Generalized Closed World
Assumption is \( \Omega^3 \)-complete*

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Generalized Closed World Assumption is \(\Pi^0_2\)-complete *

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1 Introduction

Minker [9] has defined an inference rule called the Generalized Closed World Assumption (GCWA). According to this rule, a negated ground atom \(-A\) can be inferred from a non-Horn (called also disjunctive or indefinite) logic program \(P\) iff \(A\) is not in any minimal Herbrand model of \(P\). GCWA, as opposed to CWA ([12]), does not lead to inconsistency and has been adopted as a standard rule for inferring negative information from a disjunctive logic program [16,3,15,7,11,6]).

In this note, we show that GCWA is \(\Pi^0_2\)-complete, i.e. at the second level of the arithmetical hierarchy [13]. The non-obvious part is \(\Pi^0_2\)-hardness. Therefore, GCWA is strictly harder than CWA which is \(\Pi^0_2\)-complete [2] for both Horn and non-Horn logic programs. Moreover, GCWA is strictly harder than a weaker inference rule called Weak GCWA in [10] and Disjunctive Database Rule in [14] which, like CWA, is \(\Pi^0_2\)-complete.

2 Preliminaries

Definite Horn logic programs [8] consist of universally-quantified clauses with exactly one positive literal. Non-Horn logic programs [9,5] consist of universally-quantified clauses with at least one positive literal.

Reiter [12] defined the Closed World Assumption (CWA) as the following inference rule:

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\neg A \in CWA(P) \iff P \not\models A

where \( P \) is a logic program and \( A \) is a ground atom belonging to the Herbrand base of \( P \). For any definite Horn logic program \( P \), \( CWA(P) \) can be defined equivalently as:

\neg A \in CWA(P) \iff M_P \not\models A

where \( M_P \) is the least Herbrand model of \( P \). In that case, \( P \cup CWA(P) \) is consistent. Moreover, it is maximally consistent. However, in the case of non-Horn logic programs, \( P \cup CWA(P) \) may be inconsistent.

Minker [9] defined the Generalized Closed World Assumption (GCWA) as the following inference rule:

\neg A \in GCWA(P) \iff \forall K, \ P \models A \lor K \Rightarrow P \models K

where \( P \) is a logic program, \( A \) is a ground atom, \( K \) is a disjunction of ground atoms, and \( A \) and all the disjuncts in \( K \) belong to the Herbrand base of \( P \). He also showed that the above definition has a model-theoretic counterpart:

\neg A \in GCWA(P) \iff A \text{ does not belong to any minimal Herbrand model of } P.

For any (Horn or non-Horn) logic program \( P \), \( P \cup GCWA(P) \) is consistent. Moreover, it is maximally consistent in the sense that adding more negative literals would result either in an inconsistency or in a new positive conclusion being derivable.

3 Main result

Theorem 3.1 \( GCWA(P) \) is a \( \Pi^0_2 \)-complete set.

Proof: We show first that \( GCWA(P) \) is in \( \Pi^0_2 \). The complementary problem:

\neg A \notin GCWA(P) \iff \exists K, \ P \models A \lor K \text{ and } P \not\models K

is recursively enumerable in \( \Pi^0_2 \), and therefore is in \( \Sigma^0_2 \).

We now show \( \Pi^0_2 \)-hardness. Take an arbitrary \( \Pi^0_2 \) subset \( Q \) of some finitely generated Herbrand universe \( U \) that contains at least one constant and one function symbol. We will exhibit a non-Horn logic program \( P \) over the same Herbrand universe \( U \) such that

\neg c(x) \in GCWA(P) \iff Q(x)

for all \( x \) and some predicate symbol \( c \).

By the definition of \( \Pi^0_2 \), there is a recursively enumerable relation \( R \) over \( U \) such that:

\( Q(x) \) iff \( \forall y, R(x, y) \).

For this relation \( R \), there is a definite Horn logic program \( S \) such that:

\( R(x, y) \) iff \( S \models r(x, y) \)
by a result of Andreka and Nemeti [1] (also [2, Theorem 7] and [4]).

We define the non-Horn logic program $P$ postulated at the beginning of the proof in several steps.

First, we include the program $S$ in $P$. Second, we introduce a new predicate symbol $\text{term}$, not appearing in $S$. The predicate $\text{term}(t)$ is true of any term $t$ from $U$ and can be defined by a finite set of definite Horn rules in an obvious way. Those rules are included in $P$.

Third, we introduce three new predicate symbols $a,b$ and $c$, not appearing in $S$ and defined by the following rules, the second of which is non-Horn:

\begin{align*}
    a(X,Y) & \leftarrow r(X,Y). \\
    a(X,Y) \lor b(X,Y) & \leftarrow \text{term}(X), \text{term}(Y). \\
    c(X) & \leftarrow b(X,Y).
\end{align*}

We will show now that for all $x$:

$$c(x) \text{ is not in any minimal Herbrand model of } P \text{ iff } \forall y, S \vdash r(x,y)$$

which is equivalent to:

$$c(x) \text{ is in a minimal Herbrand model of } P \text{ iff } \exists y, S \not\vdash r(x,y)$$

and establishes the hardness result.

Assume first that:

$$\forall y, S \vdash r(x,y).$$

We will assume that $c(x)$ is in a minimal Herbrand model $M_0$ of $P$ and derive a contradiction. If $c(x)$ is in a minimal Herbrand model $M_0$ of $P$, then there exists a $y$ such that $M_0 \models b(x,y)$. Now because $S$ is definite Horn, the original assumption implies:

$$\forall y, M_S \models r(x,y)$$

where $M_S$ is the least Herbrand model of $S$. Thus also

$$\forall y, M \models r(x,y)$$

for any minimal Herbrand model $M$ of $P$, because every such model has to contain $M_S$. Consequently,

$$\forall y, M \models a(x,y)$$

and because $M$ is a minimal model:

$$\forall y, M \models \neg b(x,y).$$

This contradicts the fact that:

$$M_0 \models b(x,y).$$
We prove now the opposite direction. Assume:

\[ \exists y, S \not\models r(x,y). \]

We will prove that \( c(x) \) is in a minimal Herbrand model of \( P \). From the assumption follows that:

\[ P \not\models r(x,y) \]

and there exists a Herbrand model \( M \) of \( P \):

\[ M \not\models r(x,y) \]

Consequently, there is a minimal Herbrand model \( M_0 \) of \( P \):

\[ M_0 \not\models r(x,y) \]

Then there is also a minimal Herbrand model \( M_1 \) of \( P \) such that:

\[ M_1 \models b(x,y) \]

which implies that:

\[ M_1 \models c(x). \]

It is easy to see that the transformation leading from \( S \) to \( P \) is total and recursive. Therefore, given any \( \Pi^0_2 \) set, we have shown how to construct a non-Horn logic program \( P \) such that:

\[ \neg A \in GCWA(P) \text{ iff } A \in X. \]

End of proof.

References


