NOTE: You are allowed to jointly work with other students in the class. However, please write your answer separately. If you use any idea from any other paper, book or document, please acknowledge the source.

1. **Three Point Interpolation using Hermite Curves:** Given two endpoints $\mathbf{p_0}$ and $\mathbf{p_1}$, an intermediate point $\mathbf{p_i}$ and its corresponding but unspecified parametric variable u_i, and the unit tangent vectors $\mathbf{t_0}$ and $\mathbf{t_1}$. How will compute a cubic Hermite curve that interpolates the three points and has the specified unit tangent vectors at the boundary?

2. Given a cubic Bézier curve in $\mathbf{R}^3$, $\mathbf{P}(t)$. The curve is specified with four control points, $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$. A *cusp* on the curve corresponds to a discontinuity in the unit tangent vector and a necessary condition for the existence of a cusp at $t = t_0$ is:

$$\mathbf{P}'(t_0) = 0.$$

   Can $\mathbf{P}(t)$ have a cusp? You can assume that the four control points are neither collinear nor coplanar.

3. Given a Bézier curve, $\mathbf{B}(\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n : t)$, specified in terms of the control points. Given $0 \le \bar{t} \le 1$, show that the curve can be subdivided into two Bézier curves given as:

$$\mathbf{B}(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n : \frac{t}{\bar{t}}), \quad 0 \le t \le \bar{t}$$

   and

$$\mathbf{B}(\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_n : \frac{t-\bar{t}}{1-\bar{t}}), \quad \bar{t} \le t \le 1,$$

   where $\mathbf{c}_i = \mathbf{b}_0^i(\bar{t})$ and $\mathbf{d}_i = \mathbf{b}_i^{n-i}(\bar{t})$, $0 \le i \le n$, and $\mathbf{b}_i^j(\bar{t})$ is computed using de Casteljau's algorithm.

4. A rational Bézier curve is defined as

$$\mathbf{P}(t) = \Sigma_{i=0}^{n} \mathbf{b}_i \frac{w_i B_i^n(t)}{\Sigma_{j=0}^{n} w_j B_j^n(t)}.$$

   What happens to the curve at $t = 0$ as the weight $w_0 \to 0$? Can you derive a lower degree Bézier representation the curve if $w_0 = 0$? Similarly a rational tensor product patch is defined as

$$\mathbf{P}(s,t) = \Sigma_{i=0}^{n} \Sigma_{j=0}^{m} \mathbf{b}_{i,j} \frac{w_{i,j} B_i^n(s) B_j^m(t)}{\Sigma_{k=0}^{n} \Sigma_{l=0}^{m} w_{k,l} B_k^n(s) B_l^m(t)}.$$

   What happens to the image of $(s,t) = (0,0)$ as $w_{0,0} \to 0$? What is the image of $\mathbf{P}(s,t)$ at $(s,t) = (0,0)$, given $w_{0,0} = 0$?

5. Let us define a piecewise rational curve, $\mathbf{p}(t)$ as

$$\mathbf{p}(t) = \begin{cases} \left(\frac{x_1(t)}{w_1(t)}, \frac{y_1(t)}{w_1(t)}\right) & 0 \le t \le t_1 \\ \left(\frac{x_2(t)}{w_2(t)}, \frac{y_2(t)}{w_2(t)}\right) & t_1 \le t \le 1 \end{cases}$$

   Consider the polynomial curves in the higher dimensional space:

$$\mathbf{P}_1(t) = (x_1(t), y_1(t), w_1(t)), \quad 0 \le t \le t_1$$

and

$$\mathbf{P}_2(t) = (x_2(t), y_2(t), w_2(t)), \quad t_1 \leq t \leq 1.$$

If the polynomial curves have C^k continuity at $t = t_1$ than the rational curve $\mathbf{p}(t)$ has C^k continuity at $t = t_1$ as well. However, this condition is sufficient and not necessary for the C^k continuity of $\mathbf{p}(t)$. Derive the necessary and sufficient conditions between the polynomials, $x_i(t), y_i(t), w_i(t)$ such that $\mathbf{p}(t)$ has C^k continuity at $t = t_1$.