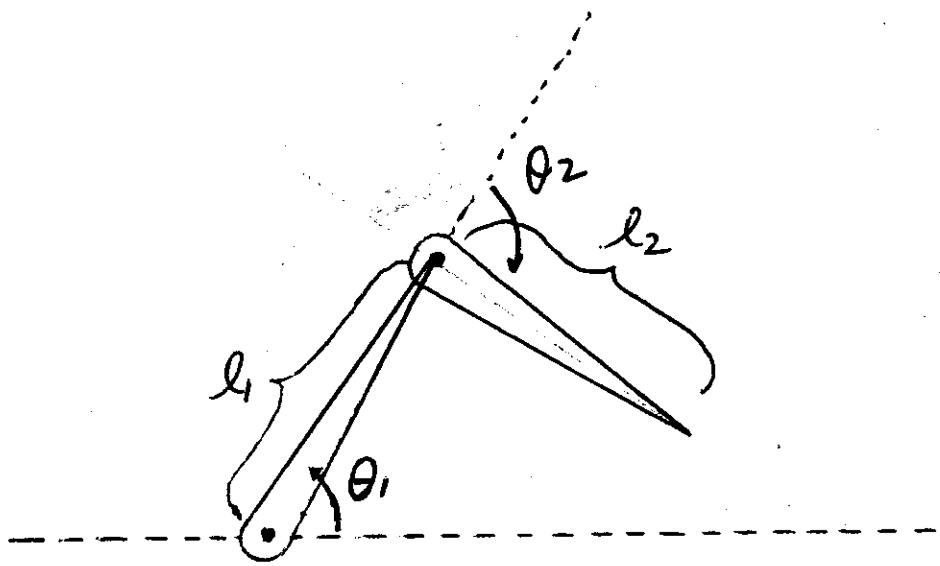
Forward vs. Inverse Kinematics



- **Forward kinematics:** $\mathbf{P} = f(\Theta)$, where $\Theta = (\theta_1, \theta_2)$ here.

- **Inverse kinematics:** $\Theta = f^{-1}(\mathbf{P})$

$$\mathbf{P} = (x, y) = (l_1 cos\theta_1 + l_2 cos(\theta_1 + \theta_2),\ l_1 sin\theta_1 + l_2 sin(\theta_1 + \theta_2))$$

$$\theta_2 = cos^{-1}\frac{(x^2 + y^2 - l_1^2 - l_2^2)}{2\, l_1\, l_2}$$

$$\theta_1 = \frac{-(l_2 sin\theta_2)x + (l_1 + l_2 cos\theta_2)y}{(l_2 sin\theta_2)y + (l_1 + l_2 cos\theta_2)x}$$

θ_1 and θ_2 are the inverse solutions solved by applying trigonometry.

Inverse Kinematics Revisit: The Jacobian

Given $\mathbf{X} = f(\mathbf{\Theta})$, where $\mathbf{X} = (x_1, ..., x_n)$ and $\mathbf{\Theta} = (\theta_1, ..., \theta_m)$, $\mathbf{X}$ is of dimension n and $\mathbf{\Theta}$ is of dimension m. The **Jacobian J** is the n x m matrix of partial derivates relating the differential changes of $\mathbf{\Theta}$, represented as $d\mathbf{\Theta}$, to differential changes in $\mathbf{X}$, written as $d\mathbf{X}$. Or it can be expressed as:

$$d\mathbf{X} = J(\mathbf{\Theta})d\mathbf{\Theta} \tag{1}$$

where the (i, j)th element of $\mathbf{J}$ is given by:

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial x_j}$$

We divide Eqn. 1 by the differential time element to get:

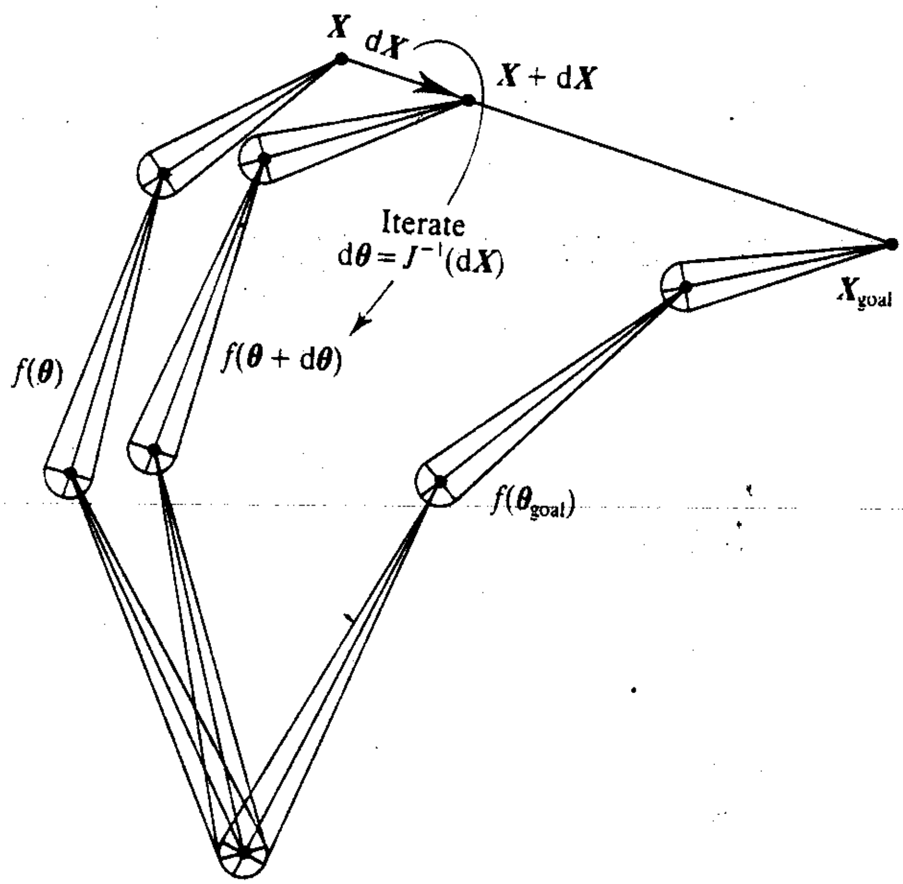
$$\dot{\mathbf{X}} = J(\mathbf{\Theta})\dot{\mathbf{\Theta}}$$

where $\dot{\mathbf{X}}$ is the velocity of the end effector which, most generally, is a vector of 6 dimensions that includes both the linear velocity $\mathbf{V}$ and the angular velocity $\mathbf{\Omega}$, and $\dot{\mathbf{\Theta}}$ is the time derivatives of the state vector. The Jacobian maps velocities in state space to velocities in cartesian space. At any given time, these two quantities are related through the linear transformation $\mathbf{J}$, which itself changes through time as $\mathbf{\Theta}$ changes. $\mathbf{J}$ is best thought of as a time-varying linear transformation.

Recall that for inverse kinematics, $\mathbf{\Theta} = f^{-1}(\mathbf{X})$, where we solve for the state vector $\mathbf{\Theta}$ given a position and orientation of the end effector $\mathbf{X}$. For most of the articulation functions $f(\cdot)$ is highly nonlinear and rapidly becomes more complex, as the number of links increases. So, the analytical inversion of $f(\cdot)$ soon becomes impossible to perform. The problem can be solved numerically by using the Jacobian:

$$d\Theta = J^{-1}(\Theta)d\mathbf{X}$$

and iterating toward the goal over a series of incremental steps as shown in the figure.



Here, we show a simple example from our previous discussion on inverse kinematics.

$$
\begin{aligned}
\mathbf{P} &= f(\boldsymbol{\Theta}) \\
\boldsymbol{\Theta} &= f^{-1}(\mathbf{P}) \\
\mathbf{P} &= (x, y) = (l_1 cos\theta_1 + l_2 cos(\theta_1 + \theta_2),\ l_1 sin\theta_1 + l_2 sin(\theta_1 + \theta_2)) \\
&= (l_1 c_1 + l_2 c_{12}, l_1 s_1 + l_2 s_{12}) \qquad \text{(using abbreviation of notions)}
\end{aligned}
$$

Differentiating using the chain rule, we get:

$$
\begin{aligned}
\dot{\mathbf{X}} &= (-l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2), l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2)) \\
&= \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
&= J(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}}
\end{aligned}
$$