First Order Logic: Quantifiers, COMP 283

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1 Lecture Notes

Definition 1

A **predicate** is a function that maps each possible input to either `True` or `False`. [Sno21]

Example 1

Here are two predicates each taking their input x from a set of days, D:
$p(x) =$ "It rained in the morning on day x," and
$q(x) =$ "I walked to campus on day x."
We can combine these to write the statement "If it did not rain in the morning on day x, then I walked to campus on day x" as $\neg p(x) \implies q(x)$ [a]

[a] In the textbook, $\implies$ is written as $\rightarrow$ and $\neg x$ is written as $\bar{x}$.

We are going to introduce the next concept with an example...

Example 2

Let's say that my previous statement only applies on Mondays, Wednesdays, and Fridays.
How can we say this using what we already learned?
$(\neg p(\text{Mon}) \implies q(\text{Mon})) \wedge (\neg p(\text{Wed}) \implies q(\text{Wed})) \wedge (\neg p(\text{Fri}) \implies q(\text{Fri}))$
We can also express this using set notation. Say $D = \{\text{Mon, Wed, Fri}\}$.
Similar to how you use a summation $\sum$ for a sequence of additions, we can use the big and $\bigwedge$ to represent a sequence of ands.
$\bigwedge_{d \in D} (\neg p(d) \implies q(d))$ We can also write this using a **quantifier**.

Definition 2

The "for all" quantifier, denoted $\forall$, is used to reason about all elements of a set.

Example 3

Back to our previous example, we already showed that for $D = \{\text{Mon, Wed, Fri}\}$,

$$(\neg p(\text{Mon}) \implies q(\text{Mon})) \wedge (\neg p(\text{Wed}) \implies q(\text{Wed})) \wedge (\neg p(\text{Fri}) \implies q(\text{Fri}))$$
$$\equiv \bigwedge_{x \in D} (\neg p(x) \implies q(x))$$

There is another way we can say this.

$\forall d \in D, (\neg p(d) \implies q(d))$.

We can do something similar with "or" statements. For this we will introduce another quantifier.

Definition 3

The "there exists" quantifier, denoted $\exists$, is used to reason about at least one element of a set.

Example 4

Now let's say that for at least one day of Monday, Wednesday and Friday, if it's not raining on day x, then I walk to campus on day x.

This can be written using logical or, big or, or with the "there exists" quantifier.

$$\begin{aligned}(\neg p(\text{Mon}) \implies q(\text{Mon})) \vee (\neg p(\text{Wed}) \implies q(\text{Wed})) \vee (\neg p(\text{Fri}) \implies q(\text{Fri}))\\ \equiv \bigvee_{x \in D} (\neg p(x) \implies q(x))\\ \equiv \exists x \in D, (\neg p(x) \implies q(x))\end{aligned}$$

Definition 4

A variable specified with a specified domain is a **bounded variable**. A variable without a specified domain is a **free variable**.

Example 5

In the preposition $\forall x \in \mathbb{Z}, f(x, y)$,

x is a bounded variable and y is a free variable.

Example 6

For $\sum_{k=0}^{10} (k+n)$, k is a bounded variable and n is a free variable.

1.1 Negation and Inference

Since 'for all' is a big 'and,' and 'exists' is a big 'or,' de Morgan's laws say that the negation of one is the other (with its statement negated.) That is:

Definition 5

$\neg(\forall x, p(x)) \equiv \exists x, \neg p(x)$ and

$\neg(\exists x, p(x)) \equiv \forall x, \neg p(x)$

Why are these true?

Example 7

$\neg(\forall x, p(x))$ in English translates to "$p(x)$ does not hold for all x". This is equivalent to saying, "There exists an x where $p(x)$ does not hold", or $\exists x, \neg p(x)$.

Similarly, $\neg(\exists x, p(x))$ in English translates to "There does not exist x such that $p(x)$ holds.

This is equivalent to saying "For all x, $p(x)$ does not hold", or $\forall x, \neg p(x)$.

Definition 6: Rules of Inference

- **Universal Generalization** says that if we know $p(x)$ is true for whatever element x of X that our adversary may challenge us with then we may conclude $\forall x \in X, p(x)$.
- **Existential generalization** says that if we can choose a specific element a $a \in X$ for which $p(a)$ is true, then we may conclude $\exists x \in X, p(x)$; it is an application of absorption.
- **Universal instantiation** says that if we know $\forall x \in X, p(x)$ then we can conclude $p(a)$ for any specific choice of $a \in X$; it is an application of simplification.
- **Existential instantiation** says that if we know $\exists x \in X, p(x)$ we can conclude $p(y)$ for a variable y that is not currently in use, but whose value now becomes fixed so that $p(y)$ is true. [Sno21]

Example 8: Nested Quantifiers

This example is to help you get some practice with nested quantifiers and to understand that the order of them matters.

Say that $loves(x, y)$ is true iff person x loves person y.

$\forall_{e \in P}\ \exists_{s \in P}, loves(e, s)$ translates to "Everybody loves somebody."

$\exists_{s \in P}\ \forall_{e \in P}, loves(e, s)$ translates to "There is somebody that everybody loves."

1.1.1 Other Notation

Here are some other ways we can write things.

If we are talking about pairs of distinct integers, $i, j \in [1..n]$ with $i < j$, we may even write $\forall_{1 \leq i < j \leq n}$ or $\exists_{1 \leq i < j \leq n}$.

If we are talking about elements x and y that are members of the same set S, we can write $\forall_{x,y \in S}$ or $\exists_{x,y \in S}$.

2 Acknowledgements

Content for these lecture notes was taken from lecture notes by Jack Snoeyink (UNC) [Sno21], Carola Wenk (Tulane) [Wen15], and Tiffany Barnes (NCSU) [Bar21].

References

[Bar21] Tiffany Barnes. Discrete mathematics lecture notes. 2021.

[Sno21] Jack Snoeyink. Discrete mathematics lecture notes. 2021.

[Wen15] Carola Wenk. Discrete mathematics lecture notes. http://www.cs.tulane.edu/~carola/teaching/cmps2170/fall15/slides/index.html, 2015.