Propositional Logic, COMP 283

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1 Learning Objectives

- Express statements in symbolic form, using the logic operations of negation, and, inclusive or, implies, if and only if (iff), and exclusive or (and their symbols: $\neg, \wedge, \vee, \implies, \iff$, and $\oplus$) to express statements without ambiguity.
- Distinguish between inclusive and exclusive 'or'.
- Create truth tables in order to recognize tautologies and logically equivalent expressions, including De Morgan's rules and conditional expressions.
- Meet properties of logic operations and rules of logical inference, both of which help us rewrite expressions while preserving their truth values.
- Encounter Boolean algebra and circuit notations for the same logical expressions.
- Demonstrate how to use at least one of these methods to solve a logic puzzle.

2 Lecture Notes

2.1 Propositions and Basic Operations

A *proposition* is a sentence to which one and only one of the terms `True` or `False` can be applied. [DL85]

Example 1

The following are propositions:

- We are currently in Chapel Hill.
- $1 + 1 = 2$

Example 2

The following are NOT propositions:

- It is hot outside.
- $2 + x = 4$

2.1.1 Negation

Negation is represented by the NOT operator $\neg$. The negation of a predicate p is denoted as $\neg p$ and has the opposite truth value of p.

A *Truth Table* for negation would look like the following:

p	$\neg p$
T	F
F	T

What this is saying is: "When p is `True`, $\neg p$ is False, and when p is `False`, $\neg p$ is `True`."

Example 3

Here is an example of a negated proposition.

- p: "Today is Monday."
- $\neg p$: "Today is not Monday."
- *ALTERNATIVELY:* $\neg p$: "It is not the case that today is Monday."

2.1.2 Conjunction

Conjunction is represented by the AND operator $\wedge$. The conjunction ("and") of propositions p and q, denoted by $p \wedge q$, is `True` when both p and q are `True` and is `False` otherwise.

A Truth Table for conjunction would look like the following:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 4

Here is an example of conjunction.

- p: "It is sunny today."
- q: "It is Monday."
- $p \wedge q$: "It is sunny today and today is Monday."

  The conjunction $(p \wedge q)$ is `True` on sunny Mondays, but it is `False` on any non-sunny day (p is `False`), and it is `False` on any day that is not Monday (q is `False`).

2.1.3 Disjunction

Disjunction is represented by the OR operator $\vee$. The disjunction ("or") of propositions p and q, denoted by $p \vee q$, is `True` when either p or q is `True` and is `False` otherwise.

A Truth Table for disjunction would look like the following:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 5

Here is an example of disjunction.

- p: "It is sunny today."
- q: "It is Monday."
- $p \vee q$: "It is sunny today or today is Monday."

  The disjunction $(p \vee q)$ is `True`on sunny Mondays, on any sunny day (p is `True`), and on any Monday (q is `True`). It is `False`on any day where it is both not sunny and not Monday.

2.1.4 Exclusive Or

Exclusive Or is represented by the XOR operator $\oplus$. The exclusive or ("xor") of p and q, denoted by $p \oplus q$, is True when exactly one of p and q is True, and False otherwise.

A Truth Table for exclusive or would look like the following:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Example 6

Here is an example of exclusive or.

- p: "It is sunny today."
- q: "It is Monday."
- $p \oplus q$: "It is sunny today or today is Monday, but not both."

  The exclusive or ($p \oplus q$) is True on any sunny day (p is True) and on any Monday (q is True), EXCEPT for on sunny Mondays (both p and q are True). Additionally, it is False on any day where it is both not sunny and not Monday.

2.1.5 Conditionals

The *conditional statement* $p \implies q$ is False when p is True and q is False, and True otherwise. p is called the hypothesis and q the conclusion. This can also be called *implication*.

Some English phrases for this would be:

- If p, then q.
- p implies q.
- q if p.

A Truth Table for implication would look like the following:

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 7

Here is an example of implication.

- p: "It is sunny."
- q: "I walk to campus."
- $p \implies q$: "If it is sunny, then I walk to campus."

  The implication ($p \implies q$) is False if it is sunny and I do NOT walk to campus. Otherwise, it is True. Note that if it is not sunny (p is False) and I still walk to campus (q is True), this implication is still True.

2.1.6 Biconditionals

The *biconditional statement* ("if and only if" or "iff") $p \iff q$ is True when p and q have the same truth value, and False otherwise.

A Truth Table for implication would look like the following:

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 8

Here is an example of implication.

- p: "It is sunny."
- q: "I walk to campus."
- $q \iff p$: "I walk to campus if and only if it is sunny."

  The biconditional ($p \iff q$) is `True` if it is sunny and I walk to campus or if it is not sunny and I don't walk to campus. Otherwise, it is `False`. Note that, unlike the previous example, if it is not sunny (p is `False`) and I still walk to campus (q is `True`), this biconditional is `False`.

2.1.7 Useful Equivalences

One useful equivalence is **DeMorgan's Law**. It states:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Another useful equivalence is:

$$p \implies q \equiv \neg p \vee q$$

2.1.8 Other Considerations + Terminology

- A truth table computes all possible combinations of n propositions, so a truth table always has 2^n rows.
- Notice that a combination of propositions using operators (e.g. $p \wedge q$ makes a new proposition. These can also be called *compound propositions.*
- If two compound propositions have the same truth tables, they are considered *logically equivalent.*
- A compound proposition with a truth table where all the values in the last column are `True`is called a *tautology.* Another way of saying this is: for any truth value of the propositions, a tautology will always be `True`. (An example of this is $p \vee \neg p$.)

2.2 Proofs Using Truth Tables

Example 9

We are going to use truth tables to prove the first part of DeMorgan's Law: $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

We make a truth table with columns for both sides of the equivalence, $\neg(p \wedge q)$ and $\neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	**F**	**F**
T	F	F	T	F	**T**	**T**
F	T	T	F	F	**T**	**T**
F	F	T	T	F	**T**	**T**

The columns for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are equal, so this means $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent!

Example 10

We are going to use truth tables to prove $p \vee \neg p$ is a tautology.

p	$\neg p$	$p \vee \neg p$
T	F	**T**
F	T	**T**

For all values, $p \vee \neg p$ evaluates to `True`! By definition, this means it is a tautology.

2.3 Translating English Sentences

To translate logic to and from English sentences, it is important to know the common phrases for the operators.

- $\neg p$ (Negation): "not p"
- $p \wedge q$ (Conjunction): "p and q"
- $p \vee q$ (Disjunction): "p or q"
- $p \oplus q$ (Exclusive Or): "p xor q"; "p or q, but not both"
- $p \implies q$ (Conditional/Implication): "p implies q"; "if p then q"; "q if p"
- $p \iff q$ or p iff q (Biconditional): "p if and only if q"

Example 11

Here's an example of breaking up an English language sentence.

Start with the sentence: *"You cannot ride the roller coaster if you are under 4 feet tall."*

Then, you break your sentence to the smallest propositions possible.

a = *"You can ride the roller coaster"* b = *"you are under 4 feet tall"*

Now your sentence is: *Not a if b.*

So you know you can write it as: $b \implies \neg a$.

If it doesn't directly match any of the phrases listed above, use truth tables to see what it matches.

2.4 Using Logic For Problem Solving

The general steps for using logic to solve a problem are the following:

1. Define convenient variables for propositions.
2. Transform what they say into statements that *always* should evaluate to `True` in our world. These are the rules of our world.

3. Make a truth table.

4. Check if a unique row makes both statements true.

Example 12: Knights and Knaves by Raymond Smullyan

On an island, every inhabitant is a knight who always tells the truth, or a knave who always lies. You meet three inhabitants, Alice, Bob, and Chris.

Alice says: Bob is a knave or Chris is a knight. Bob says: Alice is a knight if, and only if, Chris is a knave.

Can you determine uniquely what each of Alice, Bob, and Chris are?

1. Define convenient variables for propositions.

$A =$ "Alice is a knight", $B =$ "Bob is a knight", $C =$ "Chris is a knight"

$\neg A =$ "Alice is a knave", $\neg B =$ "Bob is a knave", $\neg C =$ "Chris is a knave"

2. Transform what they say into statements that *always* should evaluate to `True`. These are the rules of our world.

Alice says: $\neg B \vee C$ Bob says: $A \iff \neg C$

You might be inclined to make these statements your rules of the world, but these rules are only true if Alice and Bob are telling the truth (aka they are knights).

- $A \implies (\neg B \vee C) \leftarrow$ This holds whether or not Alice is a liar. - $B \implies (A \iff \neg C) \leftarrow$ This holds whether or not Bob is a liar.

3. Make a truth table

A	B	C	$A \implies (\neg B \vee C)$	$B \implies (A \iff \neg C)$
T	T	T	T	F
T	T	F	F	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

4. Check if a unique row makes both statements true.

All of the values in blue are assignments that could work!

Let's test one of these assignments!

Test assignment: $A = T, B = F, C = T$

So, Alice is a knight, Bob is a knave, and Chris is a knight.

Alice says: Bob is a knave or Chris is a knight.

$A \implies (\neg B \vee C)$

$T \implies (\neg(F) \vee T)$ evaluates to True!

Bob says: Alice is a knight if, and only if, Chris is a knave.

$B \implies (A \iff \neg C)$

$F \implies (T \iff \neg T)$ evaluates to True!

Both of our rules hold in this world with these assignments, so we know that this solution works!

Why this isn't a good problem...

This problem isn't explicit enough, so we found ourselves making an assumption.

When we solved it, we assumed that if someone is lying, they don't actually know whether or not what they are saying is the truth, so their claim could either be true or false. This is why we were able to use implication.

For example, from the rule "Alice says: Bob is a knave or Chris is a knight.", we get $A \implies (\neg B \vee C)$.

However, what if we assumed that if someone is lying, they know their claim is false? Then we would have to explain our rules using a biconditional.

$A \iff (\neg B \vee C)$

Therefore, our answer is correct for what we assume the rules of the world are, but that might not match the author of the puzzle's original intention. This shows the benefit of logic! We can use it to address these ambiguities!

3 Additional Resources

Khan Academy Videos
Shaun Teaches Videos

4 Acknowledgements

Content for these lecture notes was taken from lecture notes by Jack Snoeyink (UNC) [Sno21], Carola Wenk (Tulane) [Wen15], and Tiffany Barnes (NCSU) [Bar21].

References

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