Rational Bezier Curves

- Use of homogeneous coordinates
- Rational spline curve: define a curve in one higher dimension space, project it down on the homogenizing variable

- Mathematical formulation:

$$\underline{\mathbf{P}}(u) = (X(u)\ \ Y(u)\ \ W(u)) = \sum_{i=0}^{n} (\mathbf{P}_{i},\ w_i)\, B_{i,n}(u) \qquad u \in [0,1]$$

$$\mathbf{p}(u) = (x(u)\ \ y(u)) = (X(u)/W(u)\ \ Y(u)/W(u)) = \sum_{i=0}^{n} \frac{w_i \mathbf{P}_i B_{i,n}(u)}{\sum_{j=0}^{n} w_j B_{j,n}(u)}$$

Rational Bezier Curves: Properties

- w_i are the weights and they affect the shape of the curve
- All the w_i cannot be simultaneously zero
- If all the w_i are non-negative, the curve is still contained in the convex hull of the control polygon
- It interpolates the end points
- The tangents at $\mathbf{P_0}$ is given by $\mathbf{P_1 - P_0}$ and at $\mathbf{P_n}$ is given by $\mathbf{P_n - P_{n-1}}$
- The weights affect the shape in the interior. What happens when

$$w_i \rightarrow \infty$$

- A rational curve has perspective invariance

Rational Bezier Curves and Conics

- A rational Bezier curve can exactly represent a conic
- The conics are second degree algebraic curve and their segments can be represented exactly using rational quadratic curves (i.e. 3 control points and 3 weights)
- A parabola can be represented using a polynomial curve, but a circle, ellipse and hyperbola can only be represented using a rational curve

B-Spline Curves

- Consists of more than one curve segment
- Each segment is influenced by a few control points (e.g. local control)
- Degree of a curve is indpt. of total number of control points
- Use of basis function that have a local influence
- Convex hull property: contained in the convex hull of control points
- In general, it does not interpolate any control point(s)

Non-Uniform B-Spline Curves

- A B-spline curve is defined as:

- Each segment is influenced by a few control points (e.g. local control)

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i N_{i,k}(u)$$

where $\mathbf{P}_i$ are the control points, k controls the degree of the basis polynomials

Non-Uniform Basis Function

- The basis function is defined as:

$$N_{i,1}(u) = 1 \qquad if \quad t_i \le u < t_{i+1}$$
$$\qquad\quad = 0 \qquad otherwise$$

$$N_{i,k}(u) = \frac{(u - t_i) N_{i,k-1}(u)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - u) N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}}$$

where k controls the degree (k-1) of the resulting polynomial in u and also the continuity of the curve.

- t_i are the knot values, and a set knot values define a knot vector

Non-Uniform Basis Function

- The parameters determining the number of control points, knots and the degree of the polynomial are related by

$$n + k + 1 = T,$$

where T is the number of knots. For a B-spline curve that interpolates $\mathbf{P}_0$ and $\mathbf{P}_n$, the knot T is given as

$$T = \{\alpha, \alpha, \ldots, \alpha, t_k, \ldots, t_{T-k-1}, \beta, \beta, \ldots, \beta\}$$

where the end knots α and β repeat with multiplicity k.

- If the entire curve is parametrized over the unit interval, then, for most cases α =0 and β = 1. If we assign non-decreasing values to the knots, then α =0 and β = n –k + 2.

Non-Uniform Basis Function

- Multiple or repeated knot-vector values, or multiply coincident control point, induce discontinuities. For a cubic curve, a double knot defines a join point with curvature discontinuity. A triple knot produces a corner point in a curve.

- If the knots are placed at equal intervals of the parameter, it describes a UNIFORM non-rational B-spline; otherwise it is non-uniform.
- Usually the knots are integer values. In such cases, the range of the parameter variable is:

$$0 \le u \le n - k + 2$$

Locality

- Each segment of a B-spline curve is influenced by k control points
- Each control point only influences only k curve segments
- The interior basis functions (not the ones influenced by endpoints) are independent of n (the number of control points)
- All the basis functions add upto 1: convex hull property

Uniform B-Spline Basis Functions

- The knot vector is uniformly spaced
- The B-spline curve is an approximating curve and not interpolating curve
- The B-spline curve is a uniform or periodic curve, as the basis function repeats itself over successive intervals of the parametric variable
- Use of Non-uniformity: insert knots at selected locations, to reduce the continuity at a segment joint