Ridges for Image Analysis

D. Eberly[1,4,5], R. Gardner[2,5], B. Morse[1,5], S. Pizer[1,5], C. Scharlach[2,3,5]

[1] Department of Computer Science
University of North Carolina
Chapel Hill, NC 27599–3175

[2] Department of Mathematics
University of North Carolina
Chapel Hill, NC 27599–3250

[3] Fachbereich Mathematik MA 8–3
Technische Universität Berlin
Straße des 17. Juni 136
D–10623 Berlin, Germany

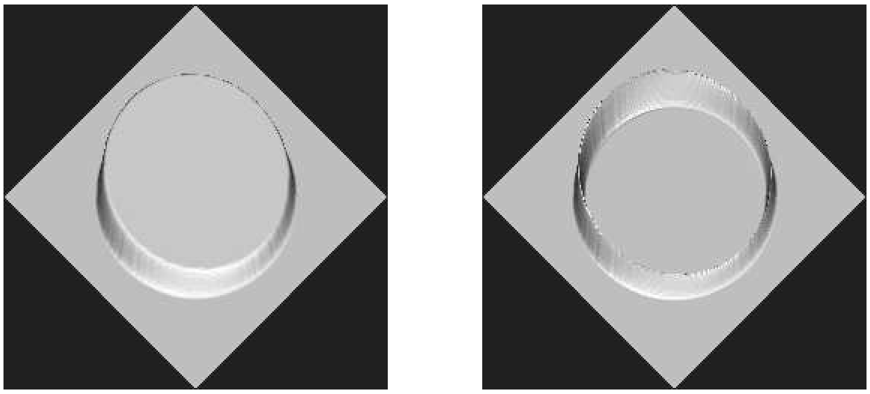
[4] Research supported by National Science Foundation Grant DMS–9003037

[5] Research supported by NIH grant # P01 CA 47982

Abstract. Representation of object shape by medial structures has been an important aspect of image analysis. Methods for describing objects in a binary image by medial axes are well understood. Many attempts have been made to construct similar medial structures for objects in gray scale images. In particular, researchers have studied images by analyzing the graphs of the intensity data and identifying ridge and valley structures on those surfaces. In this paper we review many of the definitions for ridges. Computational vision models require that medial structures should remain invariant under certain transformations of the spatial locations and intensities. For each ridge definition we point out which invariances the definition satisfies. We also give extensions of the concepts so that we can locate d–dimensional ridge structures within n–dimensional images. A comparison of the ridge structures produced by the different definitions is given both by mathematical examples and by an application to a 2–dimensional MR image of a head.

1 The Need for Ridges in Image Analysis

Methods for representing shapes of objects in gray–scale images have typically fallen into two categories: edge–based or region–based. Edge–based algorithms are developed under the assumption that large gradients of image intensity indicate the presence of an edge. The property of *edgeness* at a pixel is determined by measuring the *dissimilarity* between the pixel intensity and its neighbors' intensities, for example, by using the magnitude of the gradient of intensity. These algorithms additionally must handle edge orientation, edge strength, and edge connectivity. The method of edge detection essentially consists of following *ridges* of edgeness. Figure 1.1 illustrates this for a simple object.



graph of intensity　　　graph of magnitude of gradient of intensity

Fig. 1.1: Edges as ridges of the magnitude of gradient of intensity

Many edge–based methods are deficient since the presence of noise can make it difficult to detect an edge and determine its orientation. Moreover, the characterization of the *global* structure and shape of an object by its boundary depends greatly on the correctness of the edge connectivity scheme.

Region–based algorithms are developed under the assumption that an object is locally homogeneous. The property of *interiorness* at a pixel is determined by measuring the *similarity* between the pixel intensity and its neighbors' intensities. Regions are essentially sets of positive measure whereas edges are sets of zero measure, so region–based algorithms tend to be less sensitive to noise. However, the region growing is still stopped at ridges of edge strength, and without the use of global information, there are still problems with noise sensitivity.

A good model for representing object shape should encapsulate the ideas of both edge– and region–based methods. The Blum medial axis [1] construction for 2–dimensional binary objects was one of the first attempts at providing both region and edge information. The structure of the axis itself encodes the shape information of the object. Each axis point has an associated radius for the maximal disk centered at the point and contained entirely inside the object. The radii determine edge locations in the sense that the boundary of the union of all the maximal disks is the boundary of the object. A detailed mathematical analysis of medial axes (or more generally, symmetry sets) can be found in [4].

Given a binary object, each point inside the object can be assigned its distance to the boundary of the object. The points outside the object are assigned a value of zero. The function corresponding to this assignment is called the *Euclidean distance transform* (or *boundary distance function* [23]) of the object. Ridges on the graph of the distance transform can be projected onto the plane to obtain medial–like structures. The mathematical relationships between the Blum medial axis and ridges is discussed in [23]. In Figure 1.2 the left picture is a binary object which resembles a face. The medial axis for the object is also shown. The right image is a shaded rendering of the graph of the Euclidean distance transform for the object.

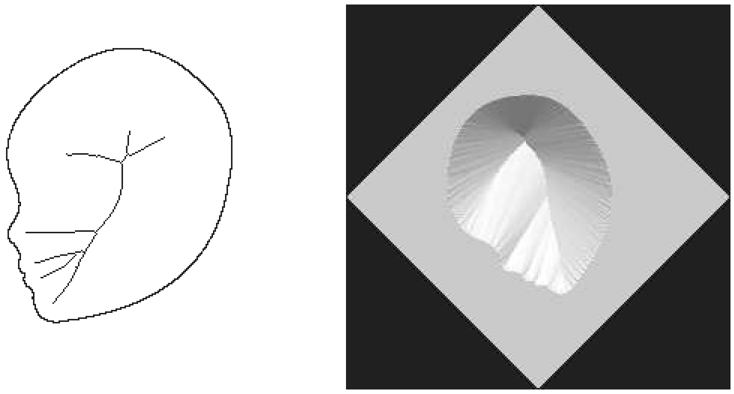


Fig. 1.2: binary object and medial axis, graph of distance function

The Blum medial axis construction requires a binary object whose edge locations are already known. Objects in gray–scale images are not binary and edge locations are not known, so the algorithm is of no practical use in this setting. However, for some types of images, the ridges of the intensity function may be medial–like. Figure 1.3 shows an MR image and its corresponding intensity graph.

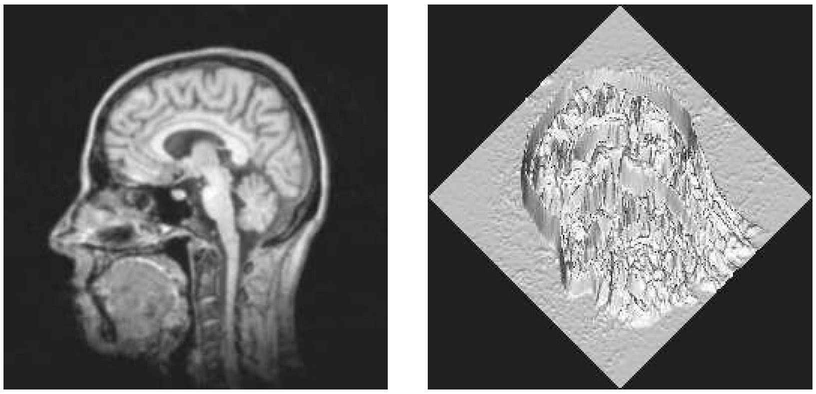


Fig. 1.3: MR image and graph of its intensity values

Note how the ridges on the graph tend to be in the "middle" of objects such as the scalp or brain stem. One still needs to decide what radius or *object width* information to assign to the ridge points. Good progress has been made in that direction with multiscale medial axes [22].

2 Ridge Definitions

Numerous attempts have been made to construct ridges for use in image and shape analysis. Consequently many definitions for ridges can be found in the literature. Each definition has its advantages and disadvantages, but it is desirable that ridges satisfy certain properties. First, the process which identifies ridges should be *local*. That is, a ridge point should be determined solely by the information in a local neighborhood of the point. Second, the ridges should be *invariant* under the following transformations:

- translations in the spatial variables,
- rotations in the spatial variables,
- uniform magnification in the spatial variables, and
- monotonic transformations of the intensity function.

The first two invariances simply mean that the operations of ridge construction and should commute. The ridges of a rotated/translated object should be the rotated/translated ridges of the original object. The third invariance means either of two things. The ridge construction should be independent of the units of measurement; a change of units is equivalent to a uniform magnification in the spatial variables. The ridge construction should also be independent of a uniform increase in size of the object. The ridges of an object which is doubled in size should be the ridges, doubled in size, of the original object. The fourth property is desired since the data, as sampled by the imaging device, may be modified in a monotonic way to meet the requirements of a display device. For example, if 12–bit data are to be displayed on an 8–bit display, the data is typically transformed using a monotonic affine transformation. We would like the ridge structures not to be affected by this transformation. A study of invariance and its applications to computer vision can be found in [8].

The definition for ridges as *slope district boundaries* is used in [5, 15] for segmenting intensity images. The graph of the intensity values is searched for maxima, minima, and saddle points. Slope lines, which are integral curves of the intensity gradient, are drawn from minima to saddles (course lines) and from saddles to maxima (ridge lines). The regions enclosed by the slope lines are called slope districts and are the primitive regions of the segmentation.

The slope district construction has also been used on hypersurfaces. The Blum medial construction for 3–dimensional objects [1, 2] yields a medial surface which encodes the shape of the object. Each surface point has an associated radius for the maximal sphere centered at the point and contained entirely inside the object. The medial surface is typically as complicated as the object itself. Attempts have been made to simplify this surface by segmenting it using slope districts [19, 20]. In this case the domain of the radius function is the medial surface, as compared to the planar domain for intensity images. The graph of the radius function is searched for maxima, minima, and saddle points. As before, ridge and course lines are drawn connecting the critical points.

The ridges from the slope district definition are clearly invariant under spatial translations, spatial rotations, and uniform spatial magnifications. They are also invariant under monotonic transformations of intensity since the gradient vector fields of the intensity and the transformed intensity have the same directions (but not necessarily the same lengths).

An undesirable consequence of the slope district construction is that the ridge and course lines are determined by a nonlocal process. Consider a ridge line whose endpoints are a maximum and a saddle. Each ridge point occurs simply because it is on the flow line connecting the maximum and saddle. No local geometric information at the point is used to establish its identity as a ridge. If the intensity function is perturbed in a small neighborhood of the saddle, the effect of the perturbation is propagated to *all* the ridge points on the flow to the maximum, even if the intensity values near the maximum have not been perturbed at all. Thus, the ridge points at the maximum have been relocated even though no change in intensity has occurred in that region. Another deficiency in the slope district construction is that it may ignore small ridges which lie on the flank of a larger ridge, because the small ridge is frequently not formed by the presence of local extrema of the intensity.

A discrete definition for ridges appears in [6], where the underlying function is the intensity convolved with a difference of low–pass transforms. Given a discrete 2–dimensional image, ridges are defined as those pixels for which the function has a positive local maximum in one of the 4 directions associated with the 8–neighborhood of the pixel. The generated ridges are invariant under spatial translations and monotonic transformations of the function. The ridges are not preserved by rotations. At any point where the function has a local maximum with respect to some direction V, that point will be labeled as a ridge in an orientation which aligns V with one of the 4 special directions, even if the point was not labeled as a ridge in its original orientation.

A more successful definition for ridges is what we call the *height definition*. A history of the attempt to define ridges of this type, dating back to 1852, is found in [17]. More recent work can be found in [12, 18]. It is based on computing local maxima for the intensity (or height) function in special directions. The ridge construction by this definition is a local process. Moreover, the ridges are invariant under spatial translations, spatial rotations, uniform spatial magnifications, but not under monotonic transformations of the height function. We discuss the height definition in section 4.

Other researchers have considered defining ridges using the ideas of differential geometry. The definitions are local and involve measurements of curvatures associated with graphs or surfaces in general. We will consider two such definitions.

The first of the differential geometric definitions, which we call the *principal direction definition*, applies to n–dimensional hypersurfaces. The standard analysis of such surfaces uses a parameterization. In typical computational vision applications we do not have a parameterization, so we develop the ridge definitions for implicitly defined hypersurfaces in $\mathbb{R}^{n+1}$. Every hypersurface (for example, graphs of functions defined on $\mathbb{R}^n$) can be described this way, thus we do not lose generality.

The ridges are constructed as local extrema of principal curvatures, where the differentiation is taken in principal directions. A geometrically intuitive discussion of this type ridge is found in [16]; a formal mathematical analysis is given in [3]. The ridge construction by the principal direction definition is a local process. For a parameterized hypersurface, the ridges are invariant under diffeomorphisms on the parameter space. Also, the ridges are invariant under spatial translations, spatial rotations, and uniform spatial magnifications, where the transformations are applied to the full space $\mathbb{R}^{n+1}$ which contains the hypersurface. In regard to graphs of functions with domain $\mathbb{R}^n$, the ridges are not invariant under monotonic changes of the defining function. An application of the principal direction definition can be found in [24]. Surfaces representing a skull were obtained from 3–dimensional data sets. Ridges on the surfaces were constructed and used as landmarks for image registration. The principal direction definition is discussed in section 5.

The second of the differential geometric definitions, which we call the *level definition*, considers curvature properties of level sets for smooth functions. The constraint of smoothness for intensity functions is consistent with the way that a scene is observed using finite width apertures [10]. A motivating example is the function $f(x, y) = 1 - (x/a)^2 - (y/b)^2$ where $a > b$, where for illustration suppose a is very much larger than b. Treating the graph as mountain terrain, a person walking at constant altitude around the mountain might label the mountain ridges as those points where the change in his direction of walking changes the most. In this case, the ridge points are identified at places where $y = 0$. The paths of constant altitude are the level curves of the function. The places where the direction (level curve tangent) changes most rapidly are those points for which the level curve curvature is locally optimum. The ridge construction is local and the ridges are invariant under spatial translations, spatial rotations, uniform spatial magnifications, and monotonic transformations of the intensity function. Applications of the level definition to computer vision problems are found in [14]. We discuss the level definition in section 6. It turns out to be exactly the principal direction definition, but applied to level surfaces of the graph instead of the entire graph itself.

Finally, we give a ridge definition, called the *nonmetric definition*, which is similar to the principal direction definition, but applies only to graphs of functions. The ridges produced by this definition are often qualitatively similar to those produced by the principal direction definition. However, unlike the principal direction definition, this ridge construction does not use the metric of the graph. We consider the nonmetric definition in section 7.

Section 8 contains the application of the height, principal direction, level, and nonmetric definitions of ridges to the MR image of Figure 1.3.

The wide range of results above clearly indicates a need for ridges in image analysis as a means of describing object shape. The purpose of this paper is to provide a formal mathematical setting for ridge definitions, to generalize the ideas to higher dimensions, and to analyze the properties of the various definitions. We also define *valley points* in a similar way to ridges. Collectively all such points will be called *crease points*. They are located on the graph of f and therefore have the form $(x, f(x))$; however, we will also refer to x as a crease point. This identification should not create any confusion in the development.

3 Directional Derivatives

Throughout the paper we use the following notations. The partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ are denoted by subscripting f with the appropriate variable. If $x = (x_1, \ldots, x_n)$, then f_{x_i} is the first partial derivative of f with respect to x_i. The *gradient* of f is the vector $\nabla f = (f_{x_1}, \ldots, f_{x_n})$. The matrix of second partial derivatives of f, called the *Hessian* of f, is denoted by $\text{Hess}\,(f) = [f_{x_i x_j}]$. The function f is said to be a C^k function if its partial derivatives through order k are continuous.

All crease definitions require finding local extrema of functions in special directions. Therefore, we need the concept of *directional derivatives* which measure how the function varies when restricted to a subset of its domain. The zeros of first–order directional derivatives will be the candidate crease points. A candidate crease point will be classified as a ridge point or valley point depending on the information obtained from second–order directional derivatives. We present two definitions for each order directional derivative. The first definition assumes that the direction vectors are constant, but the second definition allows the vectors to be variable. For first–order directional derivatives, the two definitions are equivalent. However, second–order directional derivatives are computed as iterations of differential operators which depend on the direction vectors, so the two definitions produce different results.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Let $v \in \mathbb{R}^n$ be a constant nonzero vector. The *first derivative of f at x in the direction v* is defined by

$$(v \cdot \nabla) f(x) = \sum_{i=1}^{n} v_i \frac{\partial f(x)}{\partial x_i},$$

where "$\cdot$" indicates the dot product of vectors. The definition is motivated by calculus of a function of a single variable. At a point $p \in \mathbb{R}^n$, let $\xi(t)$ be a differentiable curve such that $\xi(0) = x$ and $\xi'(0) = v$. Define the function $\phi(t) = f(\xi(t))$ and compute the ordinary derivative with respect to t. By the chain rule we obtain $\phi'(t) = \xi'(t) \cdot \nabla f(\xi(t))$. At $t = 0$, we have $\phi'(0) = v \cdot \nabla f(x)$. Note that the first directional derivative is independent of the chosen path as long as the tangent vector at $t = 0$ is v.

Now let the directions be dependent on the points x at which the derivative measurement is made. Let $v(x)$ be a nonzero vector. The *first derivative of f at x in the direction $v(x)$* is defined by

$$D_v f(x) = \sum_{i=1}^{n} v_i(x) \frac{\partial f(x)}{\partial x_i}.$$

At a fixed point y, let $u = v(y)$; then $(u \cdot \nabla) f(y) = D_v f(y)$, so the definitions are equivalent.

We can similarly define second derivatives in specified directions. Let v be a constant nonzero vector. The *second derivative of f at x in the direction v* is defined by

$$(v \cdot \nabla)^2 f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i \frac{\partial^2 f(x)}{\partial x_i \partial x_j} v_j = v^t \,\text{Hess}\,(f) v.$$

If we define $\xi(t) = x + tv$ and $\phi(t) = f(\xi(t))$, then $\phi(0) = f(x)$, $\phi'(0) = v \cdot \nabla f(x)$, and $\phi''(0) = v^t \,\text{Hess}\,(f) v$. Therefore, $(v \cdot \nabla)^2 f(x)$ measures the second derivative along a *linear* path through x in the direction v. Mixed directional derivatives are also allowed. Let u and v be constant nonzero vectors. The *second derivative of f at x in the directions u and v* is defined by

$$(v \cdot \nabla)(u \cdot \nabla) f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i \frac{\partial^2 f(x)}{\partial x_i \partial x_j} v_j = u^t \,\text{Hess}\,(f) v.$$

Since f is C^2, the order of differentiation is irrelevant (the matrix Hess (f) is symmetric), so $(v \cdot \nabla)(u \cdot \nabla)f(x) = (u \cdot \nabla)(v \cdot \nabla)f(x)$. Now define $\xi(s,t) = x + su + tv$ and $\phi(s,t) = f(\xi(s,t))$; then $\phi(0,0) = f(x)$, $\phi_s(0,0) = u \cdot \nabla f(x)$, $\phi_t(0,0) = v \cdot \nabla f(x)$, and $\phi_{st}(0,0) = u^t \text{ Hess } (f(x))v = v^t \text{ Hess } (f(x))u = \phi_{ts}(0,0)$.

The definition for constant direction vectors is characterized by the parameterization $\xi(s,t) = x + su + tv$. Other parameterizations may lead to different results. If we want to specify *variable* nonzero vector fields $u(x)$ and $v(x)$, and then take second derivatives, there are some problems to consider. We would like the parameterization $\xi(s,t)$ to have the properties $\xi(0,0) = x$, $\xi_s(s,t) = u(\xi(s,t))$, and $\xi_t(s,t) = v(\xi(s,t))$. For a smooth solution $\xi(s,t)$ to exist, it is necessary that $\xi_{st} = \xi_{ts}$, which implies that

$$0 = \frac{\partial}{\partial t} u(\xi(s,t)) - \frac{\partial}{\partial s} v(\xi(s,t)) = \frac{\partial u(\xi)}{\partial x} v(\xi) - \frac{\partial v(\xi)}{\partial x} u(\xi)$$

where $\partial u / \partial x$ is the $n \times n$ matrix whose i th row contains the partial derivatives of component u_i. Therefore, at every x it is necessary that the vector fields commute in the sense of Lie products: $[u,v](x) = 0$ for all x. If we have commuting vector fields, then the *second directional derivative of f at x in the directions $u(x)$ and $v(x)$* is defined by

$$D_u D_v f(x) = u(x)^t \text{ Hess } (f(x))v(x) + \nabla f(x)^t \frac{\partial u}{\partial x} v(x) = D_v D_u f(x).$$

Note that this definition reduces to the previous one when u and v are constant vectors.

The directional derivative definitions using constant vectors are what we use in the height definition for creases; the quadratic form involving the Hessian will be used for second derivative tests. (This is the type of derivative used in the analysis of invariants in [8].) But in the principal direction definition, we will need to consider variable vectors, so the other directional derivative definitions will be used in this case.

4 Height Definition for Creases

The height definition for ridges and valleys is based on a generalization of local extrema for real–valued functions of a vector variable. We give a brief summary of the construction of local extrema; the construction can be found in standard calculus books, for example, [11].

4.1 Extreme Points

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. A point $x_0 \in \mathbb{R}^n$ is said to be a *critical point* for f if $(v \cdot \nabla)f(x_0) = 0$ for every direction v, which is equivalent to $\nabla f(x_0) = 0$. Critical points are classified via the following. The function f has

- a minimum at x_0 if $(v \cdot \nabla)^2 f(x_0) > 0$ for every direction v;
- a maximum at x_0 if $(v \cdot \nabla)^2 f(x_0) < 0$ for every direction v.

Any such point x_0 is called an *extreme point* and the corresponding function value is called an *extreme value*. It is only necessary to perform the tests for n–linearly independent directions. Equivalently, x_0 is a critical point if $\nabla f(x_0) = 0$. The function f has

- a minimum at x_0 if Hess $(f(x_0))$ is positive definite (all eigenvalues are positive);
- a maximum at x_0 if Hess $(f(x_0))$ is negative definite (all eigenvalues are negative).

4.2 Generalized Extreme Points

We consider a more general definition for extreme points f. Let $v_1, \ldots, v_d$ be a set of constant linearly independent vectors in $\mathbb{R}^n$, where $1 \leq d \leq n$. When $d = n$, the vectors can be thought of as the columns of an $n \times n$ invertible matrix V. The matrices Hess (f) and V^t Hess $(f)V$ have the same definiteness by Sylvester's Theorem [13]. Therefore, the construction of local extrema in the previous subsection is equivalent to the following. The function f has

- a *minimum* at x_0 if $V^t \nabla f(x_0) = 0$ and V^t Hess $(f(x_0))V$ is positive definite;
- a *maximum* at x_0 if $V^t \nabla f(x_0) = 0$ and V^t Hess $(f(x_0))V$ is negative definite.

Now let us consider the cases $d < n$. Let v_1 through v_d be linearly independent vectors. Let V be the $n \times d$ matrix whose columns are these vectors. We search for local extrema of f restricted to the subspace spanned by the v_i. The function has

- a *generalized minimum of type* $n - d$ at x_0 if $V^t \nabla f(x_0) = 0$ and V^t Hess $(f(x_0))V$ is positive definite;
- a *generalized maximum of type* $n - d$ at x_0 if $V^t \nabla f(x_0) = 0$ and V^t Hess $(f(x_0))V$ is negative definite.

Such points x_0 are called *generalized extreme points of type* $n - d$ *for* f *with respect to* V. The classification has the same form as the case $d = n$, but the $x_0 \in \mathbb{R}^n$ are now solutions to d equations in n unknowns. We expect the solution sets to be $(n-d)$–dimensional manifolds, hence the use of "type $n-d$" in the definition. Also, the matrix V^t Hess $(f)V$ is $d \times d$. The ridges and valleys we define will be generalized extreme points with respect to eigenvectors of the Hessian of f.

4.3 Crease Definitions

Define $W = -\operatorname{Hess}(f)$ and let κ_i and v_i, $1 \leq i \leq n$, be its eigenvalues and eigenvectors. A positive (negative) eigenvalue corresponds to convexity (concavity) of the graph of f. Assume that the eigenvalues are ordered as $\kappa_1 \geq \cdots \geq \kappa_n$. In the definition assume that $1 \leq d \leq n$.

- A point x is a *ridge point of type* $n - d$ if $\kappa_d(x) > 0$ and x is a generalized maximum point of type $n - d$ for f with respect to $V = [v_1 \cdots v_d]$. Since V^t Hess $(f)V = -\operatorname{diag}\{\kappa_1|v_1|^2, \ldots, \kappa_d|v_d|^2\}$ and the eigenvalues are ordered, the test for a ridge point reduces to $V^t \nabla f(x) = 0$ and $\kappa_d(x) > 0$.
- A point x is a *valley point of type* $n-d$ if $\kappa_{n-d+1}(x) < 0$ and x is a generalized minimum point of type $n-d$ for f with respect to $V = [v_{n-d+1} \cdots v_n]$. Since $V^t \operatorname{Hess}(f)V = -\operatorname{diag}\{\kappa_{n-d+1}|v_{n-d+1}|^2, \ldots, \kappa_n|v_n|^2\}$ and the eigenvalues are ordered, the test for a valley point reduces to $V^t \nabla f(x) = 0$ and $\kappa_{n-d+1}(x) < 0$.

This definition disallows the existence of ridges (valleys) of type $n - d$ in regions where W has fewer than d positive (negative) eigenvalues. However, ridges and valleys can occur in hyperbolic regions, where W

has both positive and negative eigenvalues. It is possible that a ridge exists at a point where a negative eigenvalue has larger magnitude than any of the positive eigenvalues. We can refine our definition to take into account the relative magnitudes of the eigenvalues.

- A ridge point x of type $n-d$ is a *strong ridge point* if $\kappa_d(x) > |\kappa_n(x)|$; otherwise, it is a *weak ridge point*. Strong ridge points occur in regions where the convexity dominates the concavity.
- A valley point x of type $n-d$ is a *strong valley point* if $|\kappa_{n-d+1}(x)| > \kappa_1(x)$; otherwise, it is a *weak valley point*. Strong valley points occur in regions where the concavity dominates the convexity.

4.4 Examples

EXAMPLE 4.1: Let $f(x,y) = x^2y$. The eigenvalues of W are $\kappa_1 = -y+\sqrt{4x^2+y^2}$ and $\kappa_2 = -y-\sqrt{4x^2+y^2}$. Observe that $\kappa_1(x,y) \geq 0 \geq \kappa_2(x,y)$ for all (x,y). Also, $\kappa_1 > |\kappa_2|$ for $y < 0$ and $|\kappa_2| > \kappa_1$ for $y > 0$. The corresponding eigenvectors are

$$v_1 = \left\{ \begin{array}{ll} (-y+R, -2x), & y \leq 0 \\ (2x, -y-R), & y \geq 0 \end{array} \right\}, \quad v_2 = \left\{ \begin{array}{ll} (2x, -y+R), & y \leq 0 \\ (-y-R, -2x), & y \geq 0 \end{array} \right\}$$

where $R = \sqrt{4x^2+y^2}$. The two different sets of eigenvectors are used since at $x = 0$, one vector in each set degenerates to the zero vector. The first directional derivatives are

$$D_{v_1}f = \left\{ \begin{array}{ll} 2x(yR - x^2 - y^2), & y \leq 0 \\ x^2(3y - R), & y \geq 0 \end{array} \right\}, \quad D_{v_2}f = \left\{ \begin{array}{ll} x^2(3y + R), & y \leq 0 \\ -2x(yR + x^2 + y^2), & y \geq 0 \end{array} \right\}.$$

Let V be the 2×2 matrix whose columns are v_1 and v_2. Since $\text{Hess}(f)$ is a symmetric matrix, its eigenvectors v_1 and v_2 are othogonal; thus, we have $V^t\, \text{Hess}\,(f)V = \text{diag}\{-\kappa_1|v_1|^2, -\kappa_2|v_2|^2\}$.

There are no ridges of type 0 since $\kappa_2 \leq 0$ for all (x,y). However, there are ridges of type 1. Firstly, note that $\kappa_1(x,y) > 0$ as long as not both $x = 0$ and $y \geq 0$. Secondly, if $x = 0$ and $y < 0$, then $D_{v_1}f = 0$ and $D_{v_1v_1}f < 0$. Also, if $y > 0$ and $x^2 = y^2$, then $D_{v_1}f = 0$ and $D_{v_1v_1}f < 0$. Therefore, the ridges of type 1 lie on three rays with origin $(0,0)$. The ray $x = 0$, $y < 0$ is a strong ridge since $\kappa_1 > |\kappa_2|$ at those points. The rays $x^2 = 2y^2$, $y > 0$ are weak ridges since $\kappa_1 < |\kappa_2|$ at those points.

Figure 4.1 contains a contour plot with the ridges drawn as thick lines. The origin is at the center of the picture and the coordinates are right–handed.

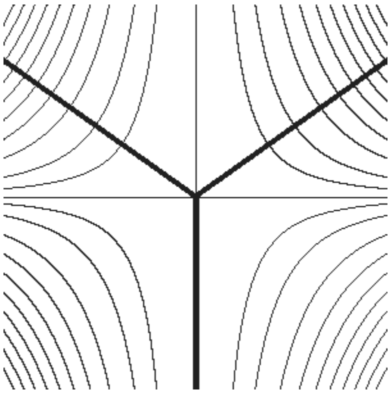


Fig. 4.1: height ridges of x^2y

EXAMPLE 4.2: Consider $f(x,y,z) = \frac{1}{2}(ax^2 + by^2 + cz^2)$ where $0 < a < b < c$. The first derivatives are $\nabla f = (ax, by, cz)$ and the second derivatives are $\text{Hess}(f) = \text{diag}(a,b,c)$. The matrix W has ordered eigenvalues $\kappa_1 = -a$, $\kappa_2 = -b$, and $\kappa_3 = -c$, with corresponding eigenvectors $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, and $v_3 = (0,0,1)$. Since the eigenvalues are all negative, we can expect valley points, but not ridge points.

The only valley point of type 0 is the local minimum point $(0,0,0)$. The valley points of type 1 consist of the x–axis since $D_{v_2}f = D_{v_3}f = 0$ imply $y = z = 0$. Intuitively this seems reasonable since the longest axes of the ellipsoidal level sets lie on the x–axis. The valley points of type 2 consist of the xy–plane since $D_{v_3}f = 0$ implies $z = 0$. This set also makes intuitive sense since the ellipsoidal level sets are flattest in the z–direction.

4.5 Invariance Properties

The ridges constructed by the height definition are invariant under spatial translations, spatial rotations, and uniform spatial magnifications. They are not invariant with respect to monotonic transformations of the intensity function. We use the following notation in the proofs. Let $x \in \mathbb{R}^n$. Let $u = u(x)$ be an invertible change of spatial variables and let $\bar{f}(u) = f(x)$. Define $\partial u/\partial x = [\partial u_i/\partial x_j]$ to be the matrix of partial derivatives of the components of u with respect to the components of x. All functions u of the class of spatial transformations mentioned above have the property $\partial^2 u_k/\partial x_i \partial x_j = 0$. Consequently, the following relationships hold:

$$\nabla f = \left(\frac{\partial u}{\partial x}\right)^t \nabla \bar{f} \quad \text{and} \quad \text{Hess}(f) = \left(\frac{\partial u}{\partial x}\right)^t \text{Hess}(\bar{f}) \frac{\partial u}{\partial x}.$$

Finally, define $W = -\text{Hess}(f)$ and $\bar{W} = -\text{Hess}(\bar{f})$. The generic eigenvalues and eigenvectors for these matrices will be denoted by κ, v, $\bar{\kappa}$, and $\bar{v}$ accordingly.

INVARIANCE UNDER SPATIAL TRANSLATIONS. Let $u = x + a$ where a is a constant vector; then $\nabla f = \nabla \bar{f}$ and $\text{Hess}(f) = \text{Hess}(\bar{f})$. Consequently, $\bar{v} = v$, $\bar{\kappa} = \kappa$, and x_0 is a solution to $D_v f = 0$ if and only if $u_0 = x_0 + a$ is a solution to $D_{\bar{v}}\bar{f} = 0$. The eigenvalue comparison in the strong ridge definition must hold since the eigenvalues have not changed magnitude.

INVARIANCE UNDER SPATIAL ROTATIONS. Let $u = Rx$ where R is a rotation matrix; then $\nabla f = R^t \nabla \bar{f}$ and $\text{Hess}(f) = R^t\,\text{Hess}(\bar{f})R$. Consequently, $\bar{v} = Rv$, $\bar{\kappa} = \kappa$, and x_0 is a solution to $D_v f = 0$ if and only if

$u_0 = Rx_0$ is a solution to $D_{\bar{v}}\bar{f} = 0$. The eigenvalue comparison in the strong ridge definition must hold since the rotation does not change the magnitudes of the eigenvalues.

Invariance under uniform spatial magnification. Let $u = cx$ where c is a positive scalar; then $\nabla f = c\nabla\bar{f}$ and $\text{Hess}\,(f) = c^2\,\text{Hess}\,(\bar{f})$. Consequently, $\bar{v} = v$, $\bar{\kappa} = \kappa/c^2$, and x_0 is a solution to $D_v f = 0$ if and only if $u_0 = cx_0$ is a solution to $D_{\bar{v}}\bar{f} = 0$. The eigenvalue comparison in the strong ridge definition must hold since dividing a set of numbers by a positive value does not change the ordering of the set.

Lack of invariance under monotonic transformations. Let $f(x, y) = 1 - x^2 - 2y^2$. The eigenvalues of $-\,\text{Hess}\,(f)$ are $\kappa_1 = 4$ and $\kappa_2 = 2$ with corresponding eigenvectors $v_1 = (0, 1)$ and $v_2 = (1, 0)$. The point $(1, 0)$ is a ridge point since $v_1 \cdot \nabla f(1, 0) = 0$ and $\kappa_1(1, 0) > 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g' > 0$. Define the composition of functions $\phi = g \circ f$. The derivatives are related by $\nabla\phi = g'(f)\nabla f$ and $\text{Hess}\,(\phi) = g'(f)\,\text{Hess}\,(f) + g''(f)\nabla f \nabla f^t$. The eigenvalues of $-\,\text{Hess}\,(\phi(1, 0))$ are $2g'(0) - 4g''(0)$ and $4g'(0)$ with corresponding eigenvectors $(1, 0)$ and $(0, 1)$, respectively. In order that $(1, 0)$ be a ridge point for ϕ we need $\kappa_1 = 4g'(0) > 2g'(0) - 4g''(0) = \kappa_2$. In this case, $v_1 = (0, 1)$ and $v_1 \cdot \nabla\phi(1, 0) = 0$. But the condition $4g'(0) > 2g'(0) - 4g''(0)$ is not satisfied by all monotonic functions, for example $g(t) = \ln(1 + t)$, where $g'(0) = 1$ and $g''(0) = -1$.

For a discussion of second–order invariance properties under general intensity transformations, see [9].

5 Principal Direction Definition for Creases

The principal direction definition for ridges and valleys is motivated by the differential geometry of n–dimensional hypersurfaces in $\mathbb{R}^{n+1}$. We define creases as loci of extrema of *principal curvatures* along associated *lines of curvature*. The curvature measurements are made with respect to the metric on the tangent hyperplanes. For a geometric motivation, compare with [23].

In standard differential geometry textbooks, hypersurfaces are described by a parameterization which is used in obtaining principal curvatures and principal directions. In computational vision applications, typically one obtains a surface as a collection of points with no underlying parameterization. Such surfaces are assumed to be implicitly defined, so we also want to construct principal curvatures and principal directions for surfaces defined as level sets of functions $F : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume that F is a C^4 function for which $\nabla F \neq 0$. The normal vectors to the surface are $N = \nabla F/|\nabla F|$.

Construction With Parameterization. Let the surface be parameterized by position $x : \mathbb{R}^n \to \mathbb{R}^{n+1}$, say $x = x(u)$. Define $J = \partial x/\partial u$, an $(n+1) \times n$ matrix which has rank n and satisfies the property $N^t J = 0$. That is, the columns of J are a basis of the tangent space and are orthogonal to N at position $x(u)$. The first and second fundamental forms are given by the $n \times n$ matrices $\mathbf{I} = J^t J$ and $\mathbf{II} = -J^t \partial N/\partial u$, respectively. The matrix representing the shape operator on the tangent space is $S = \mathbf{I}^{-1}\mathbf{II}$. Consider the eigenvector problem $Sp = \kappa p$. Each eigenvector p is a principal direction. The corresponding eigenvalue κ is a principal curvature. The vector p is an n–vector given in terms of tangent space coordinates, but its representation in $\mathbb{R}^{n+1}$ is $\xi = Jp$.

Construction without Parameterization. Define $W = -\partial N/\partial x$, an $(n+1) \times (n+1)$ matrix. We claim that if $Sp = \kappa p$, then $\xi = Jp$ satisfies $W\xi = \kappa\xi$. Firstly, we have $\mathbf{I} = J^t J$. Secondly, by the chain rule we have $\partial N/\partial u = (\partial N/\partial x)J$, so $\mathbf{II} = J^t W J$. The eigenvector problem $(\mathbf{II} - \kappa\mathbf{I})p = 0$ is therefore transformed to $J^t(W - \kappa E)\xi = 0$, where E is the $(n+1) \times (n+1)$ identity matrix and where $\xi = Jp$.

Since J^t has full rank n, its generalized inverse is given by $(J^t)^+ = J(J^t J)^{-1}$. If p is a principal direction, that is $Sp = \kappa p$, then $WJp = JSp = \kappa JP$, so $\xi = Jp$ is an eigenvector of W with corresponding eigenvalue κ. Conversely, if ξ is an eigenvector of W, that is $W\xi = \kappa\xi$, and ξ is a tangent vector, say $\xi = Jp$, then $J^+\xi = p$, $JJ^+\xi = JP = \xi$, and

$$SJ^+\xi = (J^+WJ)J^+\xi = J^+W\xi = \kappa J^+\xi,$$

so $J^+\xi$ is a principal directions with corresponding principal curvature κ. Additionally, W has an identically zero eigenvalue, but the corresponding eigenvector is not a tangent vector. This follows from the identity $W = (E - NN^t)\,\mathrm{Hess}\,(F)/|\nabla F|$ which can be derived by explicitly computing $\partial N_i/\partial x_j$ for $N = \nabla F/|\nabla F|$. The eigenvector is $\mathrm{adj}(\,\mathrm{Hess}\,(F))\nabla F$ where adj indicates the adjoint of a matrix. A short computation shows that $W\,\mathrm{adj}(\,\mathrm{Hess}\,(f))\nabla F = 0$.

5.1 Creases on Graphs

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^4 function with graph $g : \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $g(x) = (x, f(x))$. The principal curvatures κ_i and directions p_i, $1 \le i \le n$, are determined by $Sp_i = \kappa_i p_i$ where S is the shape operator described earlier. Assume that the curvatures are ordered as $\kappa_i \ge \ldots \ge \kappa_n$. In the height definition, we defined creases as generalized extrema of a single real–valued function. The principal direction definition is different in that creases will occur as extreme points of each principal curvature with respect to its principal direction. Moreover, the classification of an extreme point will depend on following the integral curves of the principal direction vector field, so we need to use the more general second directional derivative test. Like the height definition, we will characterize the creases according to the dimension of the manifold we expect when finding roots to equations. We also can refine the definitions to include the concepts of strong and weak creases. In the definition, assume that $1 \le d \le n$.

- The point x is a *ridge point of type $n-d$* if $\kappa_d(x) > 0$, and $D_{p_i}\kappa_i(x) = 0$ and $D_{p_i}D_{p_i}\kappa_i(x) < 0$ for $1 \le i \le d$. Additionally x is a *strong ridge point* if $\kappa_d(x) > |\kappa_n(x)|$; otherwise it is a *weak ridge point.*
- The point x is a *valley point of type $n-d$* if $\kappa_{n-d+1}(x) < 0$, and $D_{p_i}\kappa_i(x) = 0$ and $D_{p_i}D_{p_i}\kappa_i(x) > 0$ for $n-d+1 \le i \le n$. Additionally x is a *strong valley point* if $|\kappa_{n-d+1}(x)| > \kappa_1(x)$; otherwise it is a *weak valley point.*

We briefly contrast the height and principal direction definitions. In the height definition, we searched for the local extrema of a *single* function f whose domain was restricted to a subspace of $\mathbb{R}^n$ (so we searched in *multiple* directions). That is, if V is the $n \times d$ matrix whose columns span the desired subspace, and if s is a $d \times 1$ vector–valued parameter, then we searched for extrema of $\phi(s) = f(x+Vs)$ using the standard definition for extrema. The second derivative test involved determining the definiteness of the second derivative matrix for $\phi(s)$ when $s = 0$. In the principal direction definition, we are searching for local extrema of *multiple* functions κ_i. Each such function has a *single* direction p_i associated with it, so the construction of extrema is the usual one for functions of a single real variable. That is, if s is a real variable, then for each i we search for extrema along a path $\xi(s)$ of a function $\phi_i(s) = \kappa_i(\xi(s))$, where the path is determined by $\xi'(s) = p_i(\xi(s))$, $\xi(0) = x$. The second derivative test involves testing the sign of $\phi_i''(0)$.

5.2 Creases on Level Surfaces

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^4 function, and consider the hypersurface defined implicitly by $F(x) = 0$. As shown before, we do not need to find a parameterization of the hypersurface to construct its principal curvatures

$\kappa_i(x)$ and principal directions $\xi_i(x) \in \mathbb{R}^{n+1}$. They are the eigenvalues and non–tangential eigenvectors of the matrix $W = -\partial N/\partial x$, where $N = \nabla F/|\nabla F|$ and $\xi(x)^t N(x) = 0$. In the definition, assume that $1 \leq d \leq n$. Also, the points $x \in \mathbb{R}^n$ of interest must be solutions to $F(x) = 0$.

- The point x is a *ridge point of type* $n - d$ if $\kappa_d(x) > 0$, and $D_{\xi_i}\kappa_i(x) = 0$ and $D_{\xi_i}D_{\xi_i}\kappa_i(x) < 0$ for $1 \leq i \leq d$. Additionally x is a *strong ridge point* if $\kappa_d(x) > |\kappa_n(x)|$; otherwise it is a *weak ridge point.*
- The point x is a *valley point of type* $n - d$ if $\kappa_{n-d+1}(x) < 0$, and $D_{\xi_i}\kappa_i(x) = 0$ and $D_{\xi_i}D_{\xi_i}\kappa_i(x) > 0$ for $n - d + 1 \leq i \leq n$. Additionally x is a *strong valley point* if $|\kappa_{n-d+1}(x)| > \kappa_1(x)$; otherwise it is a *weak valley point.*

5.3 Graph Examples

EXAMPLE 5.1: In dimension $n = 1$, the matrix S is 1×1 and its single entry is $\kappa = -f_{xx}/(1 + f_x^2)^{3/2}$. The graph of $f(x)$ is a planar curve whose curvature at $(x, f(x))$ is $\kappa(x)$. Ridges (valleys) are local maxima (minima) of $\kappa(x)$. For example, let $f(x) = x^p$ where p is a positive even integer. The curvature is $\kappa = -p(p-1)x^{p-2}/(1 + p^2x^{2p-2})^{3/2}$. The solutions to $\kappa_x = 0$ are

$$x = 0,\ \pm\left(\frac{p-2}{p^2(2p-1)}\right)^{1/(2p-2)}.$$

For $p = 2$ the only solution is $x = 0$. The curvature has a negative local minimum of -2, so $x = 0$ is a valley point. For $p > 2$, κ has a local maximum of 0 at $x = 0$, so the graph of f has a flat spot which is neither a ridge nor a valley. At the other two critical points, κ has negative local minima, so the points are valley points. Note that as $p \to \infty$, the graph of x^p approaches 0 pointwise on $(-1, 1)$ and the valley points approach ± 1. Figure 5.1 shows the graphs and valley points for two different values of p.

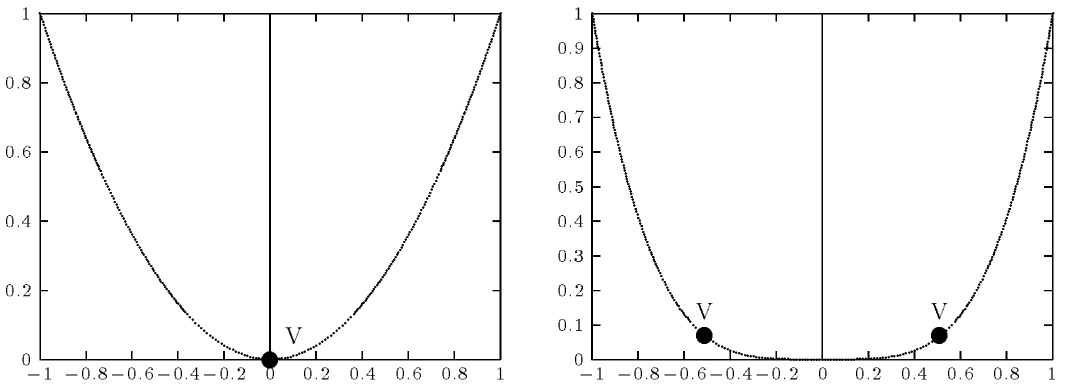


Fig. 5.1: graph of $f(x) = x^2$, graph of $f(x) = x^4$

The valley points are labeled on the graphs as V. When $p = 4$, the valleys are $\pm(1/56)^{1/6} \doteq 0.51$.

The example $f(x) = x^4$ shows that creases according to the principal direction definition are not necessarily local extrema in the function. However, the creases obtained may be better suited for functions which correspond to measurements other than intensity or for which the independent variable is not a spatial one. For example, f might be a function of time for which we are interested in knowing a first time when f has a transition between slowly decreasing and greatly decreasing. The crease points can be viewed as such transitions.

EXAMPLE 5.2: Let $f(x, y) = x^2 y$. The matrix for the shape operator is

$$S = \frac{1}{L^3} \left[\begin{array}{cc} -2y(x^4 - 1) & 2x(1 + x^4) \\ 2x(1 + 2x^2 y^2) & -4x^4 y \end{array} \right]$$

where $L = \sqrt{1 + |\nabla f|^2} = \sqrt{1 + 4x^2 y^2 + x^4}$. The principal curvatures are $\kappa = (y(3x^4 - 1) \pm R)/L^3$ where $R = \sqrt{(1 + x^4)[y^2(9x^4 + 1) + 4x^2]}$, and corresponding principal directions are

$$p_1 = \left\{ \begin{array}{ll} (-(1 + x^4)y + R, -2x(1 + 2x^2 y^2)), & y < 0 \\ (2x(1 + x^4), -(1 + x^4)y - R), & y > 0 \end{array} \right\} \text{ and } p_2 = \left\{ \begin{array}{ll} (2x(1 + x^4), -(1 + x^4)y + R), & y < 0 \\ (-(1 + x^4)y - R, -2x(1 + 2x^2 y^2)), & y > 0 \end{array} \right\}.$$

A closed form solution for the ridges is not tractable. Figure 5.2 shows numerical results for computing the ridges. The figure contains a contour plot with the ridges drawn as thick lines. The origin is at the center of the picture and the coordinates are right–handed. (Compare with the ridges in Figure 4.1).

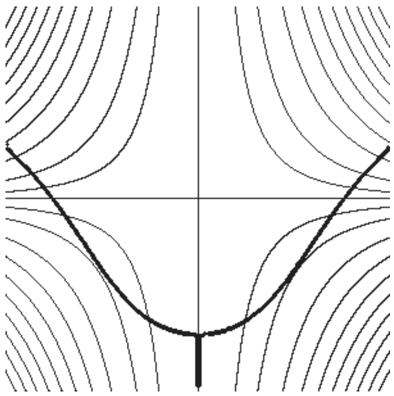


Fig. 5.2: principal direction ridges of $x^2 y$

5.4 Level Surface Example

EXAMPLE 5.3: Consider an ellipsoid defined as a level surface of the function $F(x, y, z) = (ax^2 + by^2 + cz^2)/2$, say $F(x, y, z) = p > 0$, where $0 < a < b < c$. The unit normal vectors are $N = (ax, by, cz)/L$ where $L = \sqrt{a^2 x^2 + b^2 y^2 + c^2 z^2}$. The matrix W is

$$W = \frac{1}{L^3} \left[\begin{array}{ccc} a(b^2 y^2 + c^2 z^2) & -ab^2 xy & -ac^2 xz \\ -ba^2 xy & b(a^2 x^2 + c^2 z^2) & -bc^2 yz \\ -ca^2 xz & -cb^2 yz & c(a^2 x^2 + b^2 y^2) \end{array} \right].$$

The principal curvatures of the surface are $\kappa_1 = (\alpha + \sqrt{\beta})/L^3$ and $\kappa_2 = (\alpha - \sqrt{\beta})/L^3$ where $\alpha = a^2(b + c)x^2 + b^2(a + c)y^2 + c^2(a + b)z^2$ and $\beta = a^4(b - c)^2 x^4 + b^4(a - c)^2 y^4 + c^4(a - b)^2 z^4 + 2(a - c)(b - c)a^2 b^2 x^2 y^2 +$

$2(a-b)(c-b)a^2c^2x^2z^2 + 2(b-a)(c-a)b^2c^2y^2z^2$. Corresponding principal directions are

$$\begin{aligned} p_1 &= ab(cxz, cyz, -ax^2 - by^2) + L\kappa_1(acxz, bcyz, -a^2x^2 - b^2y^2) \text{ and} \\ p_2 &= ab(-y(abx^2 + b^2y^2 + c^2z^2), x(a^2x^2 + aby^2 + c^2z^2), c(a-b)xyz) + L^3\kappa_1(-by, ax, 0). \end{aligned}$$

Clearly $\kappa_1 > 0$ for all (x, y, z), so let us attempt to locate ridges of type 1. Taking derivatives, we obtain the formula

$$L^3\nabla\kappa_1 + 3L^2\kappa\nabla L = \nabla\alpha + \frac{1}{2\sqrt{\beta}}\nabla\beta.$$

At $z = 0$, some calculations will show that $p_1 = -\alpha(0, 0, 1)$. It is easily shown that $p_1 \cdot \nabla\alpha = p_1 \cdot \nabla\beta = p_1 \cdot \nabla L = 0$ when $z = 0$. Thus, $D_{p_1}\kappa_1(x, y, 0) = 0$ for all x and y (which lie on the curve $ax^2 + by^2 = 2p$).

The second directional derivative when $z = 0$ can be shown to be $D_{p_1p_1}\kappa_1 = \alpha[\kappa_{zz} - \alpha\nabla\kappa^t(\partial p_1/\partial z)]$, where all quantities involved are evaluated at $z = 0$. Some tedious algebraic calculations lead to

$$D_{p_1p_1}\kappa_1(x, y, 0) = \frac{2c^2}{L^3\sqrt{\beta}}\left(\omega_1 a^4x^4 + \omega_2 a^2b^2x^2y^2 + \omega_3 b^4y^4\right)$$

where $\omega_1 = (c-b)[a(4b+5c) - (b+c)(b+6c)]$, $\omega_2 = \{(c-a)[ac+3(b+c)(b-6c)] + (c-b)[bc+3(a+c)(a-6c)]\}$, and $\omega_3 = (c-a)[b(4a+5c) - (a+c)(a+6c)]$. Using $0 < a < b < c$, it can be shown that all $\omega_i < 0$, so $D_{p_1p_1}\kappa_1(x, y, 0) < 0$. Therefore the points on the ellipsoid for which $z = 0$ are ridges of type 1.

Additional calculations will show that when $y = 0$ and $z = 0$, $D_{p_2}\kappa_2 = 0$ and $D_{p_2p_2}\kappa_2 < 0$. The vertices $(\pm\sqrt{2p/a}, 0, 0)$ are therefore ridges of type 0.

5.5 Invariance Properties

For parameterized hypersurfaces, the ridge construction is invariant under diffeomorphisms on the parameter space. This result follows from standard differential geometry where the principal curvatures and principal directions do not change under these transformations. With regard to transformations applied to the entire space $\mathbb{R}^{n+1}$ in which the level surface $F(x) = 0$ lives, the ridges constructed are invariant under spatial translations and spatial rotations (Euclidean motions), and under uniform spatial magnifications. The proofs are nearly identical to those in section 4.5.

In the special case of a graph defined by $F(x, z) = z - f(x) = 0$, where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, the ridge construction is not invariant to uniform magnification in x. Note that the magnification is not a reparameterization of the original surface; the transformation does change the surface. For example, if $n = 1$ and $f(x) = 1 - x^4$ for $x > 0$, a ridge is $x_0 = 56^{-1/6}$. Let $x = c\bar{x}$ for some $c > 0$ and define $\bar{f}(\bar{x}) = f(x) = 1 - c^4\bar{x}^4$. The ridge for this new function is $\bar{x_0} = (56c^8)^{-1/6} \neq x_0/c$, so the ridge is not invariant.

6 Level Definition for Creases

6.1 Creases on Level Surfaces

Let $f : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a C^4 function such that $\nabla f \neq 0$ (except at isolated points). The domain of f can be partitioned into its level sets defined by $f(x) = c$ for constants c. Note that a single level set can be viewed as a hypersurface in $\mathbb{R}^n$ implicitly defined by $F(x) = f(x) - c = 0$. Therefore, the principal direction definition may be applied to find creases on the hypersurface for each c in the range of f. The normals for the hypersurface are $N = \nabla f/|\nabla f|$ and the eigenvalues and (tangential) eigenvectors of the matrix $W = -\partial N/\partial x$ are the principal curvatures and principal directions. We will construct creases on the graph of f by applying the principal direction definition to each of its level surfaces. The set of all creases of all the level surfaces make up the creases of the graph.

The eigenvalues of W are $\kappa_1 \geq \ldots \geq \kappa_{n-1}$ and 0, with corresponding eigenvectors $\xi_1, \ldots, \xi_{n-1}$ and $\mathrm{adj}(\,\mathrm{Hess}\,(f))\nabla f$. The $\kappa_i(a)$ and $\xi_i(a)$ are the principal curvatures and principal directions for the level surface $f(x) = f(a)$. We can attempt to construct crease sets of dimension d where $1 \leq d \leq n-1$. The definition is similar to the principal direction definition, but with one subtle difference. In the principal direction definition, creases were solutions to equations of the type $D_\xi \kappa_i(x) = 0$ where $F(x) = 0$ for a single function F. In the level definition, creases are solutions to the same equations, but we now have an entire family of functions $F(x; c) = f(x) - c = 0$.

- The point x is a *ridge point of type* $n-1-d$ if $\kappa_d(x) > 0$, and $D_{\xi_i}\kappa_i(x) = 0$ and $D_{\xi_i}D_{\xi_i}\kappa_i(x) < 0$ for $1 \leq i \leq d$. Additionally x is a *strong ridge point* if $\kappa_d(x) > |\kappa_n(x)|$; otherwise it is a *weak ridge point.*
- The point x is a *valley point of type* $n-1-d$ if $\kappa_{n-d}(x) < 0$, and $D_{\xi_i}\kappa_i(x) = 0$ and $D_{\xi_i}D_{\xi_i}\kappa_i(x) > 0$ for $n-d \leq i \leq n-1$. Additionally x is a *strong valley point* if $|\kappa_{n-d}(x)| > \kappa_1(x)$; otherwise it is a *weak valley point.*

EXAMPLE 6.1: Consider the case $n = 2$. Normal and tangent vectors to the level curves are given by $N(x,y) = (f_x, f_y)/(f_x^2 + f_y^2)^{1/2}$ and $T(x,y) = (f_y, -f_x)/(f_x^2 + f_y^2)^{1/2}$, and the curvature of the level curves is $\kappa(x,y) = -(f_x^2 f_{yy} - 2f_x f_y f_{xy} + f_y^2 f_{xx})/(f_x^2 + f_y^2)^{3/2}$.

Consider the function $f(x,y) = x^2 y$ for $x > 0$ and $y > 0$. The tangents to level curves are $T = (x, -2y)/(x^2 + 4y^2)^{1/2}$. The curvature and its derivative in the T direction are $\kappa = 6xy/(x^2+4y^2)^{3/2}$ and $D_T\kappa = -24xy(x^2 - 5y^2)/(x^2+4y^2)^3$. Setting $D_T\kappa = 0$ in the first quadrant yields $x = \sqrt{5}y$, $y > 0$. Some short calculations will show that $D_{TT}\kappa < 0$, so the points are ridge points.

Figure 6.1 shows a contour plot with the ridges drawn as thick lines. The origin is at the center of the picture and the coordinates are right–handed. (Compare with the ridges in Figures 4.1 and 5.2).

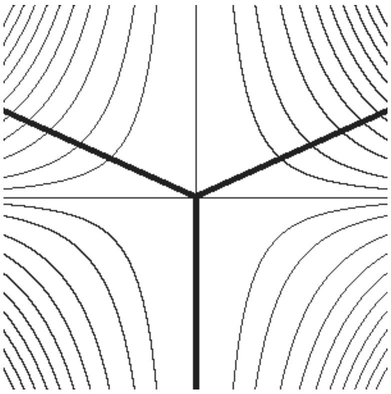


Fig. 6.1: level ridges of $x^2 y$

6.2 1–Dimensional Creases from Mean Curvature

In the numerical implementation of the level definition, one must compute the eigenvalues and eigenvectors for the matrix W at each point in an image. This process is typically time–consuming. A variation on the level definition for constructing 1–dimensional creases computes the local extrema of the *mean curvature* $\mu = \text{trace}(W)/(n-1)$ rather than computing local extrema of principal curvatures. The trace of W is more easily computed than its eigenvalues.

Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $x : \mathbb{R}^{n-1} \to \mathbb{R}^n$ denote a level surface; thus, $f(x(s)) \equiv c$ for some constant c and for all s. Let $\phi(s) = \mu(x(s))$ be the mean curvature of the level surface at position $x(s)$. In the construction we use the following abbreviations for the tensor quantities for the gradients and Hessians of the functions of interest.

$$\nabla\phi = \frac{\partial\phi}{\partial s_i}, \nabla\mu = \frac{\partial\mu}{\partial x_k}, \nabla f = \frac{\partial f}{\partial x_k}$$

and

$$\text{Hess } \phi = \frac{\partial^2\phi}{\partial s_i \partial s_j}, \text{ Hess } \mu = \frac{\partial^2\mu}{\partial x_k \partial x_m}, \text{ Hess } f = \frac{\partial^2 f}{\partial x_k \partial x_m}.$$

We also use the following abbreviations for the tensor quantitites for the first and second derivatives of position $x(s)$:

$$x'(s) = \frac{\partial x_k}{\partial s_i} \text{ and } x''(s) = \frac{\partial^2 x_k}{\partial s_i \partial s_j}.$$

The local extrema of ϕ occur when $\nabla\phi = 0$ and Hess ϕ is positive definite (local maximum) or negative definite (local minimum). We would like to determine the local extrema without having to choose a particular parameterization $x(s)$. Thus we need to select a smoothly varying basis of tangent vectors $v_i(x)$, $1 \le i \le n-1$, which can be used in the derivative tests instead of the tangent vectors $\partial x/\partial s_i$. For now let us assume that we have such a basis. Let V be the $n \times (n-1)$ matrix whose columns are $v_i(x)$. The two matrices V and $x'(s)$ are related by an invertible $(n-1) \times (n-1)$ matrix C (a change of basis), $x'(s) = VC$.

Using the chain rule, the derivatives of ϕ are

$$\nabla\phi = x'(s)^t \nabla\mu \text{ and } \text{Hess } \phi = x'(s)^t \text{ Hess } \mu x'(s) + x''(s)\nabla\mu.$$

The first derivative test is $0 = \nabla\phi = C^t V^t \nabla\mu$. Since C is invertible, the critical points are solutions to $V^t\nabla\mu = 0$. The second derivative test involves second derivatives of position, which we would like to avoid computing. Note that $V^t\nabla\mu = 0$ implies that $\nabla\mu$ is orthogonal to the tangent space (at a critical point); that is, $\nabla\mu = \rho\nabla f$ where $\rho = (\nabla\mu \cdot \nabla f)/(\nabla f \cdot \nabla f)$. Moreover, since $f(x(s)) \equiv c$, taking derivatives yields

$$0 = x'(s)^t \nabla f \text{ and } 0 = x'(s)^t \text{ Hess } f x'(s) + x''(s)\nabla f.$$

At a critical point we consequently have

$$x''(s)\nabla\mu = \rho x''(s)\nabla f = -\rho x'(s)^t \text{ Hess } f x'(s),$$

so the second derivative of ϕ at such points is

$$\text{Hess } \phi = x'(s)^t \left(\text{Hess } \mu - \rho \text{ Hess } f \right) x'(s).$$

In terms of the matrix V we have

$$C^{-t} \text{ Hess } \phi C^{-1} = V^t \left(\text{Hess } \mu - \rho \text{ Hess } f \right) V.$$

By Sylvester's Theorem [13], C^{-t} Hess ϕC^{-1} and Hess ϕ have the same definiteness, so we need only check the definiteness of $V^t(\text{ Hess}\,\mu - \rho\,\text{Hess}\,f)V$ for the second derivative test.

The remaining problem is to find a smoothly varying basis v_i for the tangent space to level surfaces which is easier to compute than the principal directions. Such a basis is given by the columns of a rotation matrix which maps the vector $e_n = (0,\ldots,0,1)$ to the normal $N = \nabla f/|\nabla f|$ of the surface. A rotation matrix is given in block form by

$$R = \left[\begin{array}{c|c} E + (N_n - 1)PP^t & Q \\ \hline -Q^t & N_n \end{array}\right]$$

where E is the $(n-1)\times(n-1)$ identity matrix, $Q = (N_1,\ldots,N_{n-1})^t$ are the first $n-1$ components of the normal vector, N_n is the last component of N, and $P = Q/|Q|$ when $Q \neq 0$. If $Q = 0$, the rotation is just the identity matrix.

The crease definitions for this variation are given below. Let V be the matrix whose columns are the v_i vectors. A point $x \in \mathbb{R}^n$ is

- a ridge point if $\mu(x) > 0$, $V^t\nabla\mu(x) = 0$, and $V^t(\text{ Hess}\,\mu - \rho\,\text{Hess}\,f)V$ is negative definite;
- a valley point if $\mu(x) < 0$, $V^t\nabla\mu(x) = 0$, and $V^t(\text{ Hess}\,\mu - \rho\,\text{Hess}\,f)V$ is positive definite,

where $\rho(x) = (\nabla\mu \cdot \nabla f)/(\nabla f \cdot \nabla f)$. The directional derivatives and eigenvalues are all evaluated at the point in question. Note that this definition is identical to the original one when $n = 2$. The qualitative differences between ridges obtained by the level definition and those obtained by the variation involving mean curvature should be minimal in convex regions (all $\kappa_i > 0$). Some noticeable differences may occur in hyperbolic regions.

6.3 Invariance Properties

The ridges constructed by the level definition are invariant with respect to spatial translations, spatial rotations, and uniform spatial magnifications, just as in the principal direction definition since the ridges are located on level surfaces using the principal definition.

The ridges are also invariant under monotonic transformations of the function $f : \mathbb{R}^n \to \mathbb{R}$. Intuitively, you can think of $\mathbb{R}^n$ as a 1–parameter family of level sets of f, each having its function value as an "attribute". Monotonic transformations on f will not change the geometric structure of the level sets; rather it will only change the attributes of the level sets. Since the level sets have the same structure, the ridges will not change.

7 Nonmetric Definition for Creases

The principal direction definition applied to graphs of functions involved finding those values κ and vectors $p \neq 0$ which solve $\mathbf{II}p = \kappa\mathbf{I}p$ where $\mathbf{I}$ is the first fundamental form and $\mathbf{II}$ is the second fundamental form. The presence of $\mathbf{I}$ means that we are making measurements in the tangent spaces to the graph.

The nonmetric definition is a variation on the principal direction definition. We can consider instead solving $\mathbf{II}v = \kappa v$ and ignore the metric $\mathbf{I}$ of the graph. This change takes us out of the realm of differential geometry. We only consider graphs of functions with this definition.

7.1 General Dimensions

The crease definitions are essentially those for principal directions. Let $f : \mathbb{R}^n \to \mathbb{R}$. Let κ_i and v_i be the eigenvalues and eigenvectors of $\mathbf{II}$ for $1 \leq i \leq n$. We assume that the eigenvalues are ordered as $\kappa_1 \geq \cdots \geq \kappa_n$. In the definition, assume that $1 \leq d \leq n$.

- The point x is a *ridge point of type* $n - d$ if $\kappa_d(x) > 0$, and $D_{v_i}\kappa_i(x) = 0$ and $D_{v_i}D_{v_i}\kappa_i(x) < 0$ for $1 \leq i \leq d$. Additionally x is a *strong ridge point* if $\kappa_d(x) > |\kappa_n(x)|$; otherwise it is a *weak ridge point.*
- The point x is a *valley point of type* $n - d$ if $\kappa_{n-d+1}(x) < 0$, and $D_{v_i}\kappa_i(x) = 0$ and $D_{v_i}D_{v_i}\kappa_i(x) > 0$ for $n - d + 1 \leq i \leq n$. Additionally x is a *strong valley point* if $|\kappa_{n-d+1}(x)| > \kappa_1(x)$; otherwise it is a *weak valley point.*

EXAMPLE 7.1: In dimension $n = 1$, the curvature formula is $\kappa = -f_{xx}/(1 + f_x^2)^{1/2}$. Consider $f(x) = \exp(-x^2/2)/\sqrt{2\pi}$, a Gaussian distribution with standard deviation 1. The function has a relatively slow decrease for small x. For larger x the graph drops off sharply and then remains close to 0 as $x \to \infty$. We will use the creases obtained from the nonmetric definition as markers of where the rate of decrease of f has transitions from slow to fast.

For this example, $\kappa_x = 0$ when $x_0 = 0$ or when x is a solution to $2\pi(x^2 - 3)\exp(x^2) = x^2 + 1$. Using Newton's method to find an approximate solution yields $x_1 \doteq 1.741$. Also, $\kappa(x_0) = 1$, $\kappa_{xx}(x_0) = -4$, $\kappa(x_1) \doteq -0.176$, and $\kappa_{xx}(x_1) \doteq 0.557$, so x_0 is a ridge point and x_1 is a valley point.

EXAMPLE 7.2: In dimension $n = 2$, we consider the same function $f(x, y) = x^2 y$ as in Example 5.2, so we can compare the two definitions. Define $L = \sqrt{1 + |\nabla f|^2} = \sqrt{1 + 4x^2y^2 + x^4}$ and $R = \sqrt{4x^2 + y^2}$. The eigenvalues of W are $\kappa_1 = (-y + R)/L$ and $\kappa_2 = (-y - R)/L$ with corresponding eigenvectors

$$v_1 = \left\{ \begin{array}{ll} (-y + R, -2x), & y \leq 0 \\ (2x, -y - R), & y \geq 0 \end{array} \right\} \text{ and } v_2 = \left\{ \begin{array}{ll} (2x, -y + R), & y \leq 0 \\ (-y - R, -2x), & y \geq 0 \end{array} \right\}.$$

The eigenvalues satisfy $\kappa_1(x, y) \geq 0 \geq \kappa_2(x, y)$ for all (x, y). The first directional derivative of κ_1 is

$$D_{v_1}\kappa_1 = \left\{ \begin{array}{ll} 2x(-y + R)g(x, y)/(RL^3), & y \leq 0 \\ 4x^2 g(x, y)/(RL^3), & y \geq 0 \end{array} \right\}$$

where $g(x, y) = 3 - (x^2 - y^2)(x^2 - 2y^2) + Ry(5x^2 + 2y^2)$.

A first set of solutions to $D_{v_1}\kappa_1 = 0$ is given by the line $x = 0$. A second set of solutions can be constructed by setting $y = mx$ in the equation $g(x, y) = 0$. The curve of solutions for $x > 0$ is given parametrically by

$$x(m) = \left[\frac{3}{(1 - m^2)(1 - 2m^2) - m(5 + 2m^2)\sqrt{4 + m^2}} \right]^{1/4}, \quad y = mx(m),$$

where $m \leq m_0$ and m_0 is the positive solution to $52m^6 + 92m^4 + 106m^2 - 1 = 0$. The solutions for $x < 0$ are obtained by reflection through the y–axis. Note that $\lim_{m \to -\infty} x(m) = 0$ and $\lim_{m \to -\infty} y(m) = -(3/4)^{1/4}$.

It can be shown by the second derivative test that the points (x, y) for which $x = 0$ and $y < -(3/4)^{1/4}$ are ridge points, and all points $(x(m), y(m))$ given above are ridge points.

Figure 7.1 shows a contour plot with the ridges drawn as thick lines. The origin is at the center of the picture and the coordinates are right–handed. (Compare with the ridges in Figures 4.1, 5.2, and 6.1).

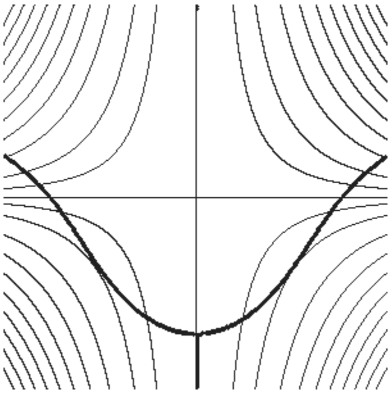


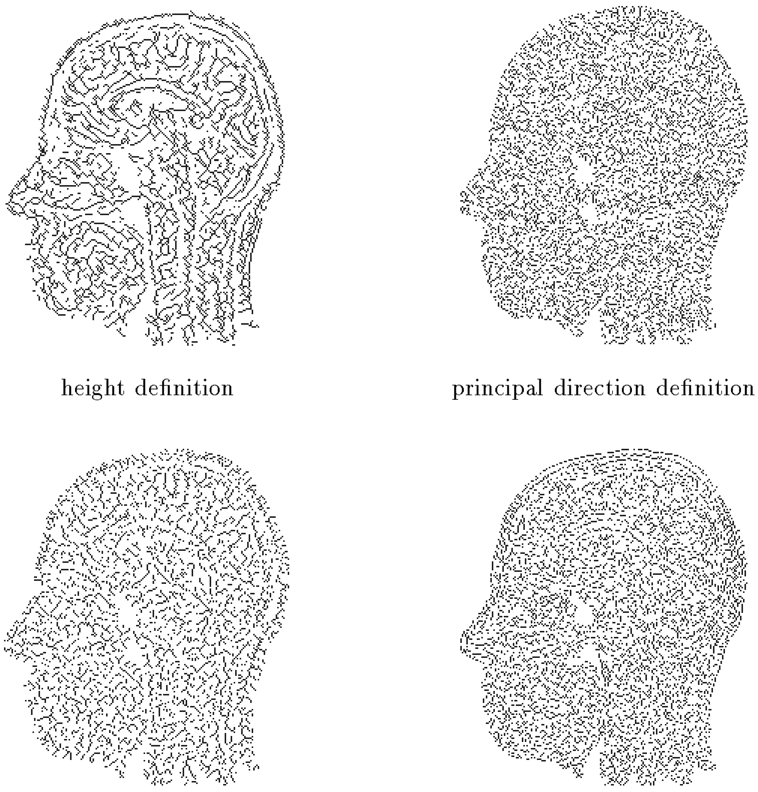
Fig. 7.1: nonmetric ridges of x^2y

7.2 Invariance Properties

The ridges constructed by the nonmetric definition have the same invariance as those of the principal direction definition; the proofs are similar. The ridges are invariant under spatial translations and spatial rotations (in $\mathbb{R}^n$). The ridges are not invariant under uniform spatial magnifications and monotonic transformations of f.

8 Experiments on Images

We tested the four different ridge definitions on a slightly blurred version of the MR image shown in the introduction. The original image was assumed to have a (inner) scale value $\sigma = 1$. The image was Gaussian blurred to scale $\sigma = 2$ to help remove small scale noise. We have ignored the valleys for clarity in the resulting images in Figure 8.1.



height definition　principal direction definition

level definition　nonmetric definition

Fig. 8.1: Ridges of MR head image

The ridges were constructed by finding zero–crossings of the appropriate directional derivatives. Both pixels involved in a zero–crossing were marked as ridges, then the resulting binary set was thinned using morphological operations. The ridges from the height definition appear to give the best qualitative information about structure of the image. The ridges from both the principal direction and nonmetric definitions seem to be overly abundant. The level definition appears to produce more dendritic–like ridge structures, but the ridges are fragmented. The reduction in detail through increasing scale seems to happen sooner for the height definition than for the other definitions. In the principal direction case, no object structures are apparent at the selected scale. In the nonmetric case you can see the scalp, corpus colosum, and brain stem.

9 Discussion

The four crease definitions: height, principal direction, level, and nonmetric, appear to produce qualitatively different structures. When applied to graphs of functions all four provide invariance under rotations and translations in the spatial variables. The level definition is additionally invariant under monotonic transformations of the function values. Based on the images of Figure 8.1, the height definition seems to capture the most large scale shape information about the image. The dendritic structures occurring from the level definition seem to be suitable for providing small scale descriptions of objects in the image.

The principal direction and nonmetric definitions applied to graphs do not provide invariance under uniform spatial magnifications and monotonic changes in intensity. Moreover, the fact that the ridges tend not to correspond to object centers makes these definitions not suitable for medical image analsis. However the principal direction definition does apply to surfaces whose coordinates all correspond to spatial information, so the definition is suitable for locating ridges and using them for image registration.

The true test of the usefulness of ridges in image analysis lies in multiscale analysis. Ridge structure at a single scale can give information about structures of an appropriate size, but the behavior of ridge structures as the scale parameter is increased allows one to capture more global object information. An application of ridges to image segmentation using a multiscale approach is found in [7].

ACKNOWLEDGMENTS: I wish to thank the first referee for pointing out references [9] and [10]. Thanks also to the second referee for providing the example which shows the height definition is not invariant to monotonic transformations of intensity, and for the help in correcting the crease definition in the context of mean curvature.

References

[1] H. Blum, *"Biological shape and visual science (Part I)"*, J. Theoret. Biol. vol. 38, pp. 205–287, 1973.

[2] H. Blum, R.N. Nagel, *"Shape description using weighted symmetric axis features"*, Pattern Recognition, vol. 10, pp. 167–180, 1978.

[3] L.A. Bucchi, *"Ridge detection on surfaces"*, M.S. dissertation, Dept. of Math., Univ. of North Carolina at Chapel Hill, 1991.

[4] J.W. Bruce, P.J. Giblin, C.G. Gibson, *"Symmetry sets"*, Proc. Roy. Soc. Edinburgh, 101A, pp. 163–186, 1985.

[5] A.C.F. Colchester, *"Network representation of 2D and 3D images"*, in 3D Imaging in Medicine, K.H. Hoehne, H. Fuchs and S. Pizer (eds.), Springer–Verlag, Berlin, pp. 45–62, 1990.

[6] J.L. Crowley, A.C. Parker, *"A representation for shape based on peaks and ridges in the difference of low–pass transform"*, IEEE Trans. PAMI, vol. PAMI–6, no. 2, pp. 156–170, 1984.

[7] D. Eberly, S.M. Pizer, *"Ridge flow models for image segmentation"*, Dept. of Comp. Sci., Univ. of North Carolina, technical report TR93–056, 1993. (Also, to appear in SPIE Proc. Med. Imag. 1994)

[8] B.M. ter Haar Romeny, L.M.J. Florak, J.J. Koenderink, M.A. Viergever, *"Scale–space: Its natural operators and differential invariants"*, in Information Processing in Medical Imaging (IPMI), Proc. 12$^{\text{th}}$ Int. Conf., Colchester and Hawkes (eds.), Springer–Verlag, Berlin, 1991.

[9] L.M.J. Florack, B.M. ter Haar Romeny, J.J. Koenderink, M.A. Viergever, *General intensity transformations and second order invariants*, in Theory & Applications of Image Analysis, World Scientific, Series in Machine Perception and Artificial Intelligence, vol. 2, pp. 22–29, P. Johansen and S. Olsen (eds.), Singapore.

[10] L.M.J. Florack, B.M. ter Haar Romeny, J.J. Koenderink, M.A. Viergever, *Images: Regular tempered distributions*, in Shape in Picture, NATO ASI Series F, A. Toet and O. Ying–Lie (eds.), Springer–Verlag, Berline, 1993.

[11] W. Fulks, *Advanced Calculus*, 3rd edition, John Wiley and Sons: New York, 1978.

[12] R. Haralick, *"Ridges and valleys in digital images"*, CVGIP, vol. 22, pp. 28–38, 1983.

[13] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press: Cambridge, England, 1991.

[14] J.M. Gauch, *"The multiresolution intensity axis of symmetry and its application to image segmentation"*, Ph.D. dissertation, Dept. of Comp. Sci., Univ. of North Carolina, technical report TR89–047, 1989.

[15] L.D. Griffin, A.C.F. Colchester, G.P. Robinson, *"Scale and segmentation of gray-level images using maximum gradient paths"*, in Lecture Notes in Computer Science 511: Information Processing in Medical Imaging, 12^{th} International Conference, Wye, UK, pp. 256–272, 1991.

[16] J. Koenderink, *Solid Shape*, MIT Press: Cambridge, MA, 1991.

[17] J. Koenderink, A.J. van Doorn, *"Local features of smooth shapes: Ridges and Courses"*, SPIE Proc. 2031: Geometric Methods in Computer Vision II, San Diego, pp. 2–13, 1993.

[18] B.S. Morse, *"Ridges of height functions in scale space"*, preprint, 1993.

[19] L.R. Nackman, *"Three-dimensional shape description using the symmetric axis transform"*, Ph.D. dissertation, Dept. of Comp. Sci., Univ. North Carolina, Chapel Hill, 1982.

[20] L.R. Nackman, *"Two-dimensional critical point configuration graphs"*, IEEE Trans. PAMI, vol. PAMI-6, no. 4, pp. 442–450, 1984.

[21] C.E. Pearson (ed.), *Handbook of Applied Mathematics*, Van Nostrand Reinhold: New York, 1983.

[22] S.M. Pizer, C.A. Burbeck, J.M. Coggins, D.S. Fritsch, B.S. Morse, *"Object shape before boundary shape: scale-space medial axis"*, Dept. of Comp. Sci., Univ. of North Carolina, technical report TR92–023, 1992.

[23] C. Scharlach, *"Ridges and the description of image structure"*, in Geometry and Topology of Submanifolds V, Proc. Conf. on Diff. Geom. and Vision, F. Dillen and L. Verstraelen (eds.), World Scientific, Singapore (to appear), 1992.

[24] J–P. Thirion, A. Gourdon, *"The 3D marching lines algorithm and its application to crest line extraction"*, INRIA Research Report, Robotique, Image et Vision, Programme 4, No. 1672, pp. 1–28, 1992.