Noise Model and Performance Analysis of Outward-looking Optical Trackers Using Lateral Effect Photo Diodes

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ABSTRACT

Electrical noise introduces uncertainty in the estimated photo-coordinates of optical flux centroids on lateral effect photo-diodes used in outward looking optical trackers. The position and orientation accuracy of such trackers is thereby limited. This report describes a noise model based on standard techniques and develops formulas for calculating the RMS uncertainty in the photo-coordinate positions. The results of these calculations can be further processed, using system sensitivity matrix methods, to estimate overall tracking system accuracy. Expressions for each of the contributing noise sources, referred to the output of the first preamplifier stages are also given, as an aid to designing the tracker electronics. An example set of calculations are included which closely represent the actual design of a tracker currently under development at the date of this report. Development of an expression for the noise properties of the signal processing used in estimating the centroid position, and a MACSYMA code listing are included as appendices.

1 INTRODUCTION

Outward-looking optical trackers use some angle-sensitive optical detection scheme to measure the relative angular disposition of fixed points in the environment. In our case, we use infra-red (IR) light emitting diodes (LEDs) as beacons, while detection is performed by a simple lens which collects IR flux from the beacons and concentrates the flux on a lateral effect photo-diode (LEPD)[Wang90]. A simple lens will map a bundle of rays incident at some angle to a corresponding position on the LEPD. While focus is not maintained for varying beacon distances, the positional centroid of the flux on the LEPD as a function of beacon direction is preserved. LEPDs have the property that the ratio of their various output currents is strictly a function of the positional centroid of the incident optical flux.

The accuracy of this kind of detector is fundamentally limited by the presence of noise on its output signals, and by any additional noise due to subsequent processing of these signals. Detector accuracy is further reflected in overall system accuracy, as described and quantitatively developed in [Gott94]. The present report develops a noise model for the anode side of a duo-lateral LEPD along with a pair of transimpedance amplifiers implemented using op amps. Noise sources are identified and means for evaluating them given. The effects of each noise source as referred to the amplifier outputs is given as an aid to design optimization. Subsequent signal manipulations modeling the digitizing, signal processing, and normalizing operations are considered in the development of an expression for an estimate of the X position of the optical flux centroid on the LEPD.

§2 develops a noise circuit model which is used to calculate the relative contributions of the signal and various noise sources to the amplifier output signals. §3 and §4 provide means for quantitatively determining the signal and noise contributions, respectively. §5 completes the noise model by accounting for signal processing in arriving at an expression for the mean and RMS uncertainty of the estimated X. Use of the MACSYMA code developed for this analysis is discussed and illustrated by examples in §6. The mathematics necessary to §5 is developed in Appendix A, and Appendix B lists MACSYMA code used to perform the calculations discussed in this report.

2 THE NOISE MODEL

The circuit illustrated in Figure 1 models noise sources arising from thermal (Johnson) noise in resistors, shot noise associated with signal and dark currents in the LEPD, and input noise of the preamps, according to the methods described in [Leac94]. Johnson noise contributed by the reverse-biased LEPD junction resistance is negligible and is not included in this model. This model represents just the LEPD anode signal paths; the cathode model is identical except for the direction of signal current i_s.

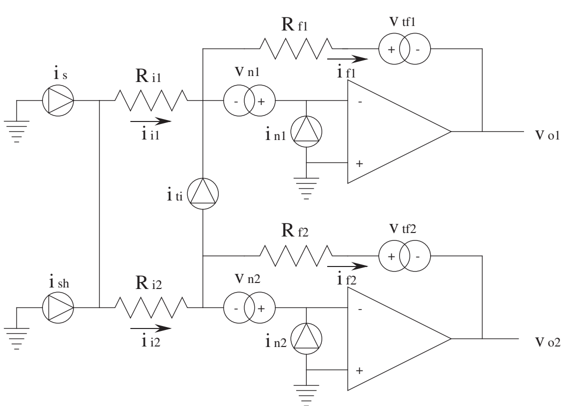

Figure 1 – noise model equivalent circuit

Under the usual assumption that the inverting input of a heavily fed-back op-amp is equipotential with the non-inverting input, we can treat the inverting inputs here as being at ground potential. Then, using Kirchoff's current and voltage laws, we write the following six equations:

$$i_{sh} - i_{i2} - i_{i1} + i_s = 0, \tag{1}$$
$$i_{ti} + i_{n1} + i_{i1} - i_{f1} = 0, \tag{2}$$
$$-i_{ti} + i_{n2} + i_{i2} - i_{f2} = 0, \tag{3}$$
$$-v_{n2} + v_{n1} + i_{i2}\, R_{i2} - i_{i1}\, R_{i1} = 0, \tag{4}$$
$$v_{tf1} + v_{o1} + v_{n1} + i_{f1}\, R_{f1} = 0, \tag{5}$$
$$v_{tf2} + v_{o2} + v_{n2} + i_{f2}\, R_{f2} = 0, \tag{6}$$

where

$$\begin{aligned}
i_s &= \text{signal current (including dark current)},\\
i_{sh} &= \text{shot noise associated with the signal current},\\
i_{ti} &= \text{Johnson noise of LEPD inter-electrode resistance } R_i = R_{i1} + R_{i2},\\
i_{n1}, i_{n2} &= \text{input current noise of amplifiers 1 and 2, respectively},\\
v_{n1}, v_{n2} &= \text{input voltage noise of amplifiers 1 and 2, respectively},\\
v_{tf1}, v_{tf2} &= \text{Johnson noise of feedback resistors } R_{f1} \text{ and } R_{f2}, \text{ respectively.}
\end{aligned}$$

Let $R_{i1} = xR_i$ and $R_{i2} = (1-x)R_i$ where $0 \le x \le 1$ represents the normalized position of the optical flux centroid on the LEPD junction. The instantaneous amplifier output signals v_{o1} and v_{o2}, comprising the desired signals along with contributions from the various noise sources, are then determined by solving eqs(1-6):

$$v_{o1} = \frac{\begin{pmatrix} -v_{tf1}\, R_i + v_{n2}\, R_{f1} - v_{n1}\, (R_{f1} + R_i) \\ -i_{ti}\, R_i\, R_{f1} + i_{sh}\, (x-1)\, R_i\, R_{f1} - i_{n1}\, R_i\, R_{f1} + i_s\, (x-1)\, R_i\, R_{f1} \end{pmatrix}}{R_i}, \tag{7}$$

$$v_{o2} = \frac{\begin{pmatrix} -v_{tf2}\, R_i - v_{n2}\, (R_{f2} + R_i) + v_{n1}\, R_{f2} \\ +i_{ti}\, R_i\, R_{f2} - i_{sh}\, x\, R_i\, R_{f2} - i_{n2}\, R_i\, R_{f2} - i_s\, x\, R_i\, R_{f2} \end{pmatrix}}{R_i}. \tag{8}$$

By inspection of eq(7) we can find the contribution of each noise source to the output signal v_{o1} of the first amplifier:

$$\begin{aligned}
v_{o1,s} &= (x-1)\, i_s\, R_{f1}\\
v_{o1,sh} &= (x-1)\, i_{sh}\, R_{f1}\\
v_{o1,in1} &= -i_{n1}\, R_{f1}\\
v_{o1,iti} &= -i_{ti}\, R_{f1}\\
v_{o1,vn1} &= -\frac{(R_{f1} + R_i)\, v_{n1}}{R_i}\\
v_{o1,vtf1} &= -v_{tf1}\\
v_{o1,vn2} &= \frac{R_{f1}\, v_{n2}}{R_i}
\end{aligned} \tag{9}$$

Similarly, by inspection of eq(8) we can find the contribution of each noise source to the output signal v_{o2} of the second amplifier:

$$
\begin{aligned}
v_{o2,s} &= -x\, i_s\, R_{f2} \\
v_{o2,sh} &= -x\, i_{sh}\, R_{f2} \\
v_{o2,in2} &= -i_{n2}\, R_{f2} \\
v_{o2,iti} &= i_{ti}\, R_{f2} \\
v_{o2,vn2} &= -\frac{(R_{f2}+R_i)\; v_{n2}}{R_i} \\
v_{o2,vtf2} &= -v_{tf2} \\
v_{o2,vn1} &= \frac{R_{f2}\, v_{n1}}{R_i}
\end{aligned}
\tag{10}
$$

Equations (9) and (10) are useful for evaluating design trade-offs in the selection of devices and resistance values in the front-end circuitry of a duo-lateral LEPD detector. Specific values for the signal current and the various noise currents and voltages can be determined according to §3 and §4 of this report, respectively.

The noise voltages and currents given in §4 are *spot noise* values, which under the assumptions made in §4 can be compared directly with each other. Comparison of signal to noise, however, is only meaningful for average noise signals calculated over a given spectrum. Assuming a single pole low-pass spectral noise envelope as a function of frequency f, having the form $f_0/(f_0 - jf)$, these average noises can be calculated by multiplying their spot noise equivalents by $\sqrt{\pi f_0}$, where f_0 is the "corner frequency" or "-3dB point" of the function.

In arriving at an estimated position X of the optical centroid on the LEPD, the preamp output voltages undergo subsequent signal processing which significantly improves the uncertainty of X over the uncertainty of these raw output voltages. Appendix A investigates this in some depth, and §5 applies those results to the the signal and spot noise values of eqs (9) and (10), evaluated according §3 and §4.

3 EVALUATING THE SIGNAL

We make the assumption that optical interference can be ignored. We treat *all* LEPD junction current as our signal current for this analysis. Note however that thermally generated dark current i_d, treated as "signal" herein, will have a small *deterministic* effect of biasing the calculated x value towards the center of the LEPD. This effect is not explored herein, but should be considered to the extent it affects overall performance.

The centroid of incident optical flux on the LEPD affects how the induced photocurrent splits between the anode electrodes, but this split does not affect the total current, which is linearly proportional to the incident flux. The proportionality constant, or responsivity ξ [amperes/watt] of the LEPD is typically available from the manufacturers' data sheets.

The optical signal source assumed is an IR LED operating at 880 [nm] wavelength. The radiant intensity I_ϕ of these devices in [watts/steradian] is typically available from manufacturers' data sheets as an on-axis specification I_0 and a directional pattern plot as a function of the off-axis angle ϕ.

The receiving optics assumed has a circular lens of diameter d with an effective transmittance η which for an uncoated singlet will typically exceed 0.9 (0.97 for coated singlets). The field of view of these optics is small so it can be assumed the projected area of the lens aperture in the direction of the optical flux is $A = \pi d^2/4$. At a distance D from the LED, this area subtends a solid angle of $A/(4\pi D^2)$. Thus, the signal current is,

$$i_s = \frac{\xi \eta I_\phi (d/D)^2}{16} + i_d. \qquad (11).$$

4 EVALUATING THE NOISE

Our noise model assumes Johnson, shot, and amplifier noise, and refers their contributions to the amplifier outputs. Notice that up to this point, the *instantaneous* value of the signal and noise sources are used. Noise calculations must be performed according to the statistical properties of these sources. We assume the signal to be constant over some sampling aperture, during which the instantaneous noise sources each vary according to their statistics. This section briefly describes the relevant noise processes and their statistical properties, then gives formulas for the algebraic manipulation of normally distributed noise signals.

4.1 Fundamental noise processes and properties

Johnson noise arises from a thermodynamic equilibrium between conduction electrons and the crystal lattice of the conductor. This lattice undergoes random thermal vibrations and exchanges energy with the conduction electrons to maintain this equilibrium. This yields a normally distributed random electrical signal with "white" (constant power per unit frequency) spectral distribution.

Shot noise arises from the Poisson distributed occurrence of uncorrelated, equally quantized impulses. These impulses represent the instantaneous creation of carrier pairs in the junction of the LEPD photodetector. The statistics of the Poisson distribution also yield a normally distributed random electrical current with a white spectrum.

Amplifier noise arises from several physical processes, but for the class of devices we consider, we can assume a normally distributed random signal having a spectral distribution that is "pink" (constant power per octave frequency) at low frequency, transitioning to white at some "corner frequency". These characteristics are typically available from manufacturers' data sheets.

All of the above processes are also bandwidth limited by other physical mechanisms such as the parasitic self-capacitance associated with diodes and resistors, and the intrinsic frequency response associated with amplifiers. In nearly all low-frequency circuits, these band limiting mechanisms may be accurately characterized as rational polynomial functions in frequency, resulting in familiar 6 [dB/octave/pole] low-pass response curves.

In this particular circuit, we assume the dominant band limiting mechanism to be the closed loop amplifier response function which acts as a low-pass filter equally for all signal and noise sources. For a sufficiently capable amplifier, this response will be determined by the parasitic capacitance in parallel with R_f. For example, 5 [pF] parasitic capacitance in parallel with a 200 [kΩ] feedback resistor would give a 159 [kHz] single pole frequency response.

This assumption is one of convenience and is valid only if the noise bandwidth of the various input-referred noise sources exceeds the amplifier bandwidth. This is easily satisfied by all the non LEPD related noise sources, and closely satisfied by LEPD related sources. This can be seen by approximating a 2 [μs] worst-case 10-90% LEPD risetime t_r, using a single-pole low-pass response function having a corner frequency $f_0 = 175$ [kHz], where $f_0 = 1/2\pi\tau$, $\tau = t_r/(ln(0.9) - ln(0.1))$.

4.2 Specific noise source models

Using the band-limiting assumption adopted in §4.1 enables us conveniently to perform appropriate statistical additions of the noise sources represented in *spectral density* form, that is in terms of [units/$\sqrt{\text{Hz}}$]. Then the bandwidth-dependent RMS calculation needs only to be calculated once for their sum. Thus, the noise contributions here are expressed in terms of voltage or current spectral density rather than in the more familiar band-limited RMS form. In general, these voltages or currents are functions of frequency, and are customarily characterized at specific frequencies as *spot noise* values. Of course for white noise, the spot noise is constant; that is, it has the same value at all frequencies, and we are free to use this value to represent the spectral density of the associated voltage or current at any frequency.

To maintain consistency with calculations based on the autocorrelation function of the noise, we adopt the convention of evaluating RMS noise over a two-sided spectrum; that is, we integrate the spectral density function from $-\infty \ldots \infty$. Since the simplified conventional formulas take bandwidth over a single-sided spectrum $0 \ldots \infty$, the coefficients conventionally used under the radical are twice what we use here.

The LEPD anode inter-electrode resistance is modeled as $R_{i1} + R_{i2}$. While the individual values of R_{i1} and R_{i2} depend on the centroid of the incident optical flux, their sum is fixed and is used to calculate the inter-electrode Johnson noise.

Our model assumes high quality resistors which do not have so-called "excess" noise. We believe this to be a valid assumption for the LEPD inter-electrode resistance, and will ensure it to be valid for the feedback resistors by using metal film SMT chip resistor technology. Each of these resistors does however contribute Johnson noise, which may be modeled as a Thevenin equivalent voltage in series with the resistor or as a Norton equivalent current in parallel with the resistor. The respective spot noise formulas are:

$$v_t = \sqrt{2\,k\,T\,R}, \tag{12}$$

$$i_t = \sqrt{\frac{2\,k\,T}{R}}; \tag{13}$$

where

$$k = 1.38 \times 10^{23} \text{ (Boltzman's constant)},$$
$$T = \text{temperature } [^\circ K],$$
$$R = \text{resistance } [\Omega],$$

The LEPD junction current comprises flows of both thermally generated and optically stimulated carrier pairs. Each such pair transports one electronic charge $q = 1.60 \times 10^{-19}$ [C] between the cathode and the anode of the junction. This Poisson distributed process results in a current having both a DC "signal" component i_s, and an AC "shot noise" component i_{sh}, as described above:

$$i_{sh} = \sqrt{q\, i_s}. \tag{14}$$

In our model, all noise sources arising within the op amps are referred to their inputs where they are represented as independent current and voltage sources which may be correlated to some degree. Since the correlation coefficient ρ is not typically available from manufacturers' data sheets, we make the worst-case assumption that they are fully correlated. Data sheets typically provide spot noise values at specific frequencies, and graphs of the noise spectra for input noise voltage and current.

We have historically adopted, and intend to continue using a signal processing technique which takes the difference of two averaged contiguous samples of LEPD currents, one with optical signal, the other without. The motivation is to cancel both the LEPD dark current and residual ambient optical background illumination, which we suppress as much as possible by the use of optical filters. This approach assumes these currents to be either DC or sufficiently slowly varying that their average values are the same across both averaging apertures. This same mechanism will suppress low frequency amplifier noise. If the corner frequency between the $1/f$ and white noise regimes is sufficiently low, the $1/f$ noise regime can be safely ignored (see Appendix A for a quantitative discussion).

For amplifiers having $1/f$ noise corner frequencies below a few hundred Hertz and contiguous sample averaging apertures not exceeding 100 [μs], we can safely treat the amplifier noise as white, and use a spot noise value from any frequency in its white noise regime.

4.3 Additive combination of noise terms

Addition of two random noise values is not linear. Under the assumption that the two values are normally distributed, band-limited random signals, they add according to:

$$z = \sqrt{z_i^2 + 2\rho z_i z_j + z_j^2}, \tag{15}$$

where $-1 \leq \rho \leq 1$ is the correlation coefficient between z_i and z_j. If these signals are fully correlated ($\rho = \pm 1$), this expression degenerates to a linear sum (or difference) of the signals, while if they are totally uncorrelated, it degenerates to a Pythagorean sum. More generally, we can calculate the statistical sum of n signals according to:

$$\sqrt{\sum_{i=1}^{n} \left({z_i}^2 + \sum_{j \neq i} 2\rho_{ij} z_i z_j \right)}. \tag{16}$$

5 CALCULATING THE RMS POSITIONAL UNCERTAINTY

The remainder of the noise model must account for the operations performed on the output voltages of the preamps in estimating the position of the optical flux centroid on the LEPD. These operations are first the digitization and cancellation of LEPD dark current by differencing repeated samples (auto-differencing) for each preamp output, and then the normalization of the difference between the signals so derived from the two preamps (cross-differencing).

5.1 Digitizing and auto-differencing operations

As mentioned in §4.2, the preamp output voltages are digitized over a "light" period (with a beacon signal being generated) and a "dark" period, and the difference of these readings is used for all subsequent processing. We assume a $\Delta\Sigma$ ADC which is believed to average its input over its conversion interval. We explicitly disregard quantization noise under the assumption it is less than the noise present on the preamp outputs. Thus, we model these operations as the difference between two averages each taken over a sample aperture of duration T, with a period T_0 between them. It is to be understood that the inputs to these operations are analog voltages, while the outputs are digital integers. This model is developed in detail in appendix A, and the results summarized here.

Let us first segregate the signal from the noise components of eqs(9)&(10), and form the statistical sum of the noise sources according to eq(16). Thus, we write,

$$s_1 = (x-1)\, i_s\, R_{f1}, \tag{17a}$$

$$n_1 = \sqrt{\begin{array}{c} \dfrac{(R_{f1}+R_i)^2\ v_{n1}^2}{R_i^2} + \dfrac{2\, i_{n1}\, R_{f1}\ (R_{f1}+R_i)\ v_{n1}}{R_i} + i_{n1}^2\, R_{f1}^2 \\ + v_{tf1}^2 + i_{ti}^2\, R_{f1}^2 + (x-1)^2\ i_{sh}^2\, R_{f1}^2 + \dfrac{R_{f1}^2\, v_{n2}^2}{R_i^2} \end{array}}, \tag{17b}$$

$$s_2 = -x\, i_s\, R_{f2}, \tag{18a}$$

$$n_2 = \sqrt{\begin{array}{c} \dfrac{(R_{f2}+R_i)^2\ v_{n2}^2}{R_i^2} + \dfrac{2\, i_{n2}\, R_{f2}\ (R_{f2}+R_i)\ v_{n2}}{R_i} + i_{n2}^2\, R_{f2}^2 \\ + v_{tf2}^2 + i_{ti}^2\, R_{f2}^2 + x^2\ i_{sh}^2\, R_{f2}^2 + \dfrac{R_{f2}^2\, v_{n1}^2}{R_i^2} \end{array}}. \tag{18b}$$

We equate m_{x1} and m_{x2} of eq(A4) respectively with s_i^{on} and s_i^{off} of eqs(17)&(18), where $i = (1, 2)$, and write,

$$S_i = s_i^{on} - s_i^{off}, \qquad i = (1, 2), \tag{19}$$

where s_i^{on} and s_i^{off} are evaluated according to eq(11) of §3.

We now evaluate n_1 and n_2 according to the methods of §4 to obtain numerical spot noise values, and use these values to define the amplitudes of the single pole low-pass voltage spectral density functions assumed in §4.1: $\mathcal{V}_i(f) = n_i f_0/(jf - f_0)$, $i = (1, 2)$. The associated positive-definite power spectral density function is obtained as $\mathcal{V}^*\mathcal{V}$, or,

$$\mathcal{S}_i(f) = \frac{n_i^2 f_0^2}{f^2 + f_0^2}, \qquad i = (1, 2).$$

From eq (6-64) of Appendix A, we can now calculate the autocorrelation functions associated with n_1 and n_2,

$$\mathcal{R}_i(\tau) = \int_{-\infty}^{\infty} \left(\frac{n_i^2 f_0^2}{f^2 + f_0^2} \right) e^{2\pi j f \tau} df,$$

which evaluates to,

$$\mathcal{R}_i(\tau) = \pi n_i^2 f_0 e^{-2\pi f_0 |\tau|}, \qquad i = (1, 2). \tag{20}$$

Finally, we use eq(A8) of Appendix A to obtain the variance (which is the square of the RMS uncertainty) of the difference between the two averages,

$$\begin{aligned} N_i^2 = \frac{2}{T^2} \Bigg(& \int_0^T 2\mathcal{R}_i(\tau)(T - \tau) d\tau \\ & - \int_{T_0}^{T_0+T} \mathcal{R}_i(\tau)(\tau - T_0) d\tau \\ & - \int_{T_0+T}^{T_0+2T} \mathcal{R}_i(\tau)(T_0 + 2T - \tau) d\tau \Bigg), \qquad i = (1, 2). \end{aligned} \tag{21}$$

Here we have omitted the m_{xi} terms since we had previously separated the signals s_i from the noises n_i which have vanishing mean values m_{xi}, subsequently using only the n_i for this calculation.

An assumption is made in using eq(A8) that the contributing noise processes remain stationary over both sample apertures. This is not strictly true for this problem under all conditions, because the shot noise is dependent on the optical flux incident on the LEPD, and this differs between the "light" and "dark" samples. However the stationary condition is well satisfied in weak signal cases which are of primary interest to this analysis, because other stationary processes dominate. To avoid unduly optimistic results for strong signal cases, we adopt the fiction that the shot noise associated with the "light" reading is stationary across the "dark" reading as well. This computational convenience is conservative insofar as it predicts slightly worse than actual performance; moreover, we are not particularly concerned with the precision of the noise model in the strong signal regime.

5.2 Cross-difference and normalizing operations

The estimated X position, $-1 \leq X \leq 1$ of the optical flux centroid on the LEPD is a statistical quantity comprising a mean and an RMS uncertainty (or *standard deviation* which is the square root of the *variance*). Conceptually, this can be calculated as the normalized difference of LEPD photocurrents associated with the axis under consideration. Accordingly, we perform this operation, but on the conditioned, digitized signals developed in §5.1 rather than on the raw outputs of the preamps. Thus, we write,

$$X = \frac{S_n + N_n}{S_d + N_d} = \frac{(S_1 + N_1) - (S_2 + N_2)}{(S_1 + N_1) + (S_2 + N_2)}, \tag{22}$$

where signals S_1, S_2 are evaluated using eq(19), noises N_1, N_2 are evaluated using eq(21), numerator terms $S_n = S_1 - S_2$, $N_n = N_1 - N_2$, and denominator terms $S_d = S_1 + S_2$, $N_d = N_1 + N_2$.

We rewrite the denominator of this expression as $S_d(1+\epsilon)$ where $\epsilon = N_d/S_d$, and use the power series definition of e^ϵ to make the approximation $e^\epsilon \approx 1 + \epsilon$ for small ϵ. Thus we can write,

$$X \approx \frac{S_n + N_n}{S_d e^\epsilon} = \frac{(S_n + N_n)e^{-\epsilon}}{S_d} \approx \frac{(S_n + N_n)(1 - N_d/S_d)}{S_d} = \frac{S_n}{S_d} + \frac{N_n}{S_d} - \frac{S_n N_d}{S_d^2} - \frac{N_n N_d}{S_d^2}.$$

Under the assumption that the noise terms are small relative to S_d, the last term of this expression can be ignored, resulting in,

$$X = \frac{S_n}{S_d} + \frac{N_n S_d - N_d S_n}{S_d^2};$$

which upon substitution gives us the instantaneous value of

$$X = \frac{S_1 - S_2}{S_1 + S_2} + \frac{2(N_1 S_2 - N_2 S_1)}{(S_1 + S_2)^2}. \tag{23}$$

The final step is to calculate the statistical sum of the noise terms. It is necessary, of course, to consider the degree of correlation ρ_n between N_1 and N_2. Taking the noise sources in our model to be statistically independent, let us see how they contribute to v_{o1} (and hence to N_1) and to v_{o2} (and hence to N_2). Shot noise i_{sh} appears covariant with an x-dependent ratio between the two channels. The LEPD interelectrode Johnson noise i_{ti} appears contravariant in equal measure, assuming $R_{f1} = R_{f2}$, because i_{ti} moves the same charge from the summing junction of preamp 2 to the summing junction of preamp 1. The preamp input current noises i_{n1} and i_{n2} appear only on their respective channels, so their contributions are uncorrelated. The same is true for the feedback resistor Johnson noises v_{tf1} and v_{tf2}. Finally, the preamp input voltage noises v_{n1} and v_{n2} each appear contravariant in nearly equal measure for $R_{f1} = R_{f2} >> R_i$. Therefore, correlation between N_1 and N_2 depends strongly on which noise source(s) dominates. We note in passing that covariant contributions tend to cancel in the difference expression, while contravariant contributions reinforce.

The estimate of X, comprising its mean X_s and RMS uncertainty X_n is thus,

$$X = X_s + X_n = \frac{S_1 - S_2}{S_1 + S_2} + \frac{2\sqrt{(N_1 S_2)^2 - 2\rho_n N_1 S_2 N_2 S_1 + (N_2 S_1)^2}}{(S_1 + S_2)^2}. \tag{24}$$

6 EXAMPLE CALCULATIONS

We use the MACSYMA code listed in Appendix B to work an example, assuming the following values of design parameters, over a range of situational parameters:

Design parameters:			
T	ADC sample aperture (averaging period)	100	[μs]
T_0	interval between ADC samples	20	[μs]
f_0	Preamp bandwidth (3dB corner frequency)	160	[kHz]
v_n	Preamp spot noise voltage @ 1 kHz (TLE2027C)	4.5	[nV/$\sqrt{\text{Hz}}$]
i_n	Preamp spot noise current @ 1 kHz	600	[fA/$\sqrt{\text{Hz}}$]
ρ	v_n, i_n cross-correlation coefficient	1	
R_f	Preamp feedback resistance	200	[kΩ]
R_i	LEPD interelectrode resistance (DLS4)	10	[kΩ]
i_d	LEPD dark current	50	[nA]
ξ	LEPD responsivity	0.6	[A/W]
I_0	On-axis radiant intensity of LED (SFH487P)	27	[mW/sr]
d	Lens diameter	17	[mm]
η	Lens transmittance	0.95	
T_a	Ambient temperature	293	[°K]
Situational parameters			
x	Centroid position on LEPD	$0 \ldots 1$	
D	distance from LED to detector	$0.6 \ldots 3$	[m]
ϕ	Detector position relative to LED optical axis	$0° \ldots 60°$	

6.1 All the gory details

This section leads one through all the functions and expressions associated with the internals of the analysis described in this report, so that one may inspect derivations, and details of the analysis. **For a simplified example** of using the results of this analysis, skip to the next section.

A global "evaluation environment" must first be established which defines the constants, expressions, and functions used to calculate the analysis results. This begins with the first three major blocks of code in appendix B, which provide symbolic expressions corresponding to eqs(1–10) in §2. Next some physical constants are defined,

```
k: 1.38e-23;            /* Boltzmann's constant [Joules/Kelvin] */
q: 1.60e-19;            /* Electronic charge [Coulombs] */
```

and values specified for design parameters which are not expected to change during the evolution of the design,

```
T_a: 293;       /* LEPD temperature */
rho: 1;         /* assume worst-case correlation of (vn, in) for op amp */
d: 1.7e-2;      /* lens diameter*/
eta: 0.95;      /* lens transmittance */
```

```
xi: 0.6;        /* UDT DLS4 LEPD responsivity */
id: 5.0e-8;     /* DLS4 dark current */
ri: 1.0e4;      /* feedback resistance for both preamps */
I0: 2.7e-2;     /* On-axis radiance of Siemens SFH487P LED */
```

A function `I(phi)` is provided which specifically fits the radiation pattern of a Siemens SFH487P infrared LED, using an on-axis radiance coefficient of I_0. The following example gives the radiance at a series of angles from the LED axis:

```
(c71) for a:0 thru 60 step 20 do display(I(a));
                                   i(0) = 0.027
                                i(20) = 0.02545008
                                i(40) = 0.02245791
                                i(60) = 0.01663158
(d71)                                  done
```

Functions `vt(r)` and `it(r)` are provided to return Johnson noise spectral density coefficients conditioned on T_a, and `isht(i)` returns a shot noise spectral density coefficient. For example, let's look at the noise voltage from a 1 [MΩ] resistor, the noise current from a 1 [kΩ] resistor, and the shot noise current associated with a 10 [μA] LEPD junction current (all, of course per $\sqrt{\text{Hz}}$):

```
(c68) vt(1.0e6);
(d68)                              8.992664e-8
(c69) it(1.0e3);
(d69)                              2.84373e-12
(c70) isht(1.0e-5);
(d70)                              1.264911e-12
```

The expression `i_o` provides the optically-stimulated LEPD junction current (eq(11), less the dark current), and evaluates to,

```
(c73) ev(i_o);
                                 1.293787e-4 i
(d73)                            -------------
                                        2
                                      ds
```

where LED radiance I, is subsequently evaluated using `I(phi)`, and distance D between LED and detector is subsequently specified.

The first major "work horse" function, `calc_v(x, phi, ds, rf, vn, in)`, numerically evaluates all the signals, the noise contributions, and the aggregate noises appearing at the preamp outputs. The calculated noise values are *voltage spectral density coefficients*, in keeping with the discussion in §4.2.

`calc_v()` performs its work by side effects; that is, it *replaces* the symbolic expressions for `v1s, v1n, v1sh,` etc. by the numerical values to which they evaluate, given the parameters and present evaluation environment. It is necessary to scope the execution of `calc_v()` within a MACSYMA block to avoid permanently binding the affected variables.

`calc_v()` was designed with one special operational mode: if `x` lies outside the range $0 \ldots 1$, it returns values associated only with the LEPD dark current irrespective of values of `phi` and `ds`. This was an attempt to provide a means for getting dark current readings in a manner intuitively consistent with the model – if the centroid falls beyond the geometric limits of the LEPD junction, we expect to see only dark current. Thus, a convention to calculate a dark current reading is to pass -1 as the `x` parameter in calling `calc_v()`.

A companion function, `show_v(f0)` displays the values calculated by `calc_v()`, integrated over a single pole low-pass voltage spectral density function as discussed at the end of §2. This function is assumed to be the same for all noise contributions, as discussed in §4.1, with corner frequency specified by the parameter `f0`. In the special case of `f0`=0, however, the raw noise spectral density coefficients themselves are displayed.

For convenience, a function `vo(x, phi, ds, rf, vn, in, f0)` is a wrapper which contains both `calc_v()` and `show_v()`, properly scoping and passing the appropriate parameters to each. Example calls to `vo()` are given in the next section.

A power spectral density function,

```
(c75) S(f);
                                           2   2
                                         f0  v0
(d75)                                    --------
                                           2    2
                                         f0  + f
```

is defined corresponding to a single pole low-pass voltage spectral density function with corner frequency `f0`. This function is used solely to calculate the autocorrelation function,

```
(c76) Rx(tau);
                                    - 2 %pi f0 abs(tau)    2
(d76)                    %pi f0 %e                        v0
```

which in turn is used solely to calculate the function `var_Dxs(v0, f0, T0, T)`, which calculates the variance of the difference of two sequential, averaged samples. Typically, a call to `calc_v()` is first made to provide suitable values for `v1n, v2n`; then these values are passed as the `v0` parameter to `var_Dxs()` on two successive calls. Assuming a call to `calc_v()` has been made such that `v1n` = 1.869576e-4, a call to `var_Dxs()` with a corner frequency `f0` = 160 [kHz], inter-sample period `T0` = 20 [μs], and sample aperture duration `T` = 100 [μs] gives,

```
(c77) var_Dxs(v1n, 1.6e5, 2.0e-5, 1.0e-4);
(d77)                                1.753133e-9
```

Finally, the function `X_est(x, phi, ds, rf, vn, in, f0, T0, T)` is provided which takes static design parameters from the evaluation environment, and calculates $X = [X_s, X_n]$ from its input parameters, as described in the next section.

6.2 Simplified analysis example

To analyze the noise performance of LEPD based tracker designs of the HiBall ilk, execute macsyma. Respond to the prompt `(c1)` by typing, `load("hiball.mac");`. This will set up an environment with some design parameters predefined (to the values listed at the beginning of this section), and others to be supplied as parameters to functions you call. There are two particularly useful functions for your use.

`vo(x, phi, ds, rf, vn, in, f0)` calculates the signal and noise voltages for each noise contribution and for the aggregate noise at the outputs of the preamps.

`X_est(x, phi, ds, rf, vn, in, f0, T0, T)` calculates the estimated position of the optical flux centroid on the LEPD in terms of its mean X_s and RMS uncertainty X_n, and the average signal and noise values S_1, S_2, N_1, N_2 as discussed in §5.2 of this report.

The parameters are:

- `x`: position of centroid $(0 < x < 1)$,
- `phi`: angular position of detector wrt LED optical axis$(0 \leq \phi \leq 90°)$,
- `ds`: distance of detector from LED $(0 < D < \infty)$ [m],
- `rf`: feedback resistor value for *both* preamps$(0 < R_f < \infty)$ [Ω],
- `vn`: spot noise voltage in white noise regime of preamp [V/$\sqrt{\text{Hz}}$],
- `in`: spot noise current in white noise regime of preamp [A/$\sqrt{\text{Hz}}$],
- `f0`: corner frequency of single pole low-pass amplifier response [Hz],
- `T0`: period between light and dark sample apertures [s], and
- `T`: period of both the light and dark sample apertures [s].

If you wish to change any of the other design parameters, you can assign new numerical values to them at any time (cf. Appendix B for examples of their default settings and syntax for setting them).

A few calls are made here to illustrate use of these two functions. First let us look at spectral density noise values at the preamp outputs for the dark current case. By setting $f_0 = 0$, we get spectral density rather than RMS voltages By setting $x = -1$, we move the optical centroid entirely off the LEPD junction so we have only the dark current (thus the values of ϕ and D have no effect).

```
(c5) vo(-1, 0, 1, 2.0e5, 4.5e-9, 6.0e-13, 0);
                                   v1s = - 0.005
                                 v1n = 2.969088e-7
                                 v1sh = 8.944272e-9
                                 v1vn1 = - 9.45e-8
                                  v1in1 = - 1.2e-7
                               v1iti = - 1.798533e-7
                               v1vtf1 = - 4.021641e-8
                                   v1vn2 = 9.0e-8
                                   v2s = - 0.005
```

```
                  v2n = 2.969088e-7
                 v2sh = 8.944272e-9
                 v2vn2 = - 9.45e-8
                  v2in2 = - 1.2e-7
                 v2iti = 1.798533e-7
               v2vtf2 = - 4.021641e-8
                   v2vn1 = 9.0e-8
(d5)                       done
```

Now let us look at the RMS noise over a bandwidth of $f_0 = 160$ [kHz], with the detector 3 [m] from the LED and 60° off axis. The signal to noise ratio $v_{1s}/v_{1n} = 281$ in this case.

```
(c6)  vo(0, 60, 3, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5);
                 v1s = - 0.05281714
                  v1n = 1.881654e-4
                 v1sh = 2.521232e-5
                v1vn1 = - 5.93761e-5
               v1in1 = - 7.539822e-5
               v1iti = - 1.130051e-4
               v1vtf1 = - 2.526872e-5
                v1vn2 = 5.654867e-5
                   v2s = - 0.005
                  v2n = 1.865533e-4
                 v2sh = 5.619852e-6
                v2vn2 = - 5.93761e-5
               v2in2 = - 7.539822e-5
                v2iti = 1.130051e-4
               v2vtf2 = - 2.526872e-5
                v2vn1 = 5.654867e-5
(d6)                       done
```

Let us compare this with the fully signal-processed case under the same conditions, but averaged over 100 [μs] apertures separated by 20 [μs]. We take three readings across the valid range of x, and see that the range of the signal is 2.0, so the signal to noise ratio is 1140. We also subsequently specify an array comprising $S_1, S_2, N_1, N_2, X_s, X_n$, to inspect their numerical values as calculated by the last call to `X_est()`.

```
(c7) X_est(0, 60, 3, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);
(d7)                      [1.0, 0.001747486]
(c8) X_est(0.5, 60, 3, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);
(d8)                      [-0.0, 0.001751273]
(c9) X_est(1, 60, 3, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);
(d9)                      [- 1.0, 0.001747486]
(c10) [s1, s2, n1, n2, x_s, x_n];
(d10)  [0.0, - 0.04781714, 4.177988e-5, 4.214092e-5, - 1.0, 0.001747486]
```

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[Dave87] Davenport, W. B. Jr. and W. L. Root, "An Introduction to the Theory of Random Signals and Noise", IEEE Press, New York, 1987.

[Leac94] Leach, W. Marshall Jr., "Fundamentals of Low-Noise Analog Circuit Design", *Proceedings of the IEEE,* v.82 no.10, October 1994, pp.1515-38.

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APPENDIX A: Difference of Averages derivation

Following Davenport and Root [Dave87] (from which all hyphenated equation numbers in this Appendix are cited), we develop an expression for the RMS uncertainty (standard deviation) of the sum$\vee$difference between two averaged readings of band-limited Gaussian noise over a pair of closely spaced sample apertures. To check results, we compare the case of the sum with eq(4-83) in the limit as the spacing between samples vanishes. The difference case provides a formula for calculating the measurement uncertainty of the difference between a pair of LEPD readings.

The variance of random variable x is defined to be,

$$\sigma_x^2 = \mu_2 = E[x-m]^2, \tag{4-13}$$

where $E[g(x)] = \int_{-\infty}^{+\infty} g(x)p(x)dx$ is called the statistical average or expected value of the continuous random variable $g(x)$, and $p(x)$ is the probability density function of x.

Let $x(t)$ be a sample function of a wide-sense stationary random process with finite mean m_x and finite variance σ_x^2. The finite interval time average is defined as,

$$A_x(T) = \frac{1}{2T}\int_{-T}^{+T} x(t)dt. \tag{4-78}$$

Taking a statistical average over an ensemble of $x(t)$ we have,

$$E[A_x(T)] = \frac{1}{2T}\int_{-T}^{+T} E[x_t]dt = m_x. \tag{4-81}$$

where $E(x_t) = m_x$ is the mean value of x.

We begin by deriving the variance of a simple average $A_x(T)$; we will subsequently generalize this to the average over a pair of sample apertures. From eq(4-13) we have,

$$\begin{aligned}
\sigma^2[A_x(T)] &= E\left[A_x(T) - E[A_x(T)]\right]^2 \\
&= E[A_x(T) - m_x]^2 && \text{from eq(4-81)} \\
&= E[A_x(T)\cdot A_x(T)] - m_x(E[A_x(T)] + E[A_x(T)]) + m_x^2 \\
&= E[A_x(T)\cdot A_x(T)] - m_x^2 && \text{from eq(4-81)} \\
&= E\left[\frac{1}{2T}\int_{-T}^{T} x(t')dt' \cdot \frac{1}{2T}\int_{-T}^{T} x(t)dt\right] - m_x^2 && \text{from eq(4-78)} \\
&= E\left[\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} x(t)x(t')dt'dt\right] - m_x^2 \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E[x'_t x_t]dt'dt - m_x^2 \\
&= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} \left(E[x'_t x_t] - m_x^2\right) dt'dt. && \text{(A1)}
\end{aligned}$$

The autocorrelation function of a sample function $x(t)$ is defined as,

$$\mathcal{R}_x(t, t-\tau) = \mathcal{R}_x(\tau) = E[x_t x^*_{t-\tau}], \tag{4-49}$$

where "*" indicates the complex conjugate. It has a property that $\mathcal{R}_x(\tau) = \mathcal{R}^*_x(-\tau)$, thus if it is real, it is an even function of τ.

The power spectral density $\mathcal{S}(f)$ of a wide-sense stationary random process is defined to be the Fourier transform of the autocorrelation function,

$$\mathcal{S}(f) = \int_{-\infty}^{\infty} \mathcal{R}_x(\tau) e^{-j\omega\tau} d\tau, \qquad \omega = 2\pi f \tag{6-63}$$

which, through the inverse Fourier transform provides us a handy formula for computing the autocorrelation function,

$$\mathcal{R}_x(\tau) = \int_{-\infty}^{\infty} \mathcal{S}(f) e^{j\omega\tau} df. \tag{6-64}$$

We now continue by letting $t' = t - \tau$ and substituting eq(4-49) into eq(A1),

$$\begin{aligned} \sigma^2[A_x(T)] &= \frac{1}{4T^2} \int_{-T-t'}^{T-t'} \int_{-T}^{T} \left(\mathcal{R}_x(\tau) - m_x^2\right) dt' d\tau \\ &= \frac{1}{4T^2} \int_{-T-t'}^{T-t'} \left(\mathcal{R}_x(\tau) - m_x^2\right) \int_{-T}^{T} dt' d\tau \end{aligned} \tag{A2}$$

The integral is taken over the area of the parallelogram shown in figure A1. Since the autocorrelation function is independent of t' here and is an even function of τ, the entire integral is just twice that of the integral taken over the right half of the parallelogram. It is evident from this figure that for $0 \le \tau \le 2T$, the limits of integration for t' are $-T$ and $T - \tau$. Therefore, eq(A2) can be rewritten in terms of these modified limits,

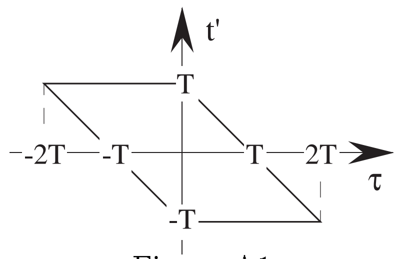

Figure A1

$$\begin{aligned} \sigma^2[A_x(T)] &= \frac{1}{2T^2} \int_0^{2T} \left(\mathcal{R}_x(\tau) - m_x^2\right) \int_{-T}^{T-\tau} dt' d\tau \\ &= \frac{1}{T} \int_0^{2T} \left(\mathcal{R}_x(\tau) - m_x^2\right) \frac{1}{2T}(2T - \tau) d\tau \\ &= \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau}{2T}\right) \left(\mathcal{R}_x(\tau) - m_x^2\right) d\tau \end{aligned} \tag{4-83}$$

We now generalize this approach to analyze the variance of the sum$\vee$difference of two averaged readings taken over two closely spaced finite sample apertures. We first define this sum$\vee$difference as,

$$D_x = \frac{1}{T_1}\int_a^b x(t)dt \pm \frac{1}{T_2}\int_c^d x(t)dt, \qquad T_1 = (b-a), \quad T_2 = (d-c). \tag{A3}$$

Taking the statistical average over an ensemble of $x(t)$ we have,

$$E[D_x] = \frac{1}{T_1}\int_a^b E[x(t)]dt \pm \frac{1}{T_2}\int_c^d E[x(t)]dt = m_{x1} \pm m_{x2}. \tag{A4}$$

We now proceed as we did for eq(A1):

$$\begin{aligned}
\sigma^2[D_x] &= E\left[D_x \cdot D_x\right] - 2(m_{x1} \pm m_{x2})E[Dx] + (m_{x1} \pm m_{x2})^2 \\
&= E\left[D_x \cdot D_x\right] - (m_{x1} \pm m_{x2})^2 \\
&= E\left[\left(\frac{1}{T_1}\int_a^b x(t')dt' \pm \frac{1}{T_2}\int_c^d x(t')dt'\right)\left(\frac{1}{T_1}\int_a^b x(t)dt \pm \frac{1}{T_2}\int_c^d x(t)dt\right)\right] \\
&\qquad\qquad - (m_{x1} \pm m_{x2})^2 \\
&= \frac{1}{T_1^2}\int_a^b\int_a^b E[x'_t x_t]dt'dt - m_{x1}^2 \pm \frac{1}{T_1T_2}\int_a^b\int_c^d E[x'_t x_t]dt'dt \mp m_{x1}m_{x2} \\
&\qquad \pm \frac{1}{T_1T_2}\int_c^d\int_a^b E[x'_t x_t]dt'dt \mp m_{x1}m_{x2} + \frac{1}{T_2^2}\int_c^d\int_c^d x(t)x(t')dt'dt - m_{x2}^2 \\
&= \frac{1}{T_1^2}\int_a^b\int_a^b \left(E[x'_t x_t] - m_{x1}^2\right)dt'dt \pm \frac{1}{T_1T_2}\int_a^b\int_c^d \left(E[x'_t x_t] - m_{x1}m_{x2}\right)dt'dt \\
&\qquad \pm \frac{1}{T_1T_2}\int_c^d\int_a^b \left(E[x'_t x_t] - m_{x1}m_{x2}\right)dt'dt + \frac{1}{T_2^2}\int_c^d\int_c^d \left(E[x'_t x_t] - m_{x2}^2\right)dt'dt \\
&= \frac{1}{T_1^2}\int_{a-t'}^{b-t'}\int_a^b (\mathcal{R}_x(\tau) - m_{x1}^2)dt'd\tau && \text{(A5.1)} \\
&\quad \pm \frac{1}{T_1T_2}\int_{a-t'}^{b-t'}\int_c^d (\mathcal{R}_x(\tau) - m_{x1}m_{x2})dt'd\tau && \text{(A5.2)} \\
&\quad \pm \frac{1}{T_1T_2}\int_{c-t'}^{d-t'}\int_a^b (\mathcal{R}_x(\tau) - m_{x1}m_{x2})dt'd\tau && \text{(A5.3)} \\
&\quad + \frac{1}{T_2^2}\int_{c-t'}^{d-t'}\int_c^d (\mathcal{R}_x(\tau) - m_{x2}^2)dt'd\tau && \text{(A5.4)}
\end{aligned}$$

These integrals are taken over the parallelograms shown in Figure A2. The situation is somewhat more complicated than for eq(4-83), especially if $T_1 \neq T_2$, as shown. In particular, integrals (A5.2) and (A5.3) require a more elaborate procedure both because $\mathcal{R}_x(\tau)$ is not symmetric about the centers of their corresponding parallelograms and because their parallelograms cannot in general be decomposed into a pair of triangles. In any case, however, integrals (A5.1) and (A5.4) can be handled just as in eq(4-83).

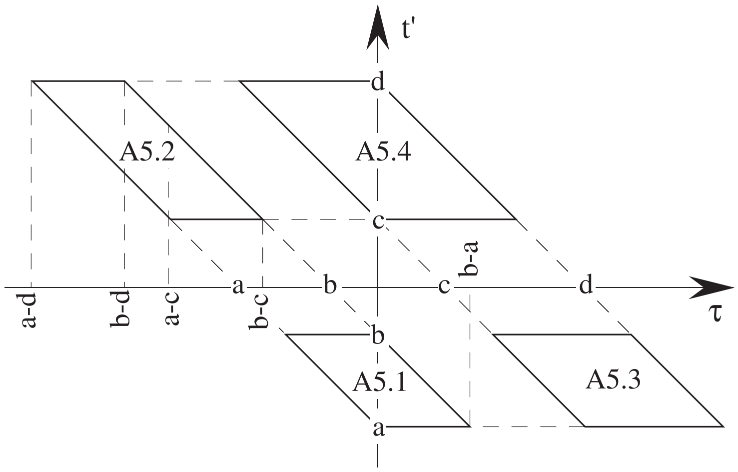

Figure A2

We take the integral (A5.1) over $0 \le \tau \le T_1 = (b-a)$ and with $a \le t' \le b-\tau$; but $Rx(\tau)$ is independent of t', so we can translate these limits to $0 \le t' \le T_1 - \tau = (b-a) - \tau$. Similarly, integral (A5.4) is taken over $0 \le \tau \le T_2$ and $0 \le t' \le T_2 - \tau$. Observing that $\mathcal{R}_x(\tau)$ is an even function, we double these values to obtain the values of the entire integrals (A5.1) and (A5.4). In the treatment of integrals (A5.2) and (A5.3), it should be noted that two distinct cases can occur, $T_1 \le T_2$, and $T_1 > T_2$. We discuss the former case, as shown in figure A2.

It is convenient to decompose the integral (A5.2) into three intervals, delimited by the points $(a-d), (b-d), (a-c)$, and $(b-c)$ along the τ axis. We now let $T_0 \equiv (c-b)$, and express these points as $(a-d) = -T_0 - T_1 - T_2$, $(b-d) = -T_0 - T_2$, $(a-c) = -T_0 - T_1$, and $(b-c) = -T_0$. We again exploit the independence of $\mathcal{R}_x(\tau)$ and t' to translate integration limits of t', to obtain the following limits of integration:

$$\begin{array}{lr}
-T_0 - T_1 - T_2 \le \tau \le -T_0 - T_2, & -\tau - T_0 - T_1 - T_2 \le t' \le 0; \\
-T_0 - T_2 \le \tau \le -T_0 - T_1, & 0 \le t' \le T_1; \\
-T_0 - T_1 \le \tau \le -T_0, & 0 \le t' \le -\tau - T_0.
\end{array}$$

We similarly decompose the integral (A5.3) and find the limits in terms of T_0, T_1, T_2:

$$\begin{array}{lr}
T_0 \le \tau \le T_0 + T_1, & -\tau + T_0 \le t' \le 0; \\
T_0 + T_1 \le \tau \le T_0 + T_2, & 0 \le t' \le T_1; \\
T_0 + T_2 \le \tau \le T_0 + T_1 + T_2, & 0 \le t' \le -\tau + T_0 + T_1 + T_2.
\end{array}$$

Notice the symmetry between these sets of integration limits. We exploit this symmetry, noting that $\mathcal{R}_x(\tau)$ is an even function, to show that the values of (A5.2) and (A5.3) are identical. Therefore, we may arbitrarily pick (A5.3), calculate its value, and double it to obtain the value of (A5.2)+(A5.3). Thus, we now write,

$$\begin{aligned}
\sigma^2[D_x] = {} & \frac{2}{T_1^2}\int_0^{T_1} (\mathcal{R}_x(\tau) - m_{x1}^2) \int_0^{T_1-\tau} dt'\,d\tau \\
& \pm \frac{2}{T_1 T_2}\int_{T_0}^{T_0+T_1} (\mathcal{R}_x(\tau) - m_{x1}m_{x2}) \int_{T_0-\tau}^{0} dt'\,d\tau \\
& \pm \frac{2}{T_1 T_2}\int_{T_0+T_1}^{T_0+T_2} (\mathcal{R}_x(\tau) - m_{x1}m_{x2}) \int_0^{T_1} dt'\,d\tau \\
& \pm \frac{2}{T_1 T_2}\int_{T_0+T_2}^{T_0+T_1+T_2} (\mathcal{R}_x(\tau) - m_{x1}m_{x2}) \int_0^{T_0+T_1+T_2-\tau} dt'\,d\tau \\
& + \frac{2}{T_2^2}\int_0^{T_2} (\mathcal{R}_x(\tau) - m_{x2}^2) \int_0^{T_2-\tau} dt'\,d\tau. \qquad \text{(A6)}
\end{aligned}$$

We can now trivially perform the integrations over t' to obtain,

$$
\begin{aligned}
\sigma^2[D_x] = & \frac{2}{T_1^2}\int_0^{T_1}(T_1-\tau)(\mathfrak{R}_x(\tau)-m_{x1}^2)d\tau \\
& \pm \frac{2}{T_1T_2}\int_{T_0}^{T_0+T_1}(\tau-T_0)(\mathfrak{R}_x(\tau)-m_{x1}m_{x2})d\tau \\
& \pm \frac{2}{T_1T_2}\int_{T_0+T_1}^{T_0+T_2}T_1(\mathfrak{R}_x(\tau)-m_{x1}m_{x2})d\tau \\
& \pm \frac{2}{T_1T_2}\int_{T_0+T_2}^{T_0+T_1+T_2}(T_0+T_1+T_2-\tau)(\mathfrak{R}_x(\tau)-m_{x1}m_{x2})d\tau \\
& + \frac{2}{T_2^2}\int_0^{T_2}(T_2-\tau)(\mathfrak{R}_x(\tau)-m_{x2}^2)d\tau.
\end{aligned}
\tag{A7}
$$

From the coefficients of these integrals, it is clear that if the magnitudes of T_1 and T_2 are even moderately disparate, the noise associated with the smaller strongly dominates. This motivates preferential use of the special case $T_1 = T_2$. Referring to figs A1 and A2, if we also let $T_1 + T_2 = 2T$, the areas over which integrals (4-83) and (A7) are taken are commensurate. In this case, eq(A7) simplifies to,

$$
\sigma^2[D_x] = \frac{2}{T^2}\left(\int_0^{T}(T-\tau)(2\mathfrak{R}_x(\tau)-m_{x1}^2-m_{x2}^2)d\tau \right. \tag{A8.1}
$$

$$
\pm\int_{T_0}^{T_0+T}(\tau-T_0)(\mathfrak{R}_x(\tau)-m_{x1}m_{x2})d\tau \tag{A8.2}
$$

$$
\left. \pm\int_{T_0+T}^{T_0+2T}(T_0+2T-\tau)(\mathfrak{R}_x(\tau)-m_{x1}m_{x2})d\tau\right). \tag{A8.3}
$$

If we further let $m_{x1} = m_{x2} = m_x$ and $T_0 = 0$, we have,

$$
\begin{aligned}
\sigma^2[D_x] &= \frac{2}{T^2}\left(\int_0^{T}(2T-2\tau)(\mathfrak{R}_x(\tau)-m_x^2)d\tau\right. \\
&\qquad \pm\int_0^{T}\tau(\mathfrak{R}_x(\tau)-m_x^2)d\tau \\
&\qquad \left.\pm\int_T^{2T}(2T-\tau)(\mathfrak{R}_x(\tau)-m_x^2)d\tau\right) \\
&= \frac{4}{T}\int_0^{2T}(1-\frac{\tau}{2T})(\mathfrak{R}_x(\tau)-m_x^2)d\tau.
\end{aligned}
\tag{A9}
$$

The value for σ^2 of eq(A9) is four times that of eq(4-83). This is to be expected, because eq(A9) represents the sum of variances of two averages, each taken over an aperture half the period of that of eq(4-83). Thus, each variance will have twice the value, and their sum four times the value of the variance of eq(4-83).

The difference case of eq(A8) is of interest for calculating the variance of the difference between a pair of LEPD readings. It is anticipated that the variance is minimized for $T_0 = 0$ due to the correlation of low frequency noise components across sample apertures. It is instructive to investigate this conjecture to see under what conditions it is significant.

Assuming a band limited Gaussian process having a power spectrum $S = f_0^2/(f_0^2 + f^2)$, where f_0 is the 3dB voltage corner frequency, we calculate the autocorrelation function as its inverse Fourier transform according to eq(6-64) to be $\mathcal{R}_x(\tau) = \pi f_0 e^{-2\pi f_0 |\tau|}$.

Figure A3 is a plot of $\mathcal{R}_x(\tau)$ for various f_0. Note that $\mathcal{R}_x$ is small for $|\tau| > 1/f_0$. This suggests that averaging apertures longer than $\tau_0 = 1/f_0$ result in variance improvement proportional to the reciprocal of the aperture duration, while shorter apertures result in a sub-linear increase of variance which approaches the unaveraged signal variance in the limit as the aperture vanishes. Moreover, for the differential case characterized by eq(A8), one would expect the "sum" integral (A8.1) to behave similarly, but the "difference" integrals (A8.2)+(A8.3) to fall off more rapidly because their integration limits fall in a range of larger $|\tau|$, especially for larger T_0 (refer again to figA2).

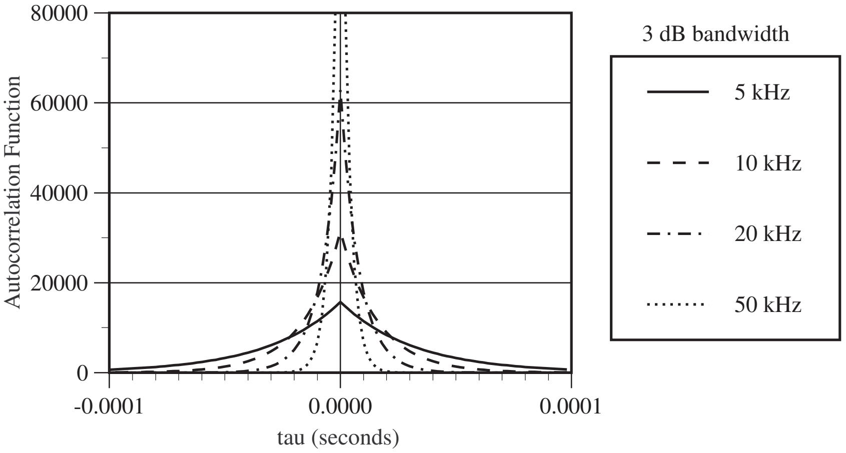

Figure A3

This is indeed the case, as illustrated in figure A4. It is also apparent that for $\tau > \tau_0$ the samples taken over T_1 and T_2 are essentially uncorrelated, so their variances add nearly linearly, irrespective of T_0, the inter-aperture period. Therefore in this case there is no noise penalty incurred by increasing the duration of the period between sample apertures.

For $\tau < \tau_0/10$, however, significant correlation occurs across the pair of sample apertures for small T_0, and the resulting noise cancellation is significant. In this regime, there is substantial noise penalty associated with increasing T_0, and it is thus desirable to take the two samples as close together as possible.

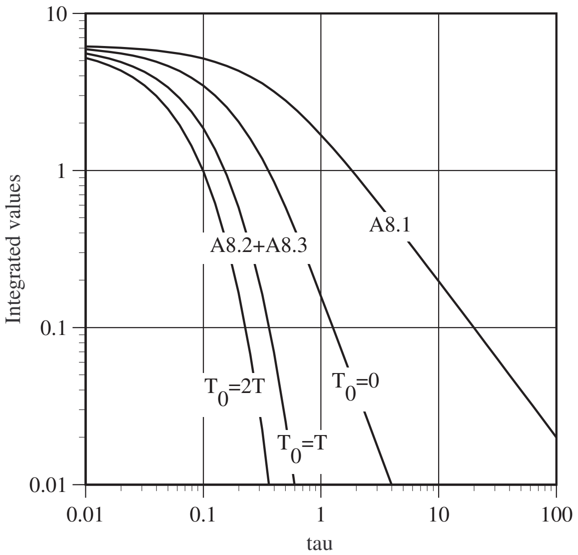

Figure A4 – integrals (A8) for $\tau_0 = 1$

APPENDIX B: MACSYMA listings

The following code provides functions implementing the math developed in this report. This code executes properly under Macsyma 419.0 for HP-UX (BSD) computers as of the date of this report as interactive keyboard input. A simplified file "hiball.mac", without derivations can be either "batched" or "loaded" to provide the functions and evaluation environment necessary for analyzing designs.

```
/*****************************************************************************
  Kirchoff's law formulation for the noise circuit model.
*****************************************************************************/
kcl1:  is + ish - ii1 - ii2 = 0;
kcl2:  ii1 + iti + in1 - if1 = 0;
kcl3:  ii2 - iti + in2 - if2 = 0;
kvl1:  -vn2 + ii2*ri2 - ii1*ri1 +vn1 = 0;
kvl2:  vo1 + vtf1 + if1*rf1 + vn1 = 0;
kvl3:  vo2 + vtf2 + if2*rf2 + vn2 = 0;

/*****************************************************************************
  Solve for preamp output voltages.
*****************************************************************************/

solve([kcl1, kcl2, kcl3, kvl1, kvl2, kvl3],
  [ii1, ii2, if1, if2, vo1, vo2]), globalsolve:true;
vo1:  expand(vo1), ri1:x*ri, ri2:(1-x)*ri;
vo2:  expand(vo2), ri1:x*ri, ri2:(1-x)*ri;


/*****************************************************************************
  Instantaneous contributions of each source, referred to amplifier outputs.
*****************************************************************************/

vo1s:  is*factor(coeff(vo1, is));
vo1sh:  ish*factor(coeff(vo1, ish));
vo1in1:  in1*coeff(vo1, in1);
vo1iti:  iti*coeff(vo1, iti);
vo1vn1:  vn1*factor(coeff(vo1, vn1));
vo1vtf1:  vtf1*coeff(vo1, vtf1);
vo1vn2:  vn2*coeff(vo1, vn2);
vo2s:  is*coeff(vo2, is);
vo2sh:  ish*coeff(vo2, ish);
vo2in2:  in2*coeff(vo2, in2);
vo2iti:  iti*coeff(vo2, iti);
vo2vn2:  vn2*factor(coeff(vo2, vn2));
vo2vtf2:  vtf2*coeff(vo2, vtf2);
vo2vn1:  vn1*coeff(vo2, vn1);


/*****************************************************************************
  Resistor and shot noise sources -- spot noise [Volts|Amps/sqrt(Hz)]
  formulas for full two-sided spectrum, (-infinity <= f <= infinity).
*****************************************************************************/

vt(r) := sqrt(2*k*T_a*r);
it(r) := sqrt(2*k*T_a/r);
isht(i) := sqrt(q*i);
```

```
/*****************************************************************************
  LEPD junction current
    d   lens diameter           [meters]
    eta lens transmittance      [1]
    ds  beacon distance         [meters]
    I   radiant intensity       [Watts/sr] (in direction of detector)
    xi  LEPD responsivity       [Amps/Watt]
*****************************************************************************/

i_o:  sfloat(%pi/4)*'xi*'eta*'I*('d/'ds)^2;


/*****************************************************************************
  Physical constants.
*****************************************************************************/

k:  1.38e-23;           /* Boltzmann's constant [Joules/Kelvin] */
q:  1.60e-19;           /* Electronic charge [Coulombs] */


/*****************************************************************************
  Design parameters
    rf  feedback resistance             [ohms]
    ri  LEPD interelectrode resistance  [ohms]
    x   centroid X position on LEPD     [1]     (0 <= x <= 1)
    id  LEPD dark current               [Amps]
    I0  radiant intensity of LED        [Watts/steradian] (on axis)
    phi angular detector position       [degrees] off LED optical axis
    ds  distance from LED to lens       [meters]
    d   diameter of lens                [meters]
    eta lens transmittance              [1]     (0 <= eta <= 1)
    xi  LEPD responsivity               [Amps/Watt]
    T_a ambient temperature             [Kelvins]
    b   bandwidth                       [Hertz]
    vn  amplifier noise voltage         [volts/root Hertz]
    in  amplifier noise current         [Amps/root Hertz]
*****************************************************************************/


/*****************************************************************************
  Design parameters expected not to be subject to frequent change.
*****************************************************************************/

T_a:  293;        /* temperature */
rho:  1;          /* assumes (worst-case) correlated vn, in for any op amp */
d:  1.7e-2;       /* lens */
eta:  0.95;
xi:  0.6;         /* UDT DLS4 LEPD (typical values) */
id:  5.0e-8;
ri:  1.0e4;
I0:  2.7e-2;      /* Siemens SFH487P LED: curve fit to manufacturer's data */
I(phi):= I0*(-1.074e-6*phi^3-2.331e-6*phi^2-2.394e-3*phi+1.000);


/*****************************************************************************
  Spot noise calculations for noise contributions to electrical outputs.
  If x is outside the range 0<=x<=1, i_o = 0 regardless of other parameters.
  Centroid of dark current is presumed to be at x = 0.5.
*****************************************************************************/
```

```
calc_v(x, phi, ds, rf, vn, in) := block([rf1,rf2,in1,in2,vn1,vn2,is],
  /* circuit constraints */
  rf1:  rf,              /* Let the feedback */
  rf2:  rf,              /* resistors be matched.  */
  in1:  in,              /* Let the amplifiers be nominally identical */
  in2:  in,              /* so their input noise currents are the same.  */
  vn1:  vn,              /* Similarly, their input */
  vn2:  vn,              /* noise voltages are the same.  */
  if 0<=x and x<=1 then is:  ev(i_o,I:I(phi)) else is:  0, /* signal current */
  /* outputs */
  v1s:  ev(vo1s) + ev(vo1s, is:id, x=0.5),
  v1sh:  sqrt(ev(vo1sh, ish:isht(is))^2 + ev(vo1sh, ish:isht(id), x=0.5)^2),
  v1in1:  ev(vo1in1),
  v1vn1:  ev(vo1vn1),
  v1iti:  ev(vo1iti, iti:it(ri)),
  v1vtf1:  ev( vo1vtf1, vtf1:vt(rf1)),
  v1vn2:  ev( vo1vn2),
  v1n:  sqrt(v1in1^2 + 2*rho*v1in1*v1vn1 + v1vn1^2
          + v1sh^2 + v1iti^2 + v1vtf1^2 + v1vn2^2),
  v2s:  ev(vo2s) + ev(vo1s, is:id, x=0.5),
  v2sh:  sqrt(ev(vo2sh, ish:isht(is))^2 + ev(vo2sh, ish:isht(id), x=0.5)^2),
  v2in2:  ev(vo2in2),
  v2vn2:  ev(vo2vn2),
  v2iti:  ev(vo2iti, iti:it(ri)),
  v2vtf2:  ev(vo2vtf2, vtf2:vt(rf2)),
  v2vn1:  ev(vo2vn1),
  v2n:  sqrt(v2in2^2 + 2*rho*v2in2*v2vn2 + v2vn2^2
          + v2sh^2 + v2iti^2 + v2vtf2^2 + v2vn1^2),
  done
  );


/***************************************************************************
  Display values of signal and noise for bandwidth f0; or spot noise if f0=0.
***************************************************************************/

show_v(f0) := block([v1n:v1n, v1sh:v1sh, v1vn1:v1vn1, v1in1:v1in1,
                    v1iti:v1iti, v1vtf1:v1vtf1, v1vn2:v1vn2,
                    v2n:v2n, v2sh:v2sh, v2vn2:v2vn2, v2in2:v2in2,
                    v2iti:v2iti, v2vtf2:v2vtf2, v2vn1:v2vn1],
  if f0=0 then f0:1 else f0:dfloat((%pi/2)*sqrt(f0)),
  display(v1s),
  v1n:  sfloat(f0*v1n), display(v1n),
  v1sh:  sfloat(f0*v1sh), display(v1sh),
  v1vn1:  sfloat(f0*v1vn1), display(v1vn1),
  v1in1:  sfloat(f0*v1in1), display(v1in1),
  v1iti:  sfloat(f0*v1iti), display(v1iti),
  v1vtf1:  sfloat(f0*v1vtf1), display(v1vtf1),
  v1vn2:  sfloat(f0*v1vn2), display(v1vn2),
  display(v2s),
  v2n:  sfloat(f0*v2n), display(v2n),
  v2sh:  sfloat(f0*v2sh), display(v2sh),
  v2vn2:  sfloat(f0*v2vn2), display(v2vn2),
  v2in2:  sfloat(f0*v2in2), display(v2in2),
  v2iti:  sfloat(f0*v2iti), display(v2iti),
  v2vtf2:  sfloat(f0*v2vtf2), display(v2vtf2),
  v2vn1:  sfloat(f0*v2vn1), display(v2vn1)
  );
```

```
/*****************************************************************************
  Calculate and show preamp output signal and noise for various parameters
*****************************************************************************/

vo(x, phi, ds, rf, vn, in, f0) := (
  calc_v(x, phi, ds, rf, vn, in),
  show_v(f0)
  );

/**** Examples ****/

vo(-1, 0, 1, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5);    /* dark current */
vo(0.5, 60, 3, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5);  /* 3 [m],  60 [deg] */
vo(0, 0, 0.6, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5);   /* 0.6 [m],  0 [deg] */


/*****************************************************************************
  Spectral density function for single-pole band-limited power spectrum case
*****************************************************************************/

block([v],
  v:  v0/(1 - %i*f/f0),                          /* [volts/sqrt(Hz)] */
  ratsimp(v*conjugate(v))
  );
S(f) := ''%;


/*****************************************************************************
  Autocorrelation function for single-pole band-limited power spectrum case
*****************************************************************************/

/* Davenport, eq(6-64) -- the autocorrelation function [volts^2]
   is the Fourier transform of the power spectral density */
tau_pos:
  integrate(S(f)*exp(%i*omega*tau), f, minf, inf), omega:2*%pi*f; pos; pos;
tau_neg:
  integrate(S(f)*exp(%i*omega*tau), f, minf, inf), omega:2*%pi*f; neg; pos;
subst(abs(tau),tau,tau_pos);
Rx(tau) := ''%;


/*****************************************************************************
  Variance of difference between two averages over apertures T1,T2, spaced T0
*****************************************************************************/

/*******************  General case *************************/

var_D(v0, f0, T0, T1, T2, mx1, mx2, sgn) := sfloat(
  2*ev(integrate((T1 - tau)*(Rx(tau) - mx1^2), tau, 0, T1)/T1^2
  + sgn*integrate((tau - T0)*(Rx(tau) - mx1*mx2), tau, T0, T0 + T1)/(T1*T2)
  + sgn*integrate(T1*(Rx(tau) - mx1*mx2), tau, T0 + T1, T0 + T2)/(T1*T2)
  + sgn*integrate((T0 + T1 + T2 - tau)*(Rx(tau) - mx1*mx2),
      tau, T0 + T2, T0 + T1 + T2)/(T1*T2)
  + integrate((T2 - tau)*(Rx(tau) - mx2^2), tau, 0, T2)/T2^2, assume_pos)
  );
```

```
/******* Difference case for T1 = T2, mx1 = mx2 = 0  *********/

var_Dx(v0, f0, T0, T) := block([ratprint:false],
  sfloat(2/T^2*(
    integrate(2*Rx(tau)*(T - tau), tau, 0, T)
  - integrate(Rx(tau)*(tau - T0), tau, T0, T0 + T)
  - integrate(Rx(tau)*(T0 + 2*T - tau), tau, T0 + T, T0 + 2*T)
  )));

/**** MACSYMA has trouble integrating if f0 > 100, but we can rescale ****/

var_Dxs(v0, f0, T0, T) := block([ratprint:false],
  v0:  v0*sqrt(f0),
  T0:  T0*f0,
  T: T*f0,
  f0:  1,
  sfloat(2/T^2*(
    integrate(2*Rx(tau)*(T - tau), tau, 0, T)
  - integrate(Rx(tau)*(tau - T0), tau, T0, T0 + T)
  - integrate(Rx(tau)*(T0 + 2*T - tau), tau, T0 + T, T0 + 2*T)
  )));


/******************************************************************************
  Mean and RMS uncertainty of estimated position -- returns [X_sig, X_noise].
  As a side effect, valid values of X_s, X_n, S1, N1, S2, and N2 are set.
******************************************************************************/

X_est(x, phi, ds, rf, vn, in, f0, T0, T) := (
  calc_v(x, phi, ds, rf, vn, in),
  S1:v1s,
  S2:v2s,
  N1:  sqrt(var_Dxs(v1n, f0, T0, T)),
  N2:  sqrt(var_Dxs(v2n, f0, T0, T)),
  calc_v(-1, 0, 1, rf, vn, in),
  S1:  S1 - v1s,
  S2:  S2 - v2s,
  X_s:  (S1-S2)/(S1+S2),
  X_n:  2*sqrt((N1*S2)^2 + 2*N1*S2*N2*S1 + (N2*S1)^2)/(S1+S2)^2,
  [X_s, X_n]
  );

/**** Example at 3 meters and 60 degrees off beacon optical axis ****/

X_est(0.0, 60, 3, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);
X_est(0.5, 60, 3, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);
X_est(1.0, 60, 3, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);

/**** Example at 0.6 meters and on beacon optical axis ****/

X_est(0.0, 0, 0.6, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);
X_est(0.5, 0, 0.6, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);
X_est(1.0, 0, 0.6, 2.0e5, 4.5e-9, 6.0e-13, 1.6e5, 2.0e-5, 1.0e-4);
```