3D Transformations

Computer Graphics
COMP 770 (236)
Spring 2007

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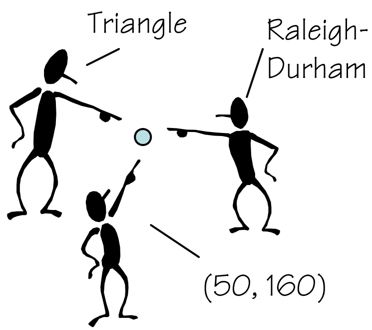
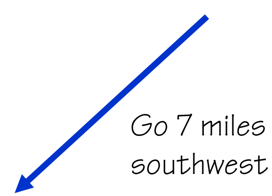
Geometry

- Geometric entities, such as points in space, exist without numbers.
- Coordinates are a naming scheme.
  - The same point can be described by different coordinates.
  - Both vectors and points expressed by coordinates, but they are very different
- Our plan

  1. understand the “things”
  2. THEN associate coordinates to them.





Scalar Field

- Definition. A set S over which addition (+) and multiplication (·) are closed.

$$\forall a, b \in S \quad a + b \in S \quad a \cdot b \in S$$

- These operators commute, associate, and distribute

$$\forall a, b, c \in S$$

$$a + b = b + a \quad a \cdot b = b \cdot a$$

$$a + (b + c) = (a + b) + c \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

- Both operators have a unique identity element

$$a + 0 = a \quad a \cdot 1 = a$$

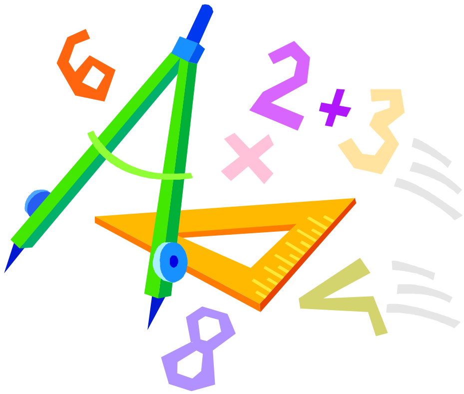
- Each element has a unique inverse under both operators

$$a + (-a) = 0 \quad a \cdot a^{-1} = 1$$

Example Scalar Fields

- Real Numbers
- Complex Numbers
  (given the standard definitions for addition and multiplication)
- Rational Functions
  (Ratios of polynomials)
- Notation: we will represent scalars by lower case letters.

  a, *b*, *c*, … are scalar variables.



Vector Spaces

- Vector space (**V**): scalars and vectors, denoted by $\vec{x}$.
- Two operations for vectors:
  - vector-vector addition $\forall \vec{u}, \vec{v} \in V \quad \vec{u} + \vec{v} \in V$
  - scalar-vector multiplication $\forall \vec{u} \in V, \forall a \in S \quad a\vec{u} \in V$
- Vector-vector addition commutes and associates.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

- There is also an additive identity, and an additive inverse for each vector

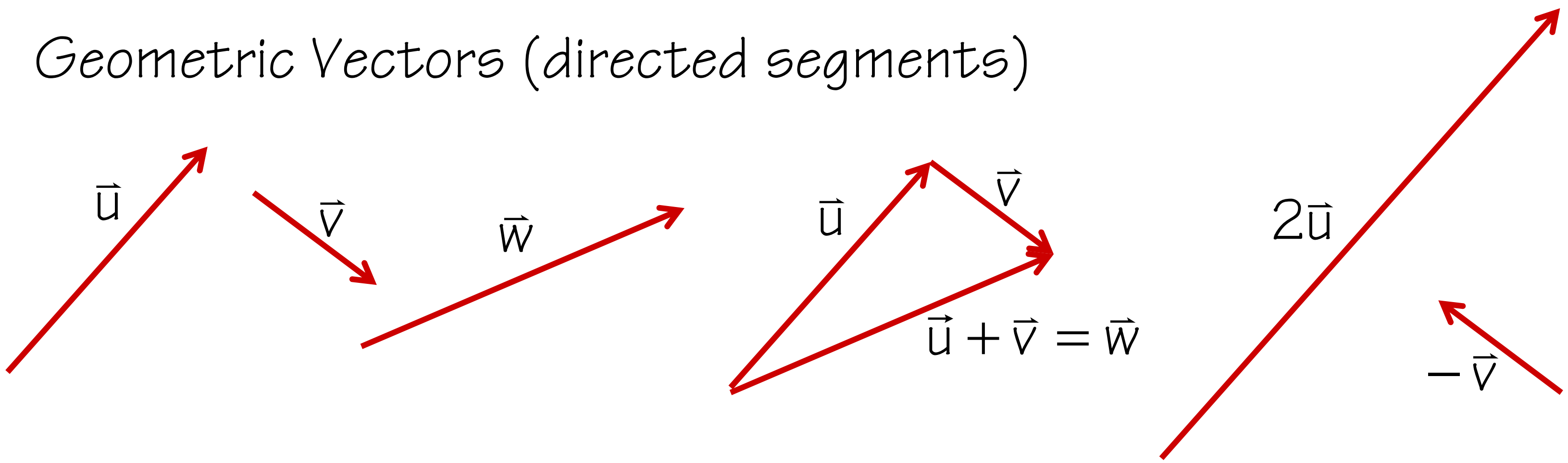
$$\vec{u} + \vec{0} = \vec{u} \quad \vec{u} + (-\vec{u}) = \vec{0}$$

- Scalar-vector multiplication distributes

$$(a+b)\vec{u} = a\vec{u} + b\vec{u} \quad a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$$

Example Vector Spaces

- Geometric Vectors (directed segments)



- N-tuples of scalars

$$\vec{u}=(1,3,7)^t \qquad \vec{u}+\vec{v}=(3,5,4)^t=\vec{w}$$

$$\vec{v}=(2,2,-3)^t \qquad 2\vec{u}=(2,6,14)^t$$

$$\vec{w}=(3,5,4)^t \qquad -\vec{v}=(-2,-2,3)^t$$

- Not coincidentally, we can use N-tuples to represent vectors

Basis Vectors

- A vector basis is a subset of vectors from **V** that can be used to generate any other element in **V**, using just additions and scalar multiplications.
- A basis set, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, is *linearly dependent* if:

$$\exists a_1, a_2, \ldots, a_n \neq 0 \quad \text{such that} \quad \sum_{i=0}^{n} a_i \vec{v}_i = 0$$

Otherwise, the basis set is *linearly independent*. A linearly independent basis set with *i* elements is said to *span* an *i-dimensional* vector space.

Basis vectors are physical things, not numbers.

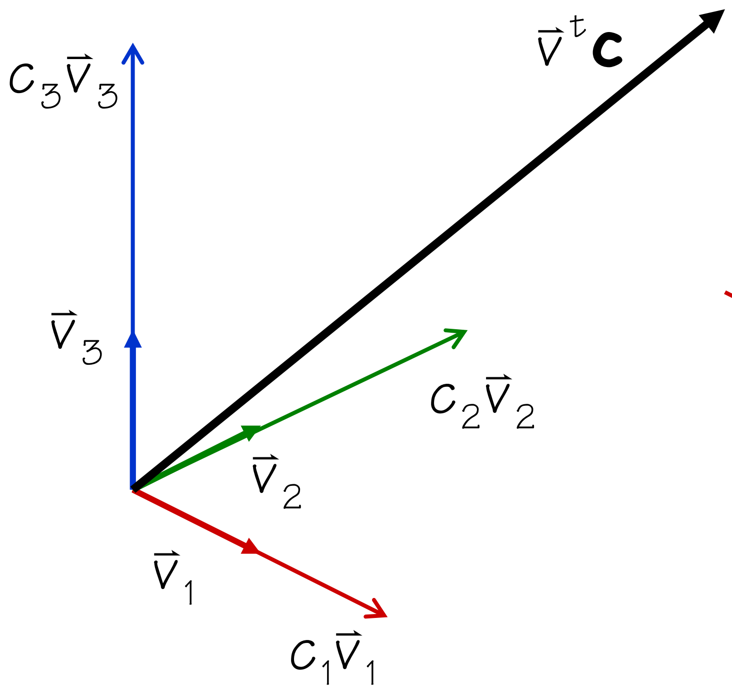
Vector Coordinates

- A linearly independent basis set can be used to uniquely name or address a vector. This is the done by assigning the vector coordinates as follows:

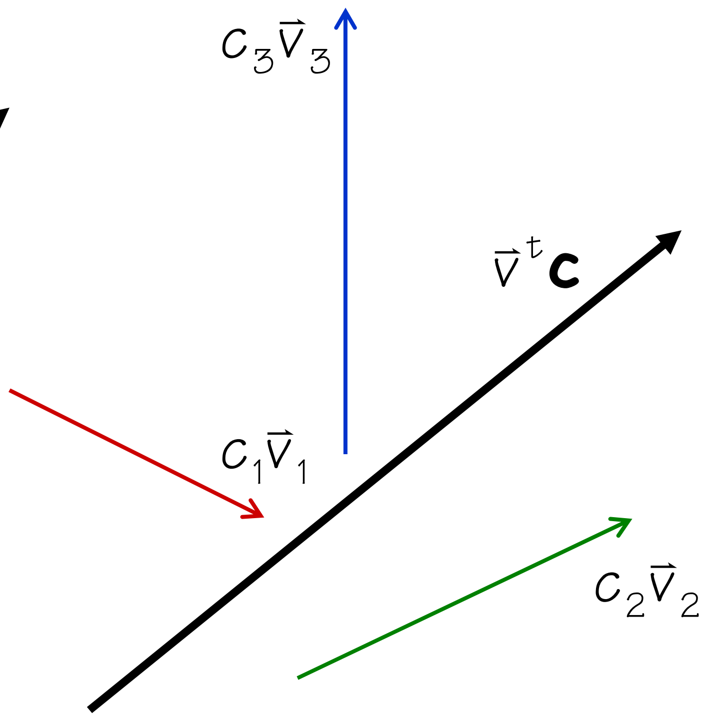
$$\vec{x} = \sum_{i=1}^{3} c_i \vec{v}_i = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{v}^{\,t} \mathbf{c}$$

- Note: we'll use **bold** letters to indicate tuples of scalars that are interpreted as coordinates
- Our vectors are still abstract entities. So how do we interpret the equation above?

Interpreting Vector Coordinates



Valid Interpretation



Equally Valid Interpretation

Remember, vectors don't have any notion of position

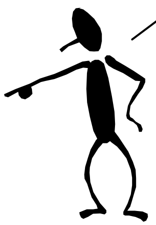
Linear Transforms

A linear transformation, **L**, is just a mapping from **V** to **V** which satisfies the following properties:

$$\mathbf{L}(\vec{u}+\vec{v}) = \mathbf{L}(\vec{u}) + \mathbf{L}(\vec{v}) \quad \text{and} \quad \mathbf{L}(a\vec{u}) = a\mathbf{L}(\vec{u})$$

Linearity implies:

$$\vec{x} \Rightarrow \mathbf{L}(\vec{x}) = \mathbf{L}\left(\sum_i c_i \vec{v}_i\right) = \sum_i c_i \mathbf{L}(\vec{v}_i)$$



Transforms my basis vectors and leaves the coordinates unchanged

Expressing $\vec{x}$ with a basis and coordinate vector gives:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{L}(\vec{v}_1) & \mathbf{L}(\vec{v}_2) & \mathbf{L}(\vec{v}_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Matrices

Linear transformations are equivalent to those that can be expressed using matrices and matrix operations.

$$\begin{bmatrix} \mathbf{L}(\vec{v}_1) & \mathbf{L}(\vec{v}_2) & \mathbf{L}(\vec{v}_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

We can interpret this expression in one of two ways

$$\left(\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

change of basis vectors

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \left(\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right)$$

change of coordinates

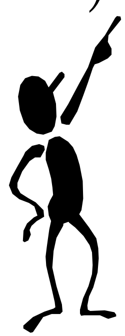
Reading Matrix Expressions

Often we desire to apply sequences of operations to vectors. For instance, we might want to rotate a particular vector, add it to some other vector, and then rotate the result back. In order to specify and interpret such sequences, you should become proficient at reading matrix expressions.

Consider the following expression:

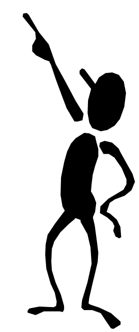
$$\vec{v}^{\,t}\mathbf{c} \Rightarrow \vec{v}^{\,t}\mathbf{Mc} = \left(\vec{v}^{\,t}\mathbf{M}\right)\mathbf{c} = \vec{m}^{\,t}\mathbf{c}$$

Think of this as changing from one space to another (i.e. world space to eye space)



$$\vec{v}^{\,t}\mathbf{c} \Rightarrow \vec{v}^{\,t}\mathbf{Mc} = \vec{v}^{\,t}\left(\mathbf{Mc}\right) = \vec{v}^{\,t}\mathbf{d}$$

Think of this as moving an vector, changing its coordinates, within a common space. (i.e. rotate a normal around some axis)



The Basis is Important!

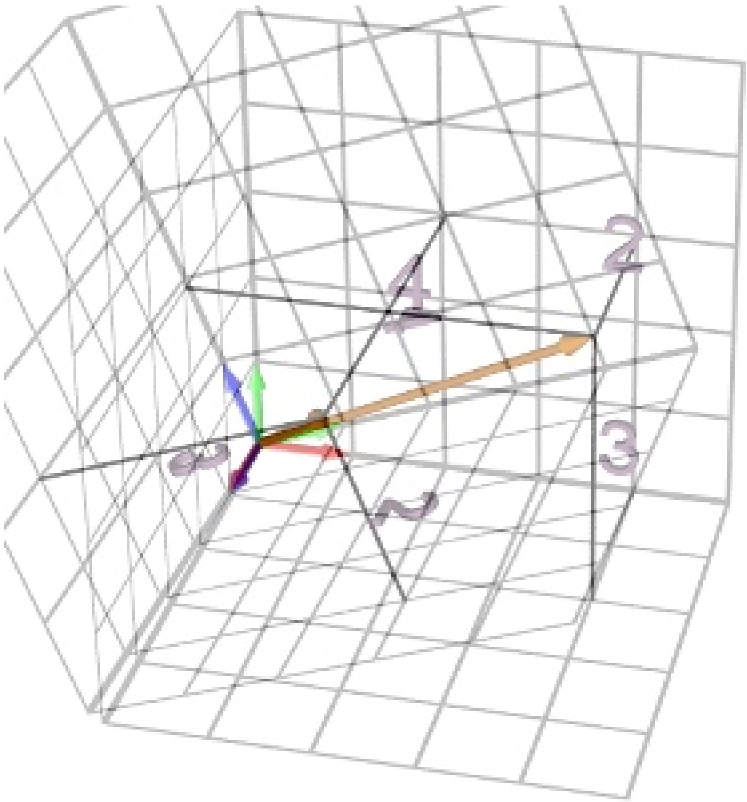
If you are given coordinates and told to transform them using a matrix, you have not been given enough information to determine the final mapping.

Consider the matrix:

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we apply this matrix to coordinates there *must* be some implied basis, because coordinates are not geometric entities (a basis is required to convert coordinates into a vector). Assume this implied basis is $\vec{w}^t$. Thus, our coordinates describe the vector $\vec{v} = \vec{w}^t \mathbf{c}$. The resulting transform, $\vec{w}^t \mathbf{c} \Rightarrow \vec{w}^t \mathbf{M}\mathbf{c}$, will stretch this vector by a factor of 2 in the direction of the first element of the basis set. Of course that direction depends entirely on $\vec{w}^t$.

Transformation Example



$$\vec{w}^t = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$$

$$\vec{n}^t = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$$

These vectors with identical initial and final coordinates are very different geometric entities

Points

Conceptually, Points and Vectors are very different.

A point is a place in space. A vector describes a direction independent of position. As mentioned previously, we will distinguish between points and vectors in our notation.

Points are denoted as $\dot{p}$ and vectors as $\vec{v}$.

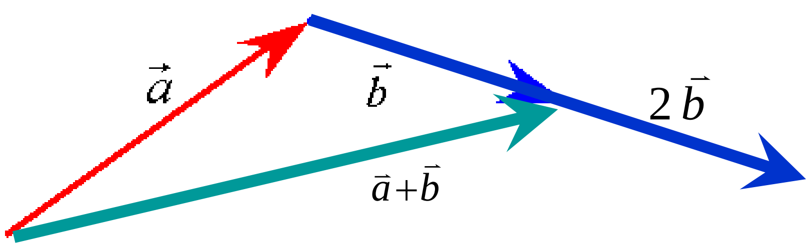
Furthermore, we will consider vectors to live in the **Linear** space $\mathbf{R}^3$ and points to to live in the **Affine** space $\boldsymbol{A}^3$.

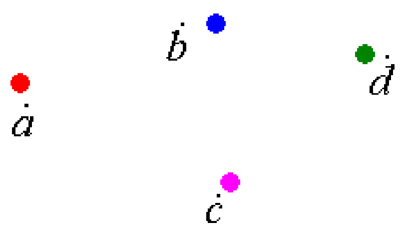
Let's clarify this distinction.

How Vectors and Points Differ

The operations of addition and multiplication by a scalar are well defined for vectors. The addition of 2 vectors expresses the concatenation of 2 "motions". Multiplying a vector by some factor scales the motion.

However, these operations don't make sense for points. What should it mean to add two points together? For example, what is Raleigh plus Durham? What does it mean to multiply a point by an arbitrary scalar? What is 7 times Chapel Hill?

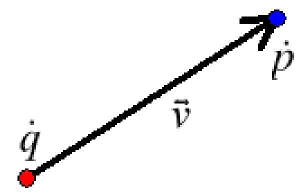




Making Sense of Points

There are some operations that do make sense for points. For instance, if you want to compute a vector that describes the motion from one point to another.

$$\dot{p} - \dot{q} = \vec{v}$$



We can also find a new point that is some vector away from a given point.

$$\dot{q} + \vec{v} = \dot{p}$$

A Basis for Points

One of the goals of our definitions is to make the subtle distinctions between points and vectors more apparent. The key distinction between vectors and points are that points are *absolute* whereas vectors are *relative*. We can capture this notion in our definition of a basis set for points. A vector space is completely defined by a set of basis vectors, however, the space that points live in requires the specification of an absolute origin.

$$\dot{p} = \dot{o} + \sum_i \vec{v}_i c_i = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

Notice how 4 scalars (one of which is 1) are required to identify a 3-D point.

Frames

We distinguish between spaces that points live in and spaces that vectors live in by our basis definition. We will call the spaces that points live in *Affine spaces*, and explain why shortly. We will also call affine-basis-sets ***frames***.

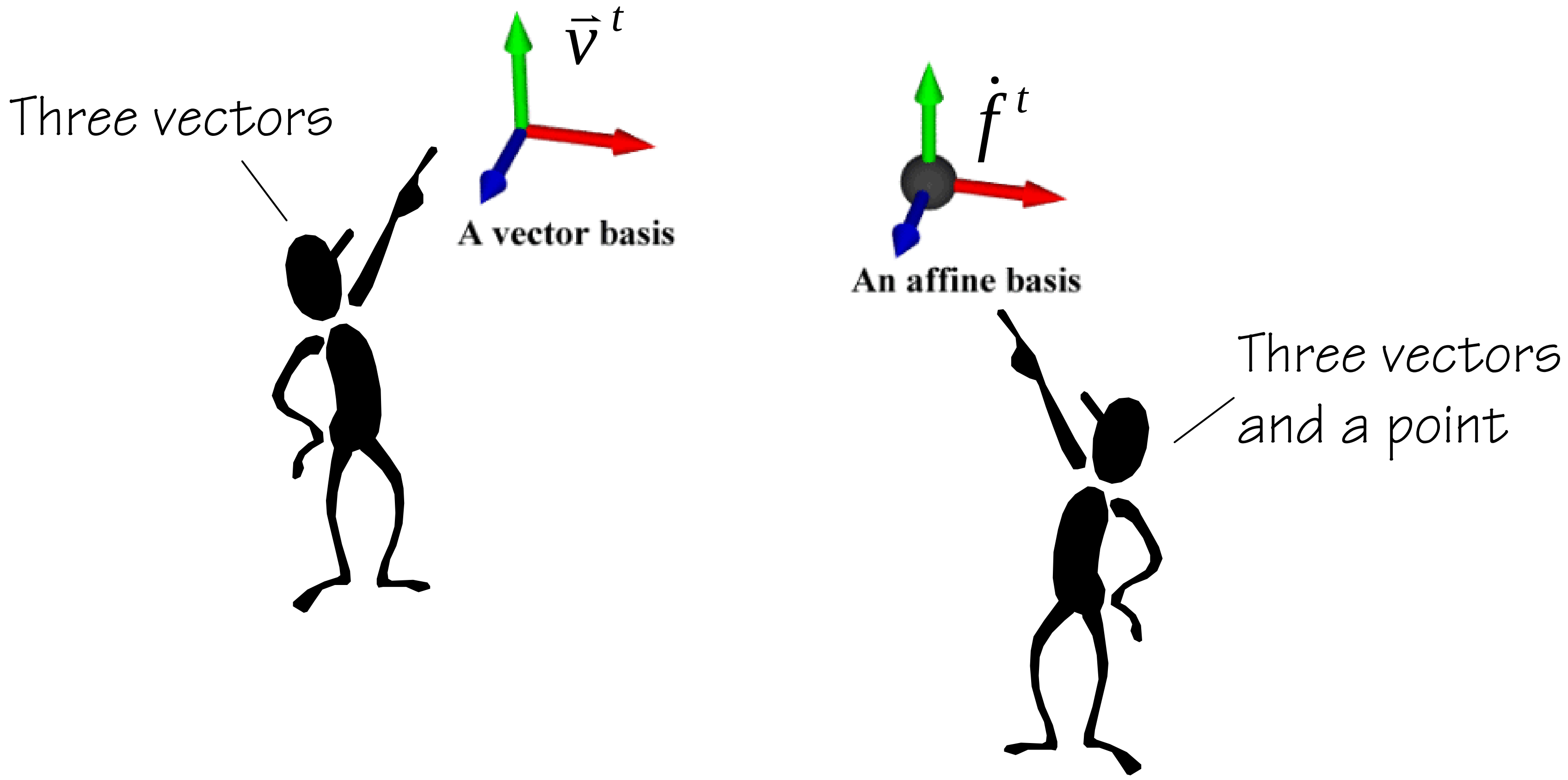
$$\dot{f}^t = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix}$$

BTW, frames can describe vectors as well as points.

$$\dot{p} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix}$$

Pictures of Frames

Graphically, we will distinguish between vector bases and affine bases (frames) using the following convention.



A Consistent Model

Note how the behavior of affine frame coordinates is completely consistent with our intuition. Subtracting two points yields a vector. Adding a vector to a point produces a point. If you multiply a vector by a scalar you still get a vector. And, in most cases, when you scale points you'll get some nonsense 4th coordinate element, which should serve to remind you that the thing you're left with is no longer a point.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + v_1 \\ a_2 + v_2 \\ a_3 + v_3 \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Notice, how we have snuck up on the idea of **Homogeneous Coordinates**, based on simple logical arguments. Keep the following in mind, coordinates are not geometric, they are just scales for basis elements. Thus, you should not be bothered by the fact that our coordinates suddenly have 4 numbers. We could have had more (no one said we have to have a linearly independent basis set).

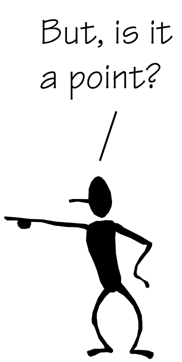
When you realize that 3-D homogeneous coordinates refer to an affine frame with its 3 basis vectors and origin point, the 4 coordinates suddenly make sense. Our 4th coordinate can have one of two values, [0,1], indicating if whether the coordinates name a vector or a point.

Affine Combinations

There are certain situations where is *does* make sense to scale and add points.

If you add scaled points together carefully, you can end up with a valid point. Suppose you have two points, one scaled by α_1 and the other scaled by α_2. If we restrict the sum of these alphas, $\alpha_1 + \alpha_2 = 1$, we can assure that the result will have *1* as it's 4th coordinate value.

$$\alpha_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 a_1 + \alpha_2 b_1 \\ \alpha_1 a_2 + \alpha_2 b_2 \\ \alpha_1 a_3 + \alpha_2 b_3 \\ \alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 a_1 + \alpha_2 b_1 \\ \alpha_1 a_2 + \alpha_2 b_2 \\ \alpha_1 a_3 + \alpha_2 b_3 \\ 1 \end{bmatrix}$$



The Points Between

This combination, defines all points that share the line connecting our two initial points. This idea can be simply extended to 3, 4, or any number of points. This type of constrained-scaled addition is called *affine combination* (hence, the name of our space).

In fact, one could define an entire space in terms of the affine combinations of elements by using the α_i's as coordinates, describing one of our constrained combination of basis points. This leads to a description called Barycentric Coordinates, but that is a topic for another day.

Affine Transformations

As with vectors, we can apply Linear transformations to points using matrices. However, we will need to use 4 by 4 matrices since our basis set has four components. However, we will initially limit ourselves to transforms that preserve the integrity of our points and vectors. Literally, those transforms that produce a point or vector when given a one of the same.

$$\dot{p} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \Rightarrow \dot{p}' = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

This subset of 4 by 4 matrices has the property that it preserves points and vectors. This subset of matrices is called, you guessed it, the *affine* subset.

Composing Transformations

We will often want to specify complicated transformations by stringing together sequences of simple manipulations. For instance, if you want to translate points and then rotate them about the origin. Suppose that the translation is accomplished by the matrix operator **T**, and the rotation is achieved using the matrix, **R**.

Given what we know now, it is a simple matter to construct this series of operations.

$$\dot{p} = \dot{w}^t \mathbf{c} \Rightarrow \dot{p}' = \dot{w}^t \mathbf{R}\mathbf{T}\mathbf{c} = \dot{w}^t(\mathbf{R}(\mathbf{T}\mathbf{c})) = \dot{w}^t(\mathbf{R}\mathbf{c}') = \dot{w}^t \mathbf{c}''$$

Each step in the process can be considered as a change of coordinates.

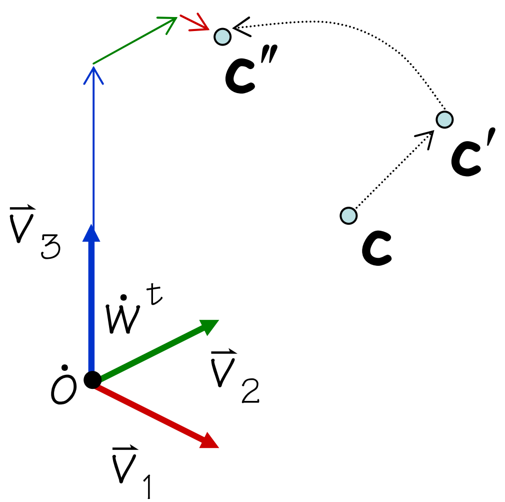
Alternatively, we could have considered the same sequence of operations as:

$$\dot{p} = \dot{w}^t \mathbf{c} \Rightarrow \dot{p}' = \dot{w}^t \mathbf{R}\mathbf{T}\mathbf{c} = ((\dot{w}^t \mathbf{R})\mathbf{T})\mathbf{c} = (\dot{m}^t \mathbf{T})\mathbf{c} = \dot{e}^t \mathbf{c}$$

Where each step is considered as a change of basis.

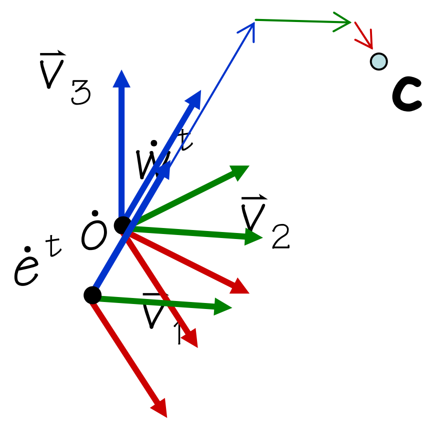
Same Points

These are alternate interpretations of the same transformations. They mean entirely different things, however they result in the same set of transformed coordinates. The first sequence is considered as a transformation about a *global* frame. The second sequence is considered as a change in *local* frames. Frequently, we will mix together these notions (moving points and changing coordinates) in a single transformation.



$\dot{w}^t(\mathbf{R}(\mathbf{T}\mathbf{c}))$

$((\dot{w}^t\mathbf{R})\mathbf{T})\mathbf{c}$



Same Point in Different Frames

Given this framework, some rather difficult problems become easy to solve. For instance, suppose you have 2 frames, and you know the coordinates of a particular point relative to one of them. How would you go about computing the coordinate of your point relative to the other frame?

$$\dot{p} = \dot{w}^t \mathbf{c} = \dot{z}^t\ ?$$

Suppose that my two frames are related by the transform $\mathbf{S}$ as shown below.

$$\dot{z}^t = \dot{w}^t \mathbf{S} \quad \text{and} \quad \dot{w}^t = \dot{z}^t \mathbf{S}^{-1}$$

Thus, the coordinate for the point in second frame is simply:

$$\dot{p} = \dot{w}^t \mathbf{c} = \dot{z}^t \mathbf{S}^{-1} \mathbf{c} = \dot{z}^t (\mathbf{S}^{-1} \mathbf{c}) = \dot{z}^t \mathbf{d}$$

Substitute for the frame

Reorganize & reinterpret

More Frame Changes

Even harder problems become simple. Suppose that you want to rotate the points describing some object (say a cow) about some arbitrary axis in space (say a merry-go-round). This is easy so long as we have the transform relating our two frames.

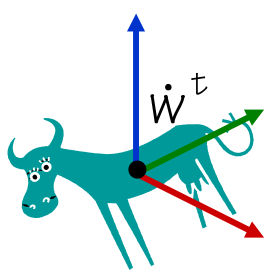
$$\dot{w}^t \mathbf{Z} = \dot{m}^t \quad \text{and} \quad \dot{w}^t = \dot{m}^t \mathbf{Z}^{-1}$$

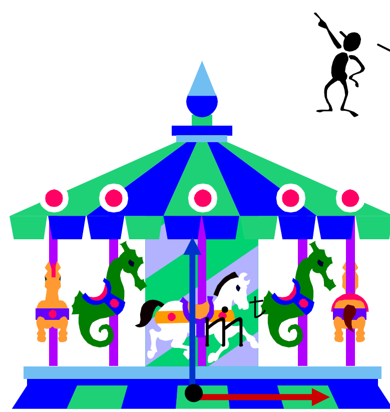
Thus,

$$\dot{w}^t = \dot{m}^t \mathbf{Z}^{-1} \Rightarrow \dot{m}^t \mathbf{R} \mathbf{Z}^{-1} = \dot{w}^t \boxed{\mathbf{Z}\mathbf{R}\mathbf{Z}^{-1}}$$

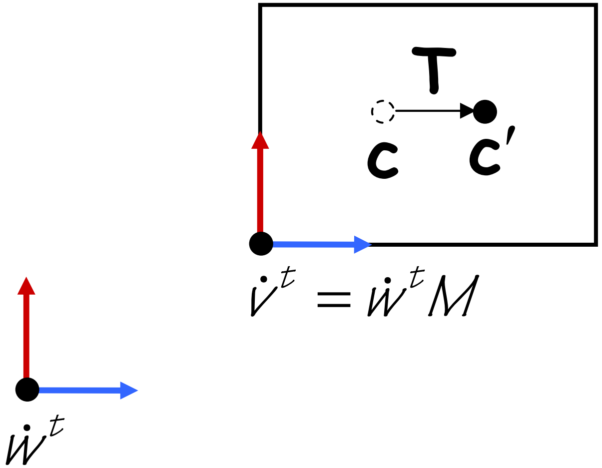
This is the transform that I need to apply to them since they are defined in the world basis

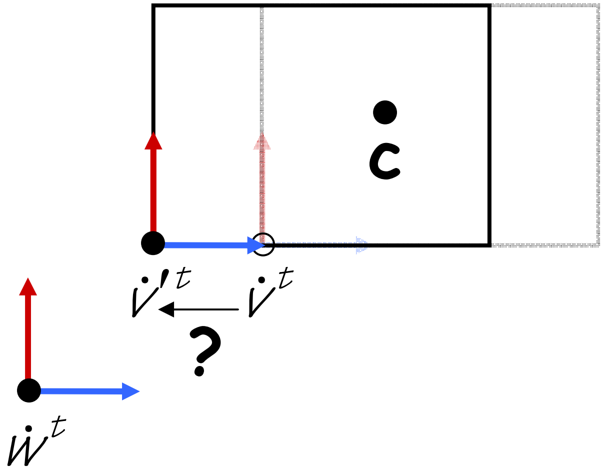
I want to rotate my cow's coordinates around the axis of my merry-go-round





More Frame Changes





$$
\begin{aligned}
\dot{w}^t(\mathbf{Tc}) &= \dot{w}^t\mathbf{c}' \\
&= (\dot{v}^t\mathbf{M}^{-1})\mathbf{c}' \\
&= \dot{v}^t(\mathbf{M}^{-1}\mathbf{c}') \\
&= \dot{v}^t\mathbf{d}
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}^t &= (\dot{w}^t\mathbf{T})\mathbf{M} \\
\dot{w}^t &= (\dot{v}^t\mathbf{T}^{-1})\mathbf{M}^{-1} \\
&= \dot{v}'^t\mathbf{M}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
(\dot{w}^t\mathbf{T})\mathbf{c} &= (\dot{v}'^t\mathbf{M}^{-1})\mathbf{c} \\
&= \dot{v}'^t(\mathbf{M}^{-1}\mathbf{c}) \\
&= \dot{v}'^t\mathbf{d}'
\end{aligned}
$$

Next Time

- 3-D Transformation Mechanics
- How to find a specific transform