Triangular Bezier Patches

- Natural generalization to Bezier curves
- Triangles are a simplex: Any polygon can be decomposed into triangles
- Formulation based on Barycentric coordinates and linear interpolation

Barycentric Coordinates

•Given a triangle with vertices **A**, **B**, **C** and a fourth point **P,**

•**P** can be expressed as a barycentric combination of **A**, **B**, and **C:**

$$\mathbf{P} = u\,\mathbf{A} + v\,\mathbf{B} + w\,\mathbf{C},$$

and $u + v + w = 1$

•The coefficients (u,v,w) are called barycentric coordinates of **P** with respect to **A**, **B**, **C**

•Given **A,B,C** and **P**, the barycentric coordinates can be computed as:

$$u = \frac{area(\mathbf{P},\mathbf{B},\mathbf{C})}{area(\mathbf{A},\mathbf{B},\mathbf{C})} \quad v = \frac{area(\mathbf{A},\mathbf{P},\mathbf{C})}{area(\mathbf{A},\mathbf{B},\mathbf{C})} \quad w = \frac{area(\mathbf{A},\mathbf{B},\mathbf{P})}{area(\mathbf{A},\mathbf{B},\mathbf{C})}$$

Barycentric Coordinates

•Barycentric coordinates are affinely invariant, i.e. an affine map or tranformations preserves the barycentric coordinates

•If a point is outside the triangle one of the Barycentric coordinate may be negative

•For all points inside the triangle, the Barycentric coordinates are non-negative

de Casteljau Algorithm for Triangular Patches

Given a triangular patch of degree n with $\frac{(n+1)(n+2)}{2}$ control points ($\mathbf{b_{i}} = \mathbf{b_{ijk}}$), where $\mathbf{i} = ijk$ and $|\mathbf{i}| = i + j + k$; $\mathbf{e1} = (1,0,0)$; $\mathbf{e2} = (0,1,0)$; $\mathbf{e3} = (0,0,1)$

The de Casteljau evaluation algorithm is:

$$\mathbf{b_i}^{r}(u) = u\mathbf{b_{i+e1}}^{r-1}(\mathbf{u}) + v\mathbf{b_{i+e2}}^{r-1}(\mathbf{u}) + w\mathbf{b_{i+e3}}^{r-1}(\mathbf{u})$$

where r = 1,…..,n and $|\mathbf{i}| =$ n – r, and $\mathbf{u} = (u,v,w)$ are the barycentric coordinates of a point, where the function is evaluated.

$\mathbf{b_i}^{0}(\mathbf{u}) = \mathbf{b_i}$ and $\mathbf{b_0}^{n}(\mathbf{u})$ is the point with parameter value $\mathbf{u}$ on the triangular Bezier patch.

Properties of Triangular Patches

- Affine invariance
- Convex hull property
- Invariance under affine parameter transformation
- Boundary curves are Bezier curves of degree n

Bernstein polynomials

•The Bernstein polynomials are defined as:

$$B_{\mathbf{i}}^{n}(\mathbf{u}) = \binom{n}{\mathbf{i}} u^i v^j w^k \qquad \text{, where } |\mathbf{i}| = n$$

and a triangular patch can be written in terms of Bernstein polynomials as:

$$B(\mathbf{u}) = \sum_{|\mathbf{j}|=\mathbf{n}} \mathbf{b_j} \ \mathbf{B_j^n}(\mathbf{u})$$