A 2D Finite Element Formulation of Maxwell's Equations for Propagation of TEM waves between PCB Planes.

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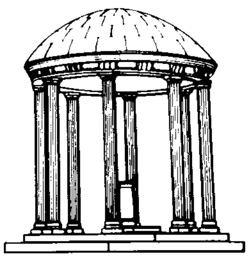

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Abstract

PCSIM is a finite element (FE) field solver for Maxwell's equations constrained to the transverse electro-magnetic (TEM) propagation of sinusoidal waves between adjacent conducting planes of a circuit board. Both conductor and dielectric losses are accounted for, including skin effect in the conductors. PCSIM is one of a suite of utilities including a mesh generator which provides a minimal simulation environment for studying the 2D properties of salphasic clock propagation. While PCSIM itself addresses a number of bookkeeping and computational details outside the scope of this report, it seemed appropriate to document its physical and mathematical basis, particularly so that other 2D FE programs can be utilized as they become available. In this report, the voltage wave equation is derived from Maxwell's equations, along with a FE formulation of this equation.

1 Introduction

PCSIM was developed as part of a simulation environment to support research in the properties of sinusoidal salphasic clock signal propagation [5]. Since lossiness is a critical factor in salphasic behavior, one of the requirements was an accurate representation of losses in both the dielectric and the conductors, including a proper accounting for skin effect. At commencement of this development, I was unable to acquire a 2D simulator capable of solving the quasi-TEM problem in a slightly lossy dielectric sandwiched between slightly lossy parallel conducting planes; thus necessitating the building of PCSIM.

The other major component developed for this simulation environment is MESHER, a program that reads a polygonal boundary description and generates a mesh of Voronoi cells which is converted by CELL2ELEM into a space filling mesh of Delaunay simplices. From CELL2ELEM, PCSIM accepts the vertex and element lists along with constitutive parameters and Neumann boundary conditions which are generated from input specifications to MESHER.

It was my original intent to document PCSIM in its entirety, consequently its mathematical development was written in the form of two appendices. But since this simulator only implements a minimal capability environment, it now seems worthwhile to separately document the mathematics representing the physics of the problem, along with the development of a corresponding FE representation. This should facilitate the retargeting of this problem to a suitable, well supported and capable 2D FE program, when one becomes available. Therefore, the entirety of this report comprises just these two appendices.

Appendix A: Derivation of Voltage Equation for Lossy 2D Planar Circuit

Let us start with Maxwell's equations and the material-dependent constitutive equations,

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}, \qquad \text{(A1)} \qquad\qquad \overline{\nabla} \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}, \qquad \text{(A2)}$$

$$\overline{\nabla} \cdot \overline{D} = \rho, \qquad \text{(A3)} \qquad\qquad \overline{\nabla} \cdot \overline{B} = 0, \qquad \text{(A4)}$$

$$\overline{D} = \epsilon \overline{E}, \quad \text{(A5)} \qquad \overline{B} = \mu \overline{H}, \quad \text{(A6)} \qquad \overline{J} = \sigma \overline{E}. \quad \text{(A7)}$$

Apply these equations to a "planar" circuit comprising two conductive plates having conductivity σ_c and permeability $\tilde{\mu}_c$, separated by a dielectric sheet having permittivity $\tilde{\epsilon}_d$ and permeability $\tilde{\mu}_d$, each layer being of uniform thickness and composition and oriented parallel to the xy plane.

The dielectric thickness h_d is assumed small relative to both signal wavelength and lateral circuit dimensions. Thus, the variation of fields with z in the dielectric can be treated as negligible, providing us the approximation $\overline{\nabla} \approx \overline{\nabla}_t = \hat{x}\partial/\partial x + \hat{y}\partial/\partial y$. Let us also treat fringing field effects at the boundary to be negligible, allowing us to approximate the $\overline{E}$ field as parallel to the z-axis and the $\overline{H}$ field as parallel to the xy-plane. This in turn allows us to use the approximation within the dielectric, $E_x = E_y = H_z = 0$.

Let the time variation of each field quantity F be restricted to a single frequency, written in "phasor" form $F(t) = F_0 e^{j\omega t}$, whereupon its time derivative is $\partial F/\partial t = j\omega F$.

Substituting the time derivative of eq(A6) into eq(A1) and taking the cross product of both sides of the resulting equation with $\hat{z}$, we obtain in the dielectric,

$$\overline{\nabla}_t E_z = j\omega\tilde{\mu}_d \overline{H}_t \times \hat{z}, \qquad \text{where} \quad \overline{H}_t = \hat{x}H_x + \hat{y}H_y \qquad \text{(A8)}$$

Similarly, substituting the time derivative of eq(A5) into eq(A2),

$$\overline{\nabla}_t \times \overline{H}_t = (j\omega\tilde{\epsilon}_d E_z + J_z)\hat{z}, \qquad \text{(A9)}$$

where J_z is a current source density term representing a "vertical" flow of current between the two conductor layers.

The voltage between the plates at location (x, y) can be calculated as $\int_{z1}^{z2} E_z dz$; but since $E_z(x, y)$ is independent of z, it is simply proportional to the dielectric thickness h_d,

$$v_d = h_d E_z. \qquad \text{(A10)}$$

The surface current density in each conducting plate is $\overline{J}_s(x, y) = \int_{z_0}^{z_\infty} \overline{J}_t(x, y, z)dz$, with z_0 taken at the surface of the plate and z_∞ taken sufficiently deep in the plate that the H field is negligible, which condition can be fulfilled for plates that are thick relative to the "skin depth". Assuming $H_z = 0$ also holds within the conducting plates and $\partial\overline{D}/\partial t$ is negligible therein, we calculate $\overline{J}_s$ from eq(A2), by taking both pre- and post-cross products of both sides with $\hat{z}$ (i.e., $\hat{z} \times \overline{A} \times \hat{z}$) and performing the integration to yield,

$$\overline{J}_s = -\overline{H}_t \times \hat{z}. \tag{A11}$$

Substituting eq(A8) into the transverse gradient of eq(A10) and then substituting eq(A11) into the result yields,

$$\overline{\nabla}_t v_d = -j\omega\tilde{\mu}_d h_d \overline{J}_s. \tag{A12}$$

Substituting eq(A9) into the transverse divergence of eq(A11), using the vector identity $\overline{\nabla}_t \cdot (\overline{A} \times \hat{z}) = \hat{z} \cdot \overline{\nabla}_t \times \overline{A}$, and substituting eq(A10) into the result yields,

$$\overline{\nabla}_t \cdot \overline{J}_s = -\frac{j\omega\tilde{\epsilon}_d}{h_d} v_d - J_z. \tag{A13}$$

The material properties of the dielectric are represented by $\tilde{\mu}_d = \mu_0\mu_d$ and $\tilde{\epsilon}_d = \epsilon_0\epsilon_d(1 - j\ tan_\delta)$ where ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space; ϵ_d is the dielectric constant, tan_δ is the dissipation factor (or tangent of the loss angle), and $\mu_d \approx 1$ is the relative permeability of the dielectric. Thus, observing that $v_d = v$ across the dielectric,

$$\overline{\nabla}_t \cdot \overline{J}_s = -bv - J_z, \qquad \text{where} \quad b = G + j\omega C, \tag{A14}$$

$$G = \omega C tan_\delta, \quad C = \epsilon_0\epsilon_d/h_d, \tag{A14a}$$

The material properties of the conductive plates are represented by ϵ_c, conductivity σ_c, and $\tilde{\mu}_c = \mu_0\mu_c$. In practical conductors of interest, we assume $\partial\overline{D}/\partial t \ll \overline{J}$, thus the value of ϵ_c becomes inconsequential in this development. From eq(A7), it is evident that a voltage gradient $\overline{E}_t$ is supported by the flow of $\overline{J}_t$ through the finite conductivity σ_c of the conductor, being of the nature of an "*IR* drop".

The *skin effect* model allows calculation of an effective surface impedance which is, paraphrased from Matick [4], $Z_s = j\omega\sqrt{\tilde{\mu}_c/j\omega\sigma_c}$. In this model, eq(A7) takes the form $\overline{J}_s = \overline{E}_s/2Z_s$ where the transverse $\overline{E}_s$ in the conducting skins can be represented as a voltage gradient $-\overline{\nabla}_t v_s$ due to the IR drops in the two conductors. Accordingly,

$$\overline{\nabla}_t v_s = -2\overline{J}_s j\omega\sqrt{\mu_0\mu_c/j\omega\sigma_c}. \tag{A15}$$

Add this to eq(A12) to obtain the total transverse voltage gradient,

$$\overline{\nabla}_t v = -\frac{1}{k}\overline{J}_s, \quad \text{where} \quad \frac{1}{k} = R + j\omega L, \tag{A16}$$

$$R = \sqrt{2\omega\mu_0\mu_c/\sigma_c}, \quad L = \mu_0\mu_d h_d + R/\omega. \tag{A16a}$$

Note that eq(A14) and eq(A16) are the equations presented by Sorrentino [3] as "a two-dimensional form of inhomogeneous telegraphists' equations", suitably generalized to account for conductor and dielectric losses. To obtain the desired voltage equation, solve eq(A16) for J_s, take the transverse divergence of both sides, and substitute eq(A14),

$$-\overline{\nabla}_t \cdot k\overline{\nabla}_t v + bv = -J_z. \tag{A17}$$

Appendix B: Finite Element Formulation for a Lossy Planar Structure

Let us follow the procedure described in Becker [1], p 140 ff (n.b., we exchange the roles of u and v as used by Becker in this treatment). Rewriting eq(A17) of Appendix A in the form of (Becker 4.2.18a)

$$-\overline{\nabla}_t \cdot (k(x,y)\overline{\nabla}_t v(x,y)) + b(x,y)v(x,y) = f(x,y), \qquad (x,y) \in \Omega \tag{B1a}$$

with boundary conditions (Becker 4.2.18c, 4.2.18d)

$$\left.\begin{aligned} v(s) &= \hat{v}(s), & s &\in \partial\Omega_1, \\ \sigma_n \equiv -k(s)\frac{\partial v(s)}{\partial n} &= p(s)(v(s) - \hat{v}(s)), & s &\in \partial\Omega_2, \end{aligned}\right\} \tag{B1b}$$

where $k = 1/(R + j\omega L)$, $b = G + j\omega C$, $f = -J_z$, $\hat{v}$ is the voltage prescribed on boundary $\partial\Omega_1$, σ_n is the flux normal to boundary $\partial\Omega_2$ having prescribed loading p, Ω is the problem domain, and both v and "test" function u are in $H^1(\Omega)$, as defined by (Becker 4.3.10). The corresponding functional (Becker 4.3.9) is,

$$\int_\Omega (k\overline{\nabla}_t u \cdot \overline{\nabla}_t v + buv)\, dx\, dy + \int_{\partial\Omega_2} puv\, ds = \int_\Omega fu\, dx\, dy + \int_{\partial\Omega_2} pu\hat{v}\, ds. \tag{B2}$$

Using Galerkin's method, substitute the following approximations for v and u:

$$\left.\begin{aligned} v_h(x,y) &= \sum_{j=1}^{N} v_j\phi_j(x,y), \\ u_h(x,y) &= \sum_{i=1}^{N} u_i\phi_i(x,y), \end{aligned}\right\} \quad \text{where} \quad \left\{\begin{array}{l} v_j, u_i \text{ are constant coefficients,} \\ \phi_j, \phi_i \text{ are real valued basis functions,} \\ N \text{ is the number of nodes in the mesh;} \end{array}\right. \tag{B3}$$

whence eq(B2) is approximated by,

$$\int_{\Omega_h} (k\overline{\nabla}_t u_h \cdot \overline{\nabla}_t v_h + bu_h v_h)\, dx\, dy + \int_{\partial\Omega_{2h}} pu_h v_h\, ds = \int_{\Omega_h} fu_h\, dx\, dy + \int_{\partial\Omega_{2h}} pu_h\hat{v}\, ds, \tag{B4}$$

where Ω_h, and $\partial\Omega_{2h}$, represent the respective domain and boundary associated with the approximating mesh.

Substituting eq(B3) into eq(B4) and reducing,

$$\sum_{j=1}^{N} K'_{ij} v_j = F'_i, \qquad i = 1 \cdots N. \tag{B5}$$

where

$$K'_{ij} = \int_{\Omega_h} (k\overline{\nabla}_t\phi_i \cdot \overline{\nabla}_t\phi_j + b\phi_i\phi_j)\, dx\, dy + \int_{\partial\Omega_{2h}} p\phi_i\phi_j\, ds, \tag{B6}$$

$$F_i' = \int_{\Omega_h} f\phi_i \, dx \, dy + \int_{\partial\Omega_{2h}} p\hat{v}\phi_i \, ds. \tag{B7}$$

To compute the values of the global K_{ij}' and F_i', the procedure outlined above for domain Ω_h is applied successively to the domain Ω_e of each element e. Each node i in Ω_h will be found in Ω_e of one or more elements (one if it is interior to the element, two if it is in a common edge between two elements, or more if it is at a common vertex of three or more elements). The values of K_{ij}' and F_i' for each node i are accumulated from the contributions by all those elements in Ω_h which contain node i. Accordingly, write

$$\sum_{j=1}^{N} K_{ij}' v_j - F_i' = \sum_{e=1}^{E} \left(\sum_{j=1}^{N} K_{ij}^e v_j - F_i^e + \sigma_i^e \right), \quad \begin{cases} i = 1 \ldots N, \\ E = \text{number of elements}; \end{cases} \tag{B8}$$

where for each element,

$$K_{ij}^e = \int_{\Omega_e} (k\overline{\nabla}_t \phi_i^e \cdot \overline{\nabla}_t \phi_j^e + b\phi_i^e \phi_j^e) \, dx \, dy, \quad (i,j) \in e, \tag{B9}$$

$$F_i^e = \int_{\Omega_e} f\phi_i^e \, dx \, dy, \quad i \in e, \tag{B10}$$

$$\sigma_i^e = \int_{\partial\Omega_e} \sigma_n \phi_i^e \, ds, \quad i \in e, \tag{B11}$$

and where the contour integrals in eq(B6) and eq(B7) are removed from eq(B9) and eq(B10) and alternatively represented in terms of σ_n in eq(B11) which can be used to introduce loads on the boundary $\partial\Omega_h$.

The contour integrals around the $\partial\Omega_e$ intersecting node i occur in two possible domains,

$$\sigma_i^e = \int_{\partial\Omega_{1h}} \sigma_n \phi_i \, ds + \int_{\partial\Omega_{2h}} \sigma_n \phi_i \, ds, \tag{B12}$$

where $\partial\Omega_{1h}$ and $\partial\Omega_{2h}$ refer to integrals along essential and natural boundaries, respectively.

For nodes i on $\partial\Omega_{1h}$ we have an essential boundary condition where we prescribe $v_i = \hat{v}_i$ rather than σ_n. The first term of eq(B12) is thus undetermined, and we designate it as a free variable σ_i^{1h}.

For nodes i on $\partial\Omega_{2h}$ we have a natural boundary condition for which we prescribe $\sigma_n(s) = p(s)v(s) - \gamma(s)$. Using the Galerkin approximation eq(B3) for v, rewrite the second term of eq(B12) as,

$$\sigma_i^{2h} = \sum_{j=1}^{N} P_{ij}^{2h} v_j - \gamma_i^{2h}, \qquad j \in \partial\Omega_{2h}, \qquad \text{where} \tag{B13}$$

$$\gamma_i^{2h} = \int_{\partial\Omega_{2h}^e} \gamma\phi_i \, ds, \tag{B14}$$

$$P_{ij}^{2h} = \int_{\partial\Omega_{2h}^e} p\phi_i \phi_j \, ds. \tag{B15}$$

Now eq(B12) becomes $\sigma_i^e = \sigma_i^{1h} + \sigma_i^{2h}$ which we substitute back into eq(B8). Collecting like terms, we produce an equation which expresses the element-wise assembly of the global K'_{ij} matrix and F'_i vector, where γ_i^{2h} and P_{ij}^{2h} are given by eq(B14) and eq(B15), respectively. It is to be understood that P_{ij}^{2h} and γ_i^{2h} vanish for $(i,j) \notin \partial\Omega_{2h}$.

$$\sum_{j=1}^{N} K'_{ij} v_j = F'_i - \Sigma_i^{1h}, \quad i = 1 \ldots N, \qquad \text{where} \tag{B16}$$

$$K'_{ij} = \sum_{e=1}^{E} (K_{ij}^e + P_{ij}^{2h}), \tag{B17}$$

$$F'_i = \sum_{e=1}^{E} (F_i^e + \gamma_i^{2h}), \tag{B18}$$

$$\Sigma_i^{1h} = \sum_{e=1}^{E} \sigma_i^{1h}, \tag{B19}$$

The free variables Σ_i^{1h} on $\partial\Omega_{1h}$ are handled as follows. Construct a matrix K_{ij} the elements of which are the same as for K'_{ij} except for rows and columns corresponding to nodes lying on $\partial\Omega_{1h}$, the elements of which are set to vanish except for the diagonal elements which are set to unity:

$$K_{ij} = \begin{cases} 1, & (i=j) \in \partial\Omega_{1h}, \\ K'_{ij}, & (i,j) \notin \partial\Omega_{1h}, \\ 0, & \text{otherwise}; \end{cases} \tag{B20}$$

and define an auxiliary matrix $M_{ij} = K'_{ij} - K_{ij}$ the elements of which are:

$$M_{ij} = \begin{cases} K'_{ij} - 1, & (i=j) \in \partial\Omega_{1h}, \\ 0, & (i,j) \notin \partial\Omega_{1h}, \\ K'_{ij}, & \text{otherwise}. \end{cases} \tag{B21}$$

Substituting eqs(B20, B21) into eq(B16), and using the summation convention over repeated indices, write:

$$K_{ij} v_j + M_{ij} v_j = F'_i - \Sigma_i^{1h}. \tag{B22}$$

Expand the left side of eq(B22):

$$F_i \equiv K_{ij} v_j = \begin{cases} v_i, & i \in \partial\Omega_{1h}, \\ \zeta_i, & i \notin \partial\Omega_{1h}, \end{cases} \qquad A_i \equiv M_{ij} v_j = \begin{cases} \zeta_i, & i \in \partial\Omega_{1h}, \\ K'_{ik} v_k, & i \notin \partial\Omega_{1h}, \end{cases}$$

where the ζ_i are undetermined, but can be solved immediately from eq(B22) to obtain,

$$K_{ij} v_j = F_i = \begin{cases} v_i, & i \in \partial\Omega_{1h}, \\ F'_i - K'_{ik} v_k, & i \notin \partial\Omega_{1h},\ k \in \Omega_{1h}; \end{cases} \quad \text{and} \tag{B23}$$

$$M_{ij} v_j = A_i = \begin{cases} F'_i - \Sigma_i^{1h} - v_i, & i \in \partial\Omega_{1h}, \\ K'_{ik} v_k, & i \notin \partial\Omega_{1h},\ k \in \Omega_{1h}. \end{cases} \tag{B24}$$

In summary, eq(B23), as constructed from eqs(B20, B17, B18) is solved for the v_i, while eq(B24), as constructed from eqs(B21, B17, B18) is used to determine the Σ_i^{1h}.

3 References

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