \ni (0 \leq \phi_{x1} \leq \phi_{21})$ we construct $U_{x1} = cos(\phi_{x1}) + \hat{\boldsymbol{u}}_{21}sin(\phi_{x1})$, from which we calculate the interpolated rotation,

$$U_{x0} = U_{10}U_{x1}.$$

APPENDIX A – Summary of formal properties

A.1 Notation

r	a scalar (real) number
$\boldsymbol{v}$	a vector
$\hat{\boldsymbol{u}}$	a unit vector, $\boldsymbol{u} \cdot \boldsymbol{u} = 1$
i, j, k	symbolic constants with special properties (section 2)
Q	a quaternion $[q_1, q_2, q_3, q_4] = q_1 + iq_2 + jq_3 + kq_4$
$\overline{Q}$	the conjugate $[q_1, -q_2, -q_3, -q_4]$ of quaternion Q
$\|Q\|$	the norm, or magnitude $\sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$ of quaternion Q
Q^{-1}	the reciprocal $Q/(Q\overline{Q})$, or multiplicative inverse of quaternion Q
Q_r	a (purely) real quaternion $[q_1, 0, 0, 0]$
Q_v	a (purely) vector quaternion $[0, q_2, q_3, q_4]$
U	a unit quaternion, $\|Q\| = 1$
$Q\rfloor_{re}$	the real part $q = q_1$ of quaternion Q
$Q\rfloor_{ve}$	the vector part $\boldsymbol{q} = [q_2, q_3, q_4]$ of quaternion Q
$Q\|\|P$	(the vector parts of) P and Q are parallel
$Q \perp P$	(the vector parts of) P and Q are perpendicular

A.2 Properties

$P + (Q + S) = (P + Q) + S$	addition is associative
$P + Q = Q + P$	addition is commutative
$P(QS) = (PQ)S$	multiplication is associative
$PQ \neq QP$	multiplication is not commutative
$pQ = Qp$	scalar multiplication is commutative
$P(Q + S) = PQ + PS$	left multiplication is distributive over addition
$(P + Q)S = PS + QS$	right multiplication is distributive over addition
$\|Q\| = \sqrt{Q\overline{Q}} = \sqrt{\overline{Q}Q}$	the norm of Q
$Q\rfloor_{re} = (Q + \overline{Q})/2$	the real part of Q
$Q\rfloor_{ve} = (Q - \overline{Q})/2$	the vector part of Q
$Q^{-1} = \overline{Q}/\|Q\|^2$	the reciprocal of Q
$U^{-1} = \overline{U}$	the reciprocal of unit U
$Q^{-1}P = \overline{Q}P/\|Q\|^2$	the left quotient
$PQ^{-1} = P\overline{Q}/\|Q\|^2$	the right quotient
$\overline{PQ} = \overline{Q}\,\overline{P}$	conjugate of a product
$P_vQ_v = -\boldsymbol{p} \cdot \boldsymbol{q} + \boldsymbol{p} \times \boldsymbol{q}$	product of vector quaternions

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