Solutions to Exercises (8/26/99)

Exercise 1

Substitution Method:

$$\begin{aligned}
T(n) &\leq c\ lg\lceil \tfrac{n}{2} \rceil + 1 \\
&\leq c\ lg\ n - c\ lg\ 2 + 1 \\
&\leq c\ lg\ n - c + 1 \\
&\leq c\ lg\ n \qquad\qquad \textit{as long as } c \geq 1 \\
T(n) &= O(lg\ n)
\end{aligned}$$

Master Method:

$a = 1$, $b = 2$, $f(n) = 1 = \theta(1)$
$O(n^{log_2 1}) = O(n^0) = O(1)$
$T(n) = \theta(n^{log_b a}\ lg\ n) = \theta(lg\ n)$
$T(n) = \theta(lg\ n)$

Exercise 2

Guess that $T(n) \leq c\ n\ lg\ n$

$$\begin{aligned}
T(n) &\leq 2\ c\ (\tfrac{n}{2} + 17)\ lg(\tfrac{n}{2} + 17) + n \\
&= c(n+34) lg(\tfrac{n+34}{2}) + n \\
&= c(n+34) lg(n+34) - c(n+34) lg 2 + n \\
&= c(n+34) lg(n+34) - cn - 34c + n \\
&\leq c\ n\ lg(n+34) + 34c lg(n+34) - cn + n \\
&= c\ n\ lg\ n + c\ n\ lg(n+34) - c\ n\ lg\ n + 34\ c\ lg(n+34) - c\ n + n \\
&= c\ n\ lg\ n + c\ n\ (lg(n+34) - lg\ n) + c(34\ lg(n+34) - n + n/c) \\
&\leq c\ n\ lg\ n
\end{aligned}$$

Therefore, $T(n) = O(n\ lg\ n)$

Exercise 3

Let $m = lg\ n \Rightarrow n = 2^m$

$T(n) = 2T(n^{\frac{1}{2}}) + 1 \Rightarrow T(2^m) = 2T(2^{\frac{m}{2}}) + 1$

Let $S(m) = T(2^m)$, then $S(m) = 2S(\frac{m}{2}) + 1$
Guess $S(m) \leq cm - b$

$$\begin{aligned} S(m) &\leq 2(c * (\tfrac{m}{2}) - b) + 1 \\ &= c\ m - 2b + 1 \\ &\leq c\ m - b \qquad\qquad if\ b \geq 1 \end{aligned}$$

$S(m) = O(m)$, so $T(n) = T(2^m) = S(m) = O(m) = O(lg\ n)$
Therefore, $T(n) = O(lg\ n)$