Surfaces

- Locally a 2D manifold: i.e. approximating a plane in the ngbd. of each point.
- A 2-parameter family of points
- **Surface representations**: used to construct, evaluate, analyze points and curves, to reveal special properties, demonstrate the relationship to other geometric objects
- Commonly used surface representations: explicit (parametric) or implicit (algebraic)
- **Explicit** formulations are bivariate parametric equations: tensor product, triangular patches or n-sided patches
- **Procedural formulations**: Generalized implicits, subdivision surfaces

Implicit Surfaces

- Implicit formulation:F(x,y,z) = 0, where F() is a polynomial in x,y and z of the form:

$$\sum_{i,j,k} a_{ijk} x^i y^j z^k = 0$$

- If F() is irreducible polynomial, the degree of the surface is: i+j+k

- Geometrically the degree refers to the maximum number of intersections any line can have the surface (assuming finite intersections)
- Any rational parametric surface can be converted into algebraic (implicit) surface: *implicitization*

Quadric Surfaces

- Given as F(x,y,z) = 0, where F() is a quadratic polynomial in x,y and z of the form:

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz + 2Gx + 2Hy + 2Jz + K = 0$$

If A = B = C = -K = 1 & D = E = F = G = H = J = 0, then it produces a unit sphere at the origin.

- Also given in matrix form as: **P Q P**^{T} = 0, where **P** = [x y z 1] &

$$Q = \begin{bmatrix} A & D & F & G \\ D & B & E & H \\ F & E & C & J \\ G & H & J & K \end{bmatrix}$$

Quadric Surfaces

- The coefficients of Q many have no direct physical or geometric meaning
- A rigid body transformation, can be directly applied to **Q** as:

$$\mathbf{Q'} = \mathbf{T^{-1} Q} [\mathbf{T^{-1}}]^T$$

- Certain properties of the matrix of quadric equation are invariant under rigid transformation, including the determinants $|\mathbf{Q}|$ and $|\mathbf{Q}_u|$,where $\mathbf{Q}_u$ is the matrix corresponding to the gradient (or normal) vectors

Parametric Surfaces

- The most common mathematical element used to model a surface is a *patch* (equivalent to a segment of spline curve).
- A patch is a curve-bounded collection of points whose coordinates are given by continuous, bivariate, single-valued polynomials of the form $\mathbf{P}(u,w) = (x \quad y \quad z)$, where:

$$x = X(u,w) \qquad y = Y(u,w) \qquad z = Z(u,w),$$

where the parametric variables u and w are typically constrained to the intervals, $u,w \in [0,1]$.

- This generates a rectangular patch, though there are other topological variations (e.g. triangular or n-sided).
- Fixing the value of one of the parametric variables results in a curve on the patch in terms of the other variable. Or generate a curve net.

Parametric Surfaces: Patches

- Each patch has a set of boundary conditions associated with it:

four corner points ($\mathbf{P}(0,0)$, $\mathbf{P}(0,1)$, $\mathbf{P}(1,0)$, $\mathbf{P}(1,1)$),

four curves defining its edges ($\mathbf{P}(u,0)$, $\mathbf{P}(u,1)$, $\mathbf{P}(0,w)$, $\mathbf{P}(1,w)$),

tangent vectors or planes ($\mathbf{P}_u(u,w)$, $\mathbf{P}_w(u,w)$)

normal vectors ($\mathbf{P}_u(u,w) \times \mathbf{P}_w(u,w)$)

twist vectors defined at the corner points

- In practice, composite arrays are put together or assembled to represent complex surfaces (with some appropriate continuity conditions at the boundary)
- Commonly used patches: Hermite patch, Bezier patch, B-spline patch (or NURBS)

Parametric Surfaces (NURBS): Drawbacks

- Division by zero (for the rational forms)
- Non-uniform parametrization of arcs (important in CAD/CAM)
- Many surfaces cannot be efficiently represented as NURBS(e.g. helicoidal surfaces)
- Irregular isoparametric surface meshes resulting from the use of nonuniform weights (for the rational forms)
- Ill-conditioned basis for surface fitting
- Lack of closure under geometric operations (composition, projection, intersection, offsets etc.)