Solutions to Exercises (9/6/05)

Summation Problem

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j} 1 = \sum_{i=1}^{n}\sum_{j=i}^{n}(j-i+1).$$

Letting $a = j - i + 1$, we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n}(j-i+1) &= \sum_{i=1}^{n}\sum_{a=1}^{n-i+1} a \\ &= \sum_{i=1}^{n}\frac{(n-i+1)(n-i+2)}{2}.\end{aligned}$$

Now, letting $b = n - i + 1$, and reversing the order of the summation,

$$\begin{aligned}\sum_{i=1}^{n}\frac{(n-i+1)(n-i+2)}{2} &= \frac{1}{2}\sum_{b=1}^{n} b(b+1) \\ &= \frac{1}{2}\sum_{b=1}^{n}(b^2+b) \\ &= \frac{1}{2}\left(\frac{2n^3+3n^2+n}{6}+\frac{n(n+1)}{2}\right) \\ &= \frac{n^2+3n^2+2n}{6}.\end{aligned}$$

Exercise 1

Substitution Method:
We guess $T(n) \leq c \lg n$ for some c and large enough n. The ceiling operator $\lceil\ \rceil$ complicates the problem, so we first give an answer disregarding it (you will be told in the instructions to a problem if this is acceptable):

$$\begin{aligned}T(n) &\leq c\lg\tfrac{n}{2}+1 \\ &\leq c\lg\ n - c\lg\ 2+1 \\ &\leq c\lg\ n - c + 1 \\ &\leq c\lg\ n \qquad\qquad \textit{as long as } c \geq 1 \\ T(n) &= O(\lg\ n)\end{aligned}$$

Now we *do* consider the ceiling operator:

$$T(n) \;\leq\; c\lg\left\lceil\frac{n}{2}\right\rceil + 1$$

$$
\begin{aligned}
&\leq c\lg\frac{n+1}{2}+1 \\
&\leq c\lg(n+1)-c\lg 2+1 \\
&\leq c\lg(n+1)-c+1
\end{aligned}
$$

Now this is the tricky part. We want to have something in terms of $\lg n$, not $\lg(n+1)$. To do that, we need to change what's inside the logarithm to some kind of product, instead of a sum. So we write

$$
\begin{aligned}
\lg(n+1) &= \lg\left(n\frac{n+1}{n}\right) \\
&= \lg n+\lg\left(\frac{n+1}{n}\right)
\end{aligned}
$$

Using this fact,

$$
\begin{aligned}
c\lg(n+1)-c+1 &\leq c\lg n+\lg\left(\frac{n+1}{n}\right)c-c+1 \\
&= c\lg n-1-c\left(1-\lg\left(\frac{n+1}{n}\right)\right) \\
&\leq c\lg n \qquad\qquad \text{for large enough } c \text{ and } n.
\end{aligned}
$$

This completes the solution.
Finally, we solve the problem a third way using the Master Method:
$a=1$, $b=2$, so $n^{\log_b a}=n^0=1$.
$f(n)=1=\Theta(n^{\log_b a})\Rightarrow$ Case 2.
$T(n)=\Theta(n^{\log_b a}\lg n)=\theta(\lg n)$

Exercise 2

Guess that $T(n)\leq cn\lg n$

$$
\begin{aligned}
T(n) &\leq 2c(\tfrac{n}{2}+17)\lg(\tfrac{n}{2}+17)+n \\
&= c(n+34)\lg(\tfrac{n+34}{2})+n \\
&= c(n+34)\lg(n+34)-c(n+34)\lg 2+n \\
&= c(n+34)\lg(n+34)-cn-34c+n \\
&\leq cn\lg(n+34)+34c\lg(n+34)-cn+n \\
&= cn\lg n+cn\lg(n+34)-cn\lg n+34c\lg(n+34)-cn+n \\
&= cn\lg n+cn(\lg(n+34)-\lg n)+c(34\lg(n+34)-n+n/c)
\end{aligned}
$$

$$\leq cn \lg n$$

Therefore, $T(n) = O(n \lg n)$

Exercise 3

Let $m = \lg n \Rightarrow n = 2^m$

$T(n) = 2T(n^{\frac{1}{2}}) + 1 \Rightarrow T(2^m) = 2T(2^{\frac{m}{2}}) + 1$

Let $S(m) = T(2^m)$, then $S(m) = 2S(\frac{m}{2}) + 1$
Guess $S(m) \leq cm - b$

$$\begin{aligned} S(m) &\leq 2(c * (\tfrac{m}{2}) - b) + 1 \\ &= cm - 2b + 1 \\ &\leq cm - b \qquad\qquad \textit{if } b \geq 1 \end{aligned}$$

$S(m) = O(m)$, so $T(n) = T(2^m) = S(m) = O(m) = O(\lg n)$
Therefore, $T(n) = O(\lg n)$