THE SHOELACE PROBLEM

by

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ABSTRACT

The problem is to find the order of lacing a shoe, with two parallel rows of equally-spaced lace-holes (eyelets), which requires the least total length of lace. This paper determines the total length required by the three most popular styles of lacing (for any shoe parameters), and optimizes over all possible lacings.

INTRODUCTION

In a number of discussions of how shoes should be laced, it became apparent that no one seemed to have the definitive answer. Shoes were laced and re-laced, passions flared, and shoes were even thrown.... The author decided that an appeal to mathematics was indicated.

This problem is a restriction of the Traveling Salesman Problem. We are given a set of $2(n + 1)$ points (the *lace-holes* **oreyelets**) arranged in a bi-partite lattice, as shown in Figure 1 below.

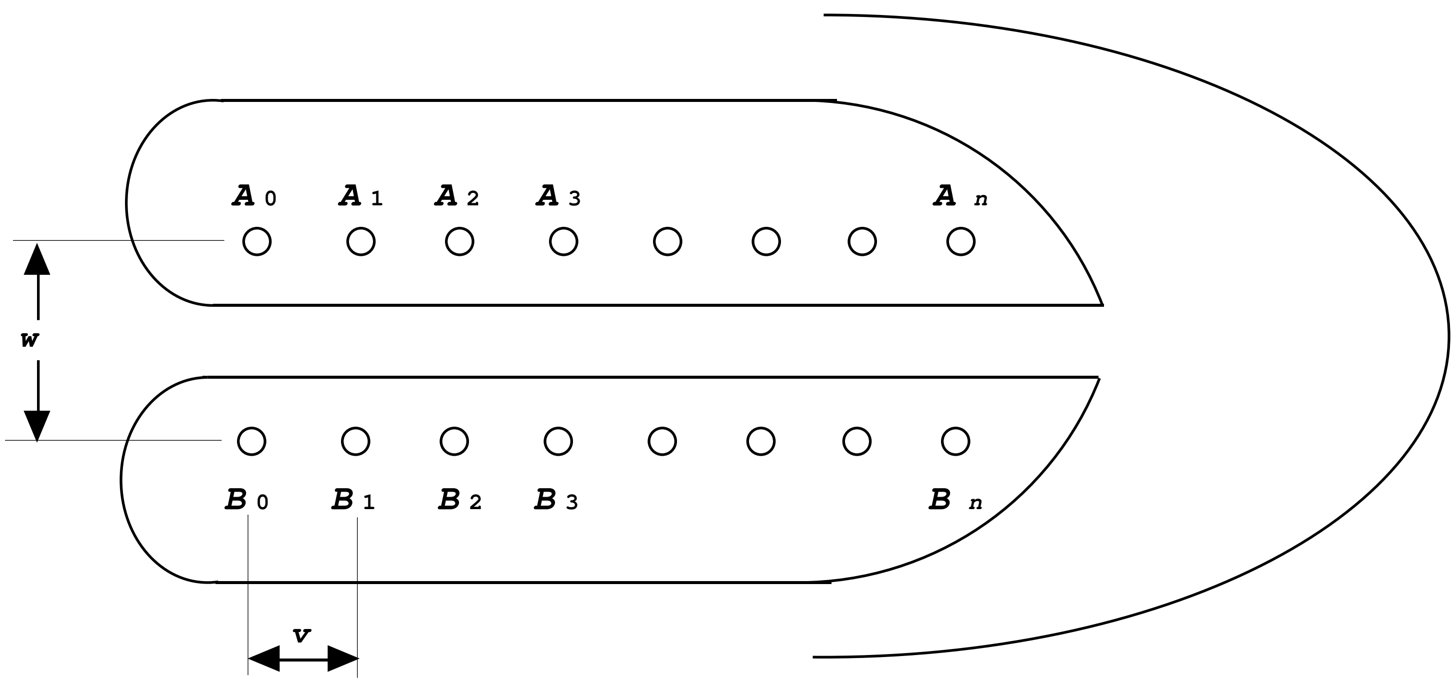


Figure 1

The shoe...

The problem is to find the shortest path from $A_0$to B_0, passing through every eyelet just once, in such a way that points of the subsets

$$A = \{A_0, A_1, A_2, \ldots, A_n\} \quad \text{and} \quad B = \{B_0, B_1, B_2, \ldots, B_n\} \tag{1}$$

alternate in the path.

Three standard lacing strategies are shown in Figures 2—4 below.

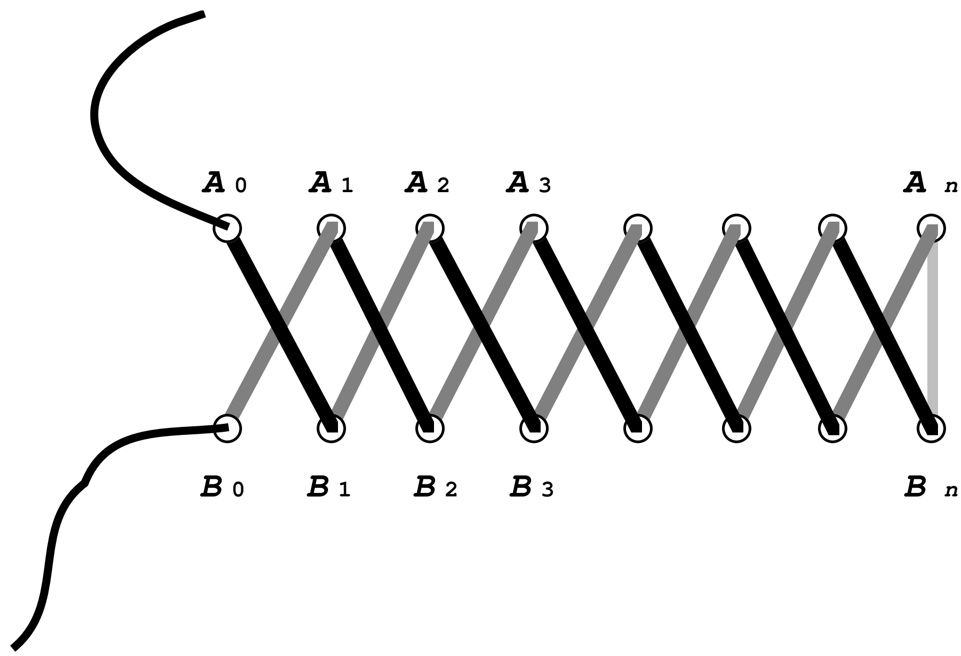


Figure 2
American, zig-zag, lacing.

Here, if *n* is *odd*, as in Figure 2, the lacing is

$$A_0 \ B_1 \ A_2 \ B_3 \ A_4 \ \ldots \ A_{n-1} \ B_n$$
$$A_n \ B_{n-1} \ A_{n-2} \ B_{n-1} \ \ldots \ A_3 \ B_2 \ A_1 \ B_0; \qquad (2)$$

if *n* is *even*, the lacing is, similarly,

$$A_0 \ B_1 \ A_2 \ B_3 \ A_4 \ \ldots \ A_{n-2} \ B_{n-1} \ A_n$$
$$B_n \ A_{n-1} \ B_{n-2} \ \ldots \ A_3 \ B_2 \ A_1 \ B_0; \qquad (3)$$

and it is easily verified that, in either case, the total length used is

$$L_{\mathrm{AM}} = L_{\mathrm{AM}}(n, v, w) = w + 2n\sqrt{v^2 + w^2}. \qquad (4)$$

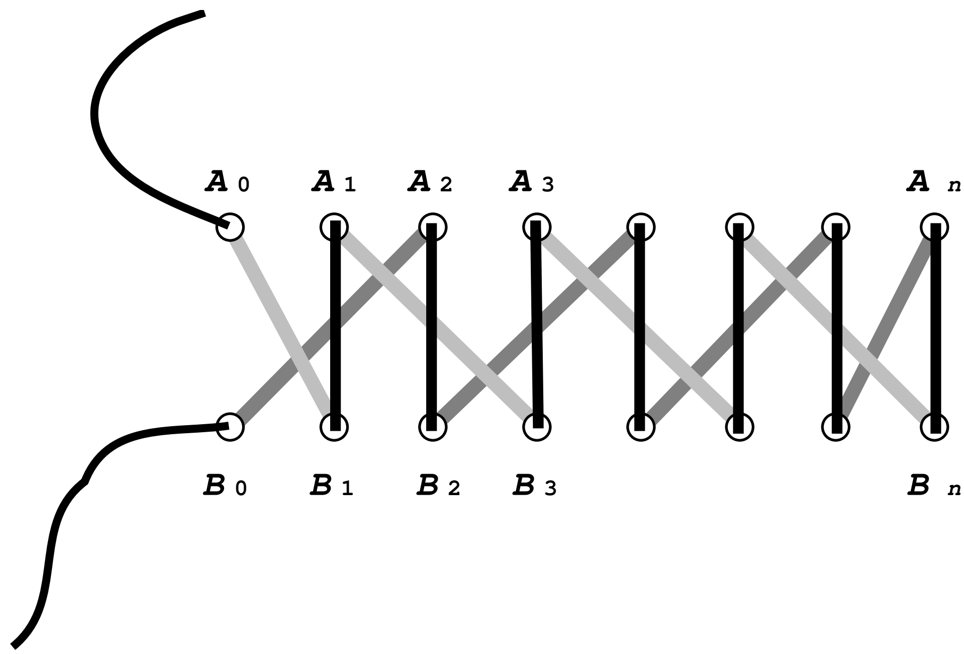


Figure 3
European, straight, lacing.

Here, when *n* is *odd*, as in Fig. 3, the lacing is

$$A_0 \;\; B_1 \;\; A_1 \;\; B_3 \;\; A_3 \;\; \ldots \;\; A_{n-2} \;\; B_n$$
$$A_n \;\; B_{n-1} \;\; A_{n-1} \;\; B_{n-3} \;\; \ldots \;\; B_2 \;\; A_2 \;\; B_0; \qquad (5)$$

when *n* is *even*, the lacing is, similarly,

$$A_0 \;\; B_1 \;\; A_1 \;\; B_3 \;\; A_3 \;\; \ldots$$
$$A_{n-1} \;\; B_n \;\; A_n \;\; B_{n-2} \;\; A_{n-2} \;\; B_{n-4} \;\; \ldots \;\; B_2 \;\; A_2 \;\; B_0; \qquad (6)$$

and, with a little more thought, we see that, in both cases, the t length
of lace is

$$L_{EU} = L_{EU}(n, v, w) = nw + 2\sqrt{v^2+w^2} + (n-1)\sqrt{4v^2+w^2}. \qquad (7)$$

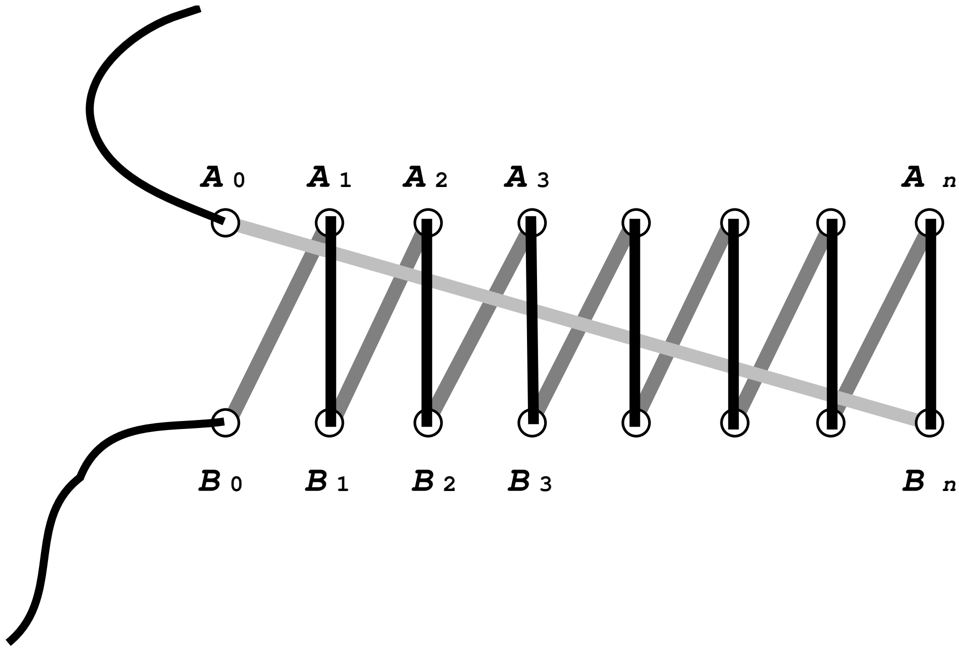


Figure 4
Shoe-shop, quick, lacing.

Here, the lacing is

$$A_0 \quad B_n \quad A_n \quad B_{n-1} \quad A_{n-1} \quad \ldots \quad B_3 \quad A_3 \quad B_2 \quad A_2 \quad B_1 \quad A_1 \quad B_0 \tag{8}$$

and we find that the total length is

$$L_{SS} = L_{SS}(n, v, w) = nw + n\sqrt{v^2+w^2} + \sqrt{n^2v^2+w^2}. \tag{9}$$

We can generalize the situation as follows. Let α and β denote permutations of $\{1, 2, 3, \ldots, n\}$:

$$\left.\begin{aligned} \alpha &= \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \\ \beta &= \{\beta_1, \beta_2, \ldots, \beta_n\}. \end{aligned}\right\} \tag{10}$$

To them will correspond the lacing

$$A_0 \quad B_{\beta_1} \quad A_{\alpha_1} \quad B_{\beta_2} \quad A_{\alpha_2} \quad B_{\beta_3} \quad \ldots \quad A_{\alpha_{n-1}} \quad B_{\beta_n} \quad A_{\alpha_n} \quad B_0, \tag{11}$$

and this will have total length

$$L = \sqrt{\beta_1^2v^2+w^2} + \sqrt{(\alpha_1-\beta_1)^2v^2+w^2} + \sqrt{(\beta_2-\alpha_1)^2v^2+w^2}$$

$$+ \sqrt{(\alpha_2 - \beta_2)^2 v^2 + w^2} + \ldots + \sqrt{(\beta_n - \alpha_{n-1})^2 v^2 + w^2} + \sqrt{\alpha_n^2 v^2 + w^2}. \tag{12}$$

For the three special lacings shown above, the particula permutations are:

$$\left.\begin{aligned}\alpha_{\mathrm{AM}}&=\{\mathit{all even numbers increasing; then all odd numbers decreasing}\},\\ \beta_{\mathrm{AM}}&=\{\mathit{all odd numbers increasing; then all even numbers decreasing}\};\end{aligned}\right\} \qquad (13)$$

$$\left.\begin{aligned}\alpha_{\mathrm{EU}}&=\{\mathit{all odd numbers increasing; then all even numbers decreasing}\},\\ \beta_{\mathrm{EU}}&=\{\mathit{all odd numbers increasing; then all even numbers decreasing}\};\end{aligned}\right\} \qquad (14)$$

$$\left.\begin{aligned}\alpha_{\mathrm{SS}}&=\{\mathit{all numbers decreasing}\},\\ \beta_{\mathrm{SS}}&=\{\mathit{all numbers decreasing}\}.\end{aligned}\right\} \qquad (15)$$

The simplicity of these permutations is indeed remarkable.

THE THREE STANDARD LACINGS

THEOREM 1. ***If $v = 0$ or $w = 0$, for all positive n,***

$$L_{AM} = L_{EU} = L_{SS}. \tag{16}$$

If $v \neq 0$ and $w \neq 0$,

$$L_{AM}(1, v, w) = L_{EU}(1, v, w) = L_{SS}(1, v, w); \tag{17}$$

and, if $v > 0$ and $w > 0$,

$$L_{AM}(2, v, w) < L_{EU}(2, v, w) = L_{SS}(2, v, w). \tag{18}$$

Finally, if $v > 0$ and $w > 0$ and $n > 2$,

$$L_{AM} < L_{EU} < L_{SS}. \tag{19}$$

Proof. **We use (4), (7), and (9), and successively prove (16)—(19).**

(i) First, by direct substitution, we see that, for $v = 0$, or $w = 0$, respectively,

$$L_{AM}(n, 0, w) = L_{EU}(n, 0, w) = L_{SS}(n, 0, w) = (2n + 1)w \tag{20}$$

and

$$L_{AM}(n, v, 0) = L_{EU}(n, v, 0) = L_{SS}(n, v, 0) = 2nv, \tag{21}$$

proving the equation (16).

(ii) Again, for all non-negative v and w,

$$L_{AM}(1, v, w) = L_{EU}(1, v, w) = L_{SS}(1, v, w) = w + 2\sqrt{v^2 + w^2}, \tag{22}$$

proving the equation (17).

(iii) Similarly, for all non-negative v and w,

$$L_{AM}(2, v, w) = w + 4\sqrt{v^2 + w^2} \tag{23}$$

and

$$L_{EU}(2, v, w) = L_{SS}(2, v, w) = 2w + 2\sqrt{v^2 + w^2} + \sqrt{4v^2 + w^2}. \tag{24}$$

Now, if $v > 0$ and $w > 0$, we get the following succession of true inequalities.

$$
\begin{aligned}
\{\, v^2w^2 > 0 \,\} &\Leftrightarrow \{\, (4v^4 + 5v^2w^2 + w^4) - (4v^4 + 4v^2w^2 + w^4) > 0 \,\} \\
&\Leftrightarrow \{\, 4v^4 + 5v^2w^2 + w^4 > 4v^4 + 4v^2w^2 + w^4 \,\} \\
&\Leftrightarrow \{\, (v^2 + w^2)(4v^2 + w^2) > (2v^2 + w^2)^2 \,\} \\
&\Rightarrow \left\{ \sqrt{(v^2+w^2)(4v^2+w^2)} > 2v^2+w^2 \right\} \quad \text{[take square root]} \\
&\Leftrightarrow \left\{ 2v^2+w^2 - \sqrt{(v^2+w^2)(4v^2+w^2)} < 0 \right\} \qquad (25) \\
&\Leftrightarrow \left\{ w^2 > w^2 + 4\left[2v^2+w^2 - \sqrt{(v^2+w^2)(4v^2+w^2)} \right] \right\} \\
&\Leftrightarrow \left\{ w^2 > (4v^2+w^2) - 4\sqrt{(v^2+w^2)(4v^2+w^2)} + 4(v^2+w^2) \right\} \\
&\Rightarrow \left\{ w > 2\sqrt{v^2+w^2} - \sqrt{4v^2+w^2} \right\} \quad \text{[take square root]} \\
&\Leftrightarrow \left\{ 2w + 2\sqrt{v^2+w^2} + \sqrt{4v^2+w^2} > w + 4\sqrt{v^2+w^2} \right\},
\end{aligned}
$$

which, with (23) and (24), yields the relation (18).

Having proved the special cases of our theorem, henceforth, we assume that $n > 2$, $v > 0$, and $w > 0$.

(iv) We first prove, for this general case, that

$$L_{EU} < L_{SS}\,. \qquad (26)$$

We proceed much as before, beginning with the true result (25), remembering that $n > 2$.

$$
\begin{aligned}
&\left\{ 2v^2+w^2 - \sqrt{(v^2+w^2)(4v^2+w^2)} < 0 \right\} \\
&\quad\Leftrightarrow \left\{ 2(n-1)(n-2)(2v^2+w^2) - 2(n-1)(n-2)\sqrt{(v^2+w^2)(4v^2+w^2)} < 0 \right\} \\
&\quad\Leftrightarrow \Big\{ (n-2)^2(v^2+w^2) - 2(n-1)(n-2)\sqrt{(v^2+w^2)(4v^2+w^2)} \\
&\qquad\qquad + (n-1)^2(4v^2+w^2) < n^2v^2+w^2 \Big\} \\
&\quad\Rightarrow \left\{ (n-1)\sqrt{4v^2+w^2} - (n-2)\sqrt{v^2+w^2} < \sqrt{n^2v^2+w^2} \right\} \quad \text{[take square root]} \\
&\quad\Leftrightarrow \left\{ nw + 2\sqrt{v^2+w^2} + (n-1)\sqrt{4v^2+w^2} < nw + n\sqrt{v^2+w^2} + \sqrt{n^2v^2+w^2} \right\},
\end{aligned}
$$

which, with (7) and (9), proves the inequality (26).

(v) Finally, we prove that

$$L_{\mathrm{AM}} < L_{\mathrm{EU}}\,. \tag{27}$$

Again, we begin with (25):

$$\left\{2v^2+w^2-\sqrt{(v^2+w^2)(4v^2+w^2)}<0\right\}$$

$$\Leftrightarrow \left\{4(v^2+w^2)-4\sqrt{(v^2+w^2)(4v^2+w^2)}+(4v^2+w^2)<w^2\right\}$$

$$\Rightarrow \left\{2\sqrt{v^2+w^2}-\sqrt{4v^2+w^2}<w\right\} \quad \text{[take square root]}$$

$$\Leftrightarrow \left\{(n-1)\left[2\sqrt{v^2+w^2}-\sqrt{4v^2+w^2}\right]<(n-1)\,w\right\}$$

$$\Leftrightarrow \left\{w+2n\sqrt{v^2+w^2}<nw+2\sqrt{v^2+w^2}+(n-1)\sqrt{4v^2+w^2}\right\},$$

$$w + 2n\sqrt{v^2+w^2} \;<\; nw + 2\sqrt{v^2+w^2} + (n-1)\sqrt{4v^2+w^2};$$

which, with (4) and (7), proves the inequality (27), thus completi the inequality (19), and the proof of our theorem. ○

THE LATTICE REPRESENTATION

Let us make a lattice of alternating parallel, equidistant s ***A*** **and** ***B*****, as shown in Figure 5. Given any lacing** ***L*****, we can represent it, as is shown for our three standard examples, by a polygonal [piecewise straigh line** ***L*** **moving always downward across the new lattice, visiting the eyelet points only once each.**

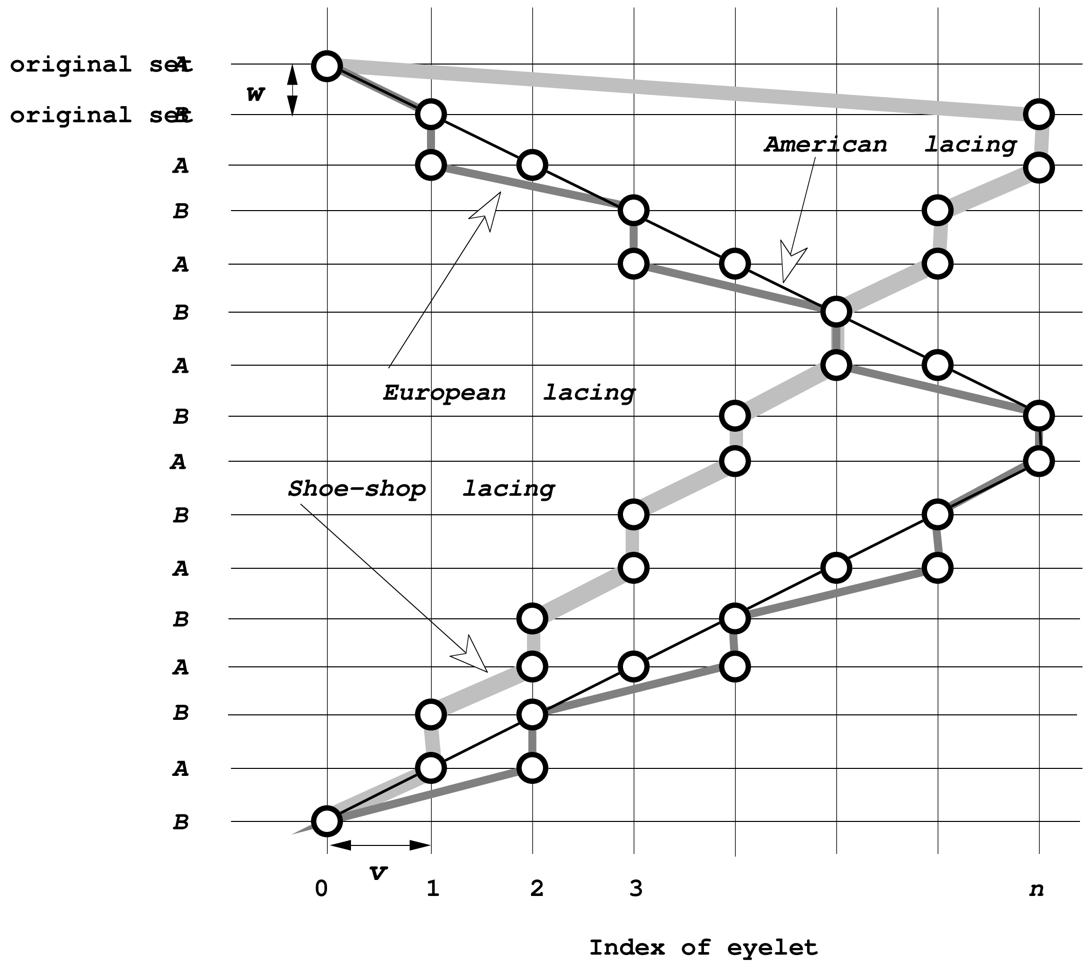
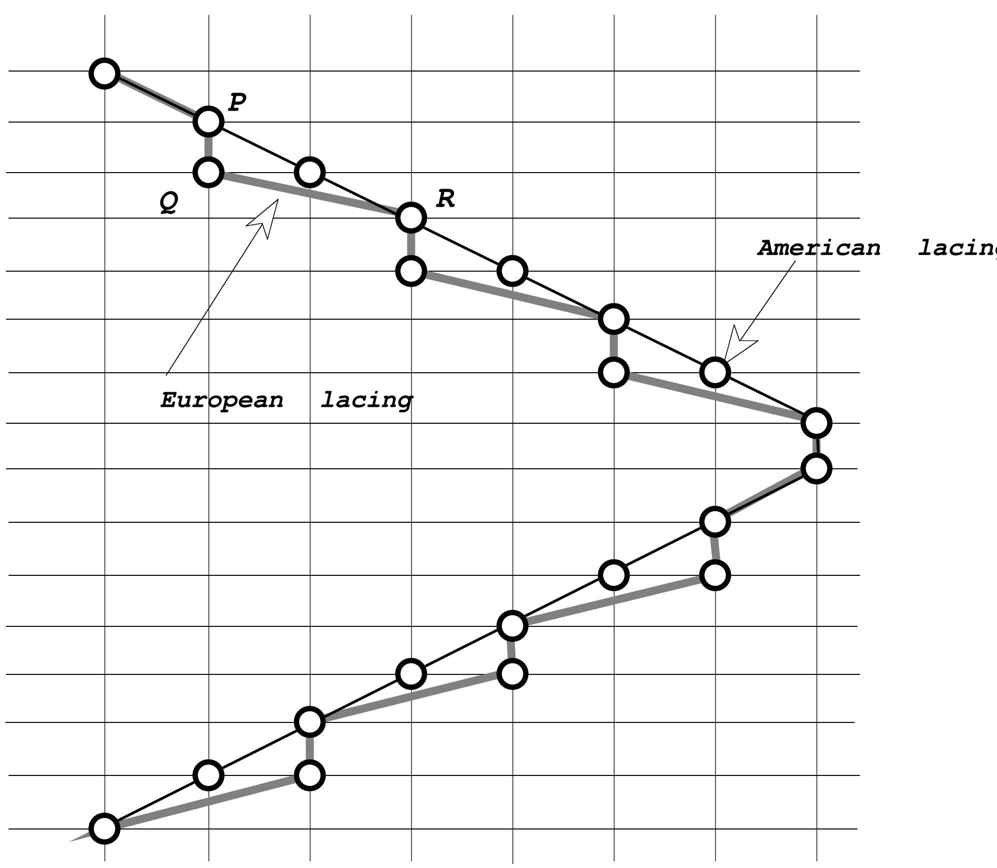


Figure 5

Lattice-representation of the three standard lacings

The first line segment in the order of lacing, is $A_0 B_{\beta_1}$ unchanged;
the next, $B_{\beta_1} A_{\alpha_1}$, is replaced by its mirror-image in the *B*-original line;
the next, $A_{\alpha_1} B_{\beta_2}$, is moved downward by two lattice-intervals, parallel to itself (i.e., it is a twice-repeated mirror-image); and so on
last segment, $A_{\alpha_n} B_0$, returns to the image of B_0 in the *B*-line displaced downward by 2*n* intervals. Clearly, the total length of representation ***L*** will equal the original total length *L* of the lacing ***L*** itself.

That the American [AM] lacing is better than the European [EU] lacing is now immediately apparent, by a straightforward application of the triangle inequality (see Figure 6).



<u>Figure 6</u>
Comparison of AM and EU lacing

The two representations, L_{AM} and L_{EU} coincide in several places. Where they differ, replicas of a triangle PQR occur, and it is clear that $PR < PQ + QR$, so that (27) follows, without further algebra!

That the EU lacing is better than the SS lacing is a little harder to prove (see Figure 7).

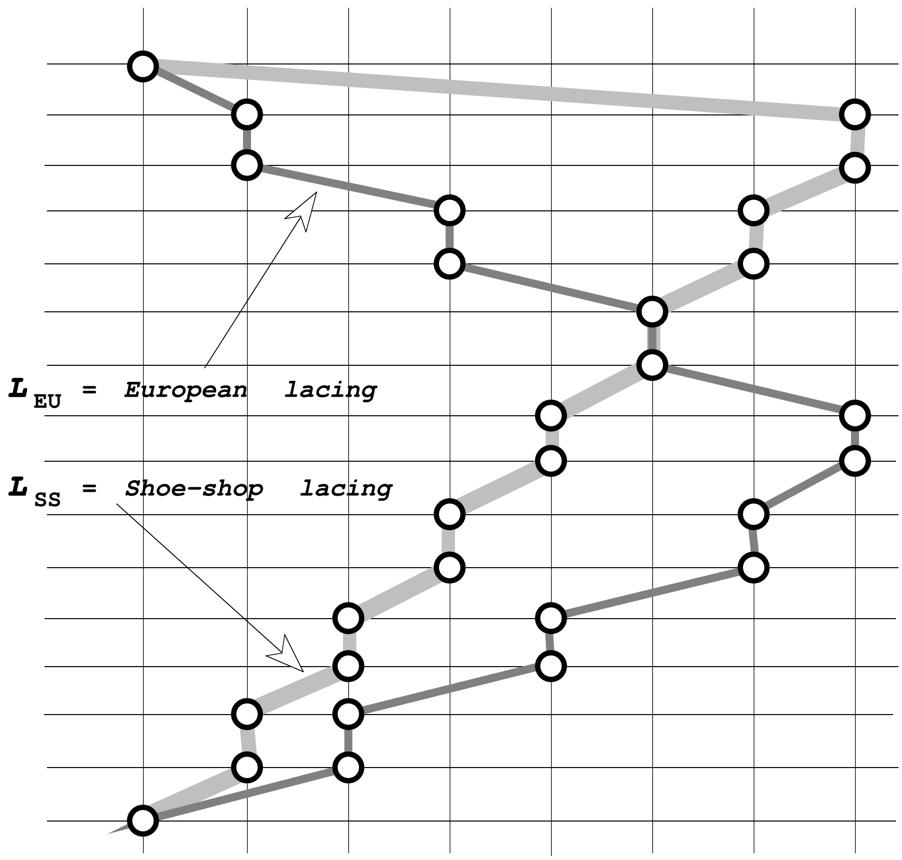


Figure 7
Comparison of EU and SS lacing

First, we observe that *both* representations L_{EU} and L_{SS} have in common just *two* diagonal segments, moving by one lattice interval in both directions (slopes −*w*/*v*), and *n* (vertical) segments, moving by one vertical lattice interval *w* only. If we omit all of common intervals, shifting the separated lower segment upwards (and in the first two cases, sideways also), parallel to themselves, to rejoin the upper segmen and thus *subtracting equal lengths from each representation*, we obtain reduced representations L^*_{EU} and L^*_{SS}. The result is shown below in Figure 8. Each representation now consists of a singl broken line (just two successive line-segments a *zig* and a *zag*).

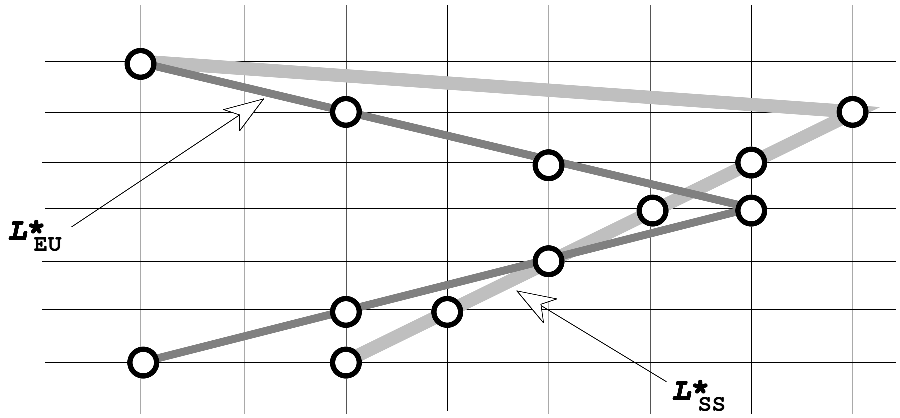


Figure 8
Comparison of EU and SS lacing reduced representations

Now perform the reflection trick again, this time in the horizo
coordinate direction, so that the leftward segment of each
representation
is reflected about the vertical. The resulting representation :
are denoted by L^{**}_{EU} and L^{**}_{SS} (see Figure 9).

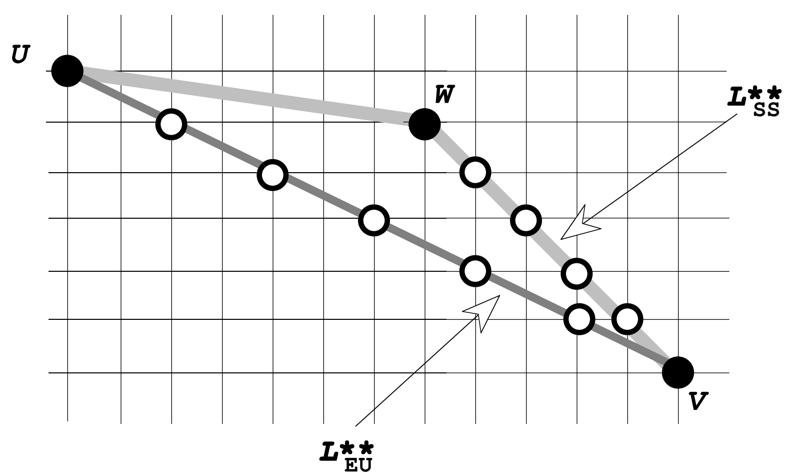


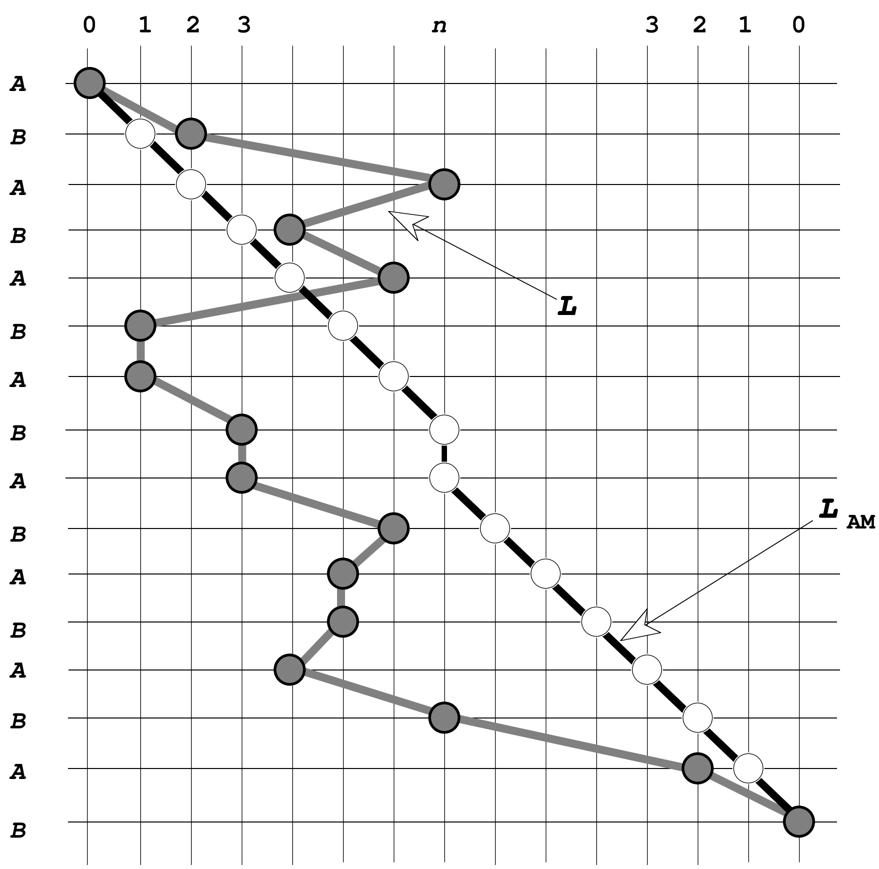
Figure 9
Comparison of EU and SS lacing reflected representations

We can now simply observe that L^{**}_{EU} is just a single straight segment *UV*, while L^{**}_{SS} consists of two straight segments, *UW* and

W V, so that, again by the triangle inequality, (26) clearly holds.

OPTIMIZATION

We adopt the lattice representation described above (see Figures 5—7), and apply the reflection trick to the part of the from B_n to B_0. The form of the path corresponding to an a typical general lacing is illustrated in Figure 10. The path L_{AM} corresponding to the AM lacing is also shown.



<u>Figure 10</u>
General lacing reflected representations

In this particular example, as before, n = 7 and the lacing is

$$A_0 \;\; B_2 \;\; A_7 \;\; B_4 \;\; A_6 \;\; B_1 \;\; A_1 \;\; B_3 \;\; A_3 \;\; B_6 \;\; A_5 \;\; B_5 \;\; A_4 \;\; B_7 \;\; \mathbf{A_2} \;\; B_0. \tag{28}$$

Its length is [compare (12)]

$$L = \sqrt{4v^2+w^2} + \sqrt{25v^2+w^2} + \sqrt{9v^2+w^2} + \sqrt{4v^2+w^2} + \sqrt{25v^2+w^2} + w$$
$$+ \sqrt{4v^2+w^2} + w + \sqrt{9v^2+w^2} + \sqrt{v^2+w^2} + w + \sqrt{v^2+w^2}$$
$$+ \sqrt{9v^2+w^2} + \sqrt{25v^2+w^2} + \sqrt{4v^2+w^2}$$
$$= 3w + 2\sqrt{v^2+w^2} + 4\sqrt{4v^2+w^2} + 3\sqrt{9v^2+w^2} + 3\sqrt{25v^2+w^2}. \tag{29}$$

In general, let the lacing have total length

$$L = \sum_{k=-n}^{n} N_k \sqrt{k^2v^2+w^2}, \tag{30}$$

where, clearly,
$$\sum_{k=-n}^{n} N_k = 2n+1 \tag{31}$$

is the net total number of downward displacements (i.e., the number of steps, since each step has a downward displacement by one lattice interval *w*), and

$$\sum_{k=-n}^{n} kN_k = 2n \tag{32}$$

is the net total number of rightward displacements by one lattice interval *v*. For the AM lacing, it is clear that

$$N_0 = 1, \quad N_1 = 2n, \quad \text{all other } N_k = 0. \tag{33}$$

THEOREM 2. ***The AM lacing has the shortest possible total length L, and it is the unique optimum lacing.***

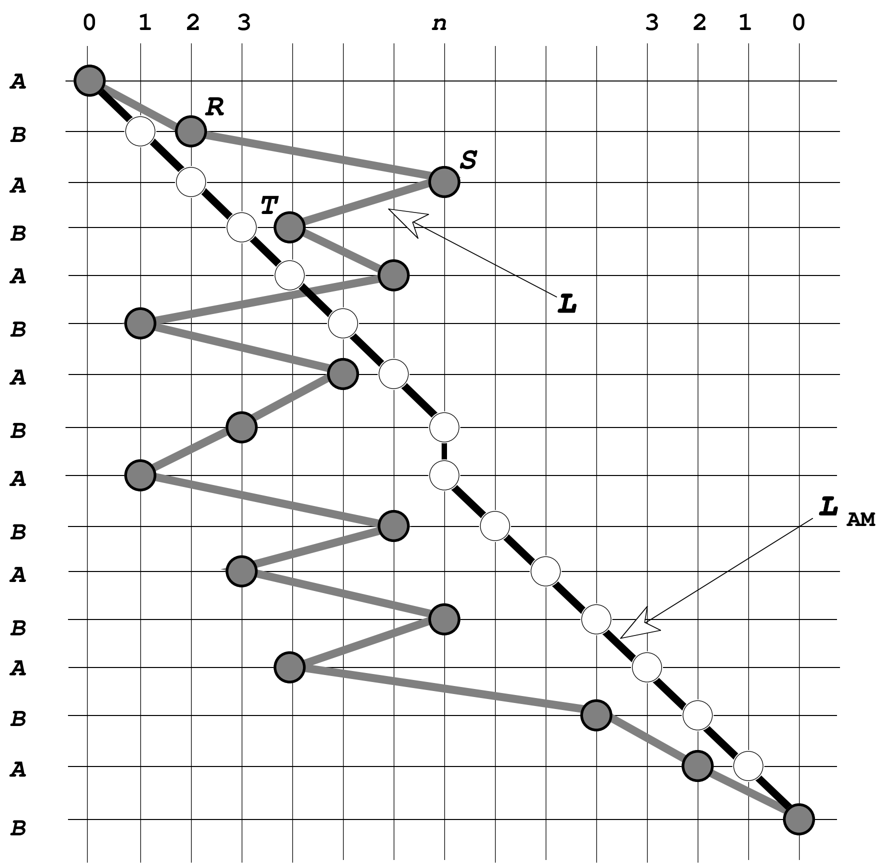
Proof. **Let *L* be the reflected representation of an arbitrary lacing *L*, and let *L* be its total length.**

(i) If $N_0 \neq 1$, let us remove any one corresponding (vertical) step from *L*, and let us remove the sole vertical step from L_{AM}, rejoining the separated pieces of the representations by parallel displacement,

as before; then the two new representations, L_{AM} and L still share their end points, and both lengths are just w less than they were. No L_{AM} is clearly minimal, being the straight line connecting these points. Therefore, for all L,

$$L_{AM} \leq L. \tag{34}$$

(ii) Suppose now that $N_0 = 0$. This is illustrated in Figure 11.



<u>Figure 11</u>
Case of $N_0 = 0$ no vertical segment

It cannot be that $N_k > 0$ only for positive values of k; for then, by (31) and (32), we would have that

$$\sum_{k=1}^{n} kN_k - \sum_{k=1}^{n} N_k = N_2 + 2N_3 + \ldots + (n-1)N_n = -1, \qquad (35)$$

which is impossible, since all $N_k \geq 0$. Therefore, that there is at least one step with a negative (*leftward*) horizontal displacement, and thus there is a first leftward step, *ST*, in the downward ord It obviously *cannot* be either the *first* or the *last* step of the representation. Hence, it preceded
by a *rightward* step, *RS*, forming an angle pointing to the right.

Now (see the enlarged detail of Figure 12) let *F* and *G* be the respective lattice points in which the vertical lines throu and *T* meet the horizontal line through *S*. Then

$$|FR| = |GT| = w \tag{35}$$

and there will be positive integers *p* ‡ 1 and *q* ‡ 1, such that

$$|FS| = pv \geq v \quad \text{and} \quad |GS| = qv \geq v\,. \tag{36}$$

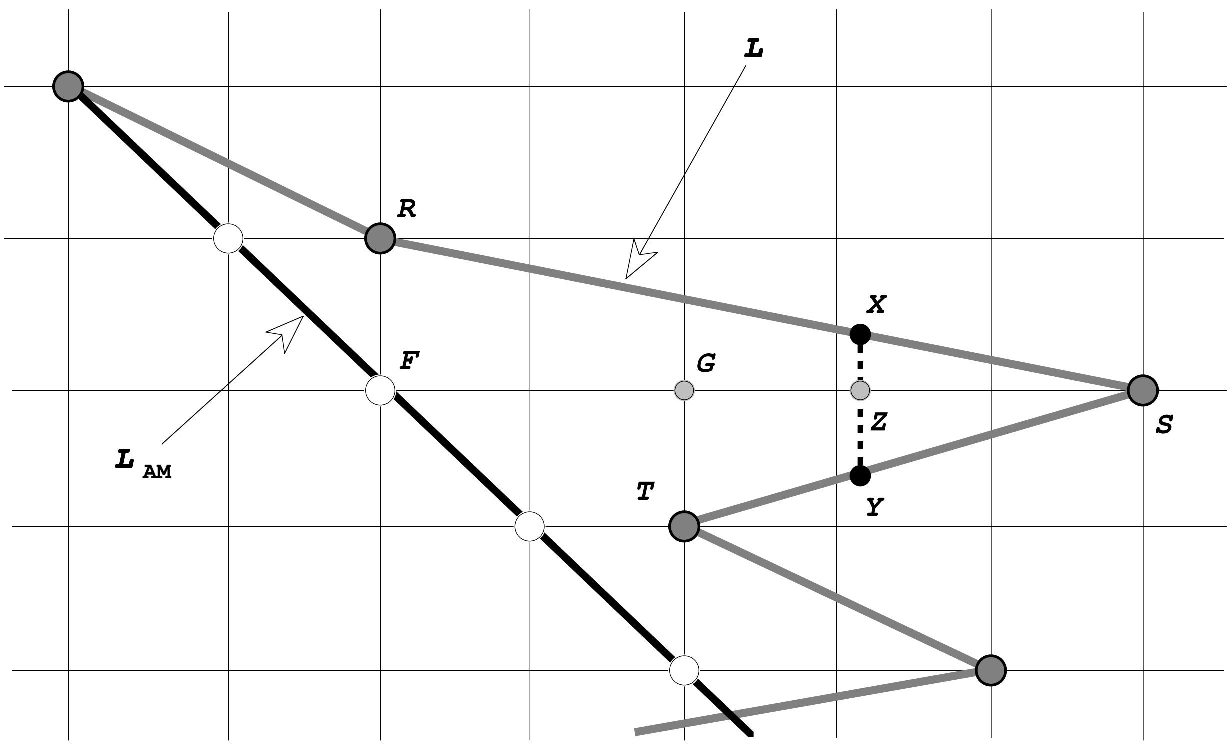


Figure 12
Magnified detail of Figure 11

If we take an arbitrary point *Z* on the horizontal line *FGS*, to the left of *S*, and if we let the vertical line through *Z* cut *RS* in *X* and *ST* in *Y*, then

$$|XY| = |ZX| + |ZY| = \frac{|ZS|}{|FS|}\,|FR| + \frac{|ZS|}{|GS|}\,|GT|$$

$$= |ZS|\left(\frac{1}{|FS|} + \frac{1}{|GS|}\right)w = |ZS|\left(\frac{1}{p} + \frac{1}{q}\right)\frac{w}{v}\,. \tag{37}$$

It follows that we can solve the equation

$$|XY| \quad = \quad w \tag{38}$$

by

$$|ZS| = \frac{v}{\frac{1}{p}+\frac{1}{q}} = \left(\frac{pq}{p+q}\right)v\,. \tag{39}$$

If we put

$$\alpha = \min\{p, q\} \quad \text{and} \quad \beta = \max\{p, q\}, \tag{40}$$

then we see that

$$|ZS| = \left(\frac{\alpha\beta}{\alpha+\beta}\right)v = \left(\frac{\alpha}{\frac{\alpha}{\beta}+1}\right)v \leq \alpha v, \tag{41}$$

so that Z is closer to S than either R or T, and X lies inside RS and Y lies inside ST.

Thus we can replace the polygonal segment RST of the representation $\mathcal{L}$ by the polygonal segment $RXYT$. By the triangle inequality,

$$|XY| < |XS| + |SY|; \tag{42}$$

so that the modified representation $\mathcal{L}^{\perp}$, say, is *shorter* than $\mathcal{L}$. But now $\mathcal{L}^{\perp}$ has a vertical segment of length w; so, by the same argument as in case (i), the inequality (34) applies.

NOTE: The representative polygonal line $\mathcal{L}^{\perp}$ is, generally, *not a representation of any lacing*, since it does not, in general, lattice points; but *this does not matter*, since, at this stage of the argument, we are only concerned with the *length* of the line.

We have now proved that, if $\mathcal{L}_{MIN}$ is any lacing of minimal length, then it and its (horizontally reflected) representation $\mathcal{L}_{MIN}$ will have a total length equal to that of the AM lacing, i.e., by (4),

$$L_{MIN} = L_{AM} = w + 2n\sqrt{v^2+w^2}. \tag{43}$$

(iii) Finally, we prove the *uniqueness* of the optimal lacing $\mathcal{L}_{MIN}$.

The arguments presented in (i) and (ii) above show that any minima lacing L_{MIN} will satisfy (33); that is, its (horizontally refl representation L_{MIN} will have n 2 straight segments, moving diagonally down-and-to-the-right by one lattice interval, and one vertical segment. However, t *position* of this vertical segment in the chain does not matter to the tot length L_{MIN}, as is indicated in (43).

Nevertheless, since L_{MIN} is not just any lattice-polygon, but the representation of a *lacing*, it *must* pass through the vertical lattice line corresponding to index n just twice (corresponding to the eyelets A_n and B_n), and this is the *only* lattice line *not* duplicated by the reflection transformation, since it is the reflection-line. Therefore, since the representation moves monotonely right (i.e. never to the left), the solitary vertical segment is constrained to be precisely in index-n position, as in L_{AM}. This completes the proof of Theorem 2. ❍