COMP 455
Models of Languages and Computation
Fall 2013
An Unsolvability Result

Unsolvability results for Turing machines arise because Turing machines can accept Turing machine descriptions as inputs. Paradoxes arise because a Turing machine can read its own description.

These unsolvability results require Turing machines to be encoded as Turing machine inputs. This can easily be done as follows:

We consider an integer $i$ as an encoding of a Turing machine $T_i$. This is done by considering $i$ as a binary number and breaking this binary number into 8 bit bytes. Each byte is interpreted as a character, and then $i$ is read as a Turing machine description in some language (such as the book gives). If $i$ does not encode a Turing machine then $T_i$ is some fixed Turing machine, possibly the Turing machine that immediately halts when it starts.

In this formalism, a *universal Turing machine* $U$ takes as input $i$ and $j$ and simulates $T_i$ on input $j$.

It's actually easier conceptually to let $encode(M)$ be an encoding of a Turing machine as a string, and $encode(x)$ be the encoding of a string $x$. Then the universal Turing machine takes as input $encode(M)encode(x)$ and simulates $M$ on input $x$.

We can use this formalism to show that some problems are undecidable. Consider the language $L = \{encode(M) : M \text{ loops on input } encode(M)\}$. A simple argument shows that there is no Turing machine that partially decides $L$. For suppose $T$ partially decided $L$. What would $T$ do on input $encode(T)$? If $T$ halts on input $encode(T)$ then (since $T$ partially decides $L$), $encode(T) \in L$, so by definition of $L$, $T$ does not halt on input $encode(T)$. If $T$ does not halt on input $encode(T)$ then (since $T$ partially decides $L$) $encode(T) \notin L$, so $T$ halts on input $encode(T)$ by definition of $L$. Either choice leads to a contradiction. So $L$ is not partially decidable.

The *halting problem* is, given a Turing machine and an input, to decide whether the Turing machine halts on the input. If the halting problem were decidable, $L$ would be decidable. Since $L$ is not even partially decidable, $L$ is not decidable, so the halting problem is not decidable, either.