Properties of Bezier Curves

- Invariance under affine parameter transformation

$$\sum_{i=0}^{n} \mathbf{P}_i B_{i,n}(u) = \sum_{i=0}^{n} \mathbf{P}_i B_{i,n}((u-a)/(b-a))$$

- Invariance under barycentric combinations (weighted average):

$$\sum_{i=0}^{n} (\alpha \mathbf{Q}_i + \beta \mathbf{R}_i) B_{i,n}(u) = \alpha \sum_{i=0}^{n} \mathbf{Q}_i B_{i,n}(u) + \beta \sum_{i=0}^{n} \mathbf{R}_i B_{i,n}(u),$$

$$\alpha + \beta = 1$$

- **Pseudo-local control**: $B_{i,n}(u)$ has a max at $u = i/n$. If we move the control point $\mathbf{P}_i$, then the curve is most affected in the region around the parameter value i/n.

Derivatives of Bezier Curve

- Derivative of a Bezier curve:

$$\frac{d}{du}\mathbf{P}(u) = n \sum_{0}^{n-1} \Delta\mathbf{P}_i B_{i,n-1}(u) = \mathbf{P'}(u),$$

where $\Delta\mathbf{P}_i = \mathbf{P}_{i+1} - \mathbf{P}_i$.

$\mathbf{P'}(u)$ is also called the *hodograph* curve

- Higher order derivatives can also be defined in terms of lower order Bezier curves
- Based on the derivatives, we can place constraints on the control points for C^1 or G^1 continuity.

Degree Elevation

- Geometric representation of a degree n curve in terms of n+1 degree curve
  - Compute the control points ($\underline{\mathbf{P}}_i$) of the elevated curve

$$\sum_{0}^{n+1} \underline{\mathbf{P}}_i B_{i,n+1}(u) = \sum_{i=0}^{n} \mathbf{P}_i B_{i,n}(u)$$

where $\underline{\mathbf{P}}_i = \left(i/(n+1)\right) \mathbf{P}_{i-1} + \left(1 - i/(n+1)\right) \mathbf{P}_i$, where i=0,….,n+1

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- What happens if degree elevation is applied repeatedly?

Truncating a Bezier Curve

- *Truncation* and subsequent reparametrization: Given a Bezier curve, find the new set of control points of a Bezier curve that define a segment of this curve in the parametric interval: $u \in [u_i, u_j]$

- *Subdivision*: Given a Bezier curve, $\mathbf{P}(u)$, subdivide at a parameter value u_i. Compute the control points of two Bezier curves: $\mathbf{P}_1(s)$ and $\mathbf{P}_2(t)$, so that $\mathbf{P}_1(s)$, $s \in [0,1]$ corresponds to $\mathbf{P}(u)$, $u \in [0,u_i]$, and $\mathbf{P}_2(t)$, $t \in [0,1]$ corresponds to $\mathbf{P}(u)$, $u \in [u_i,1]$.

- Subdivision can be used to truncate a curve. The control points of the subdivided curve are computed using de Casteljau's algorithm.

Subdividing a Bezier Curve

- Subdivision doesn't change the shape of a Bezier curve
- It can be used for *local* refinement: subdivide a curve and change the control point(s) of one of the subdivided curve
- The union of convex hulls of the subdivided curve is a subset of the convex hull of the original curve (i.e. the convex hulls are a better approximation of the Bezier curve).
- Asymptotically the control polygons of the subdivided curve converge to the actual curve (at a quadratic rate)
- Subdivision and convex hulls are frequently used for intersection computations