1. (12 points) (a) Give an upper bound on the height of a red black tree having \( n \) internal nodes. \( \text{__________} \) (b) If a red black tree has black height 15, what is its maximum height? \( \text{__________} \) (c) If a red black tree has black height 15, what is the minimum number of internal nodes in the tree? \( \text{__________} \)

2. (10 points) Give a sequence of seven elements in the range \( \{1, 2, 3, \ldots, 7\} \) which, when inserted into an initially empty binary search tree, will produce a perfectly balanced tree. \( \text{__________} \)

3. (10 points) Suppose \( T \) is a height balanced binary tree. Suppose that \( T_1, T_2, T_3, \) and \( T_4 \) are height-balanced binary trees of height \( h \). (a) Assume that the following tree is a binary search tree. Is this tree necessarily height balanced? Why or why not? \( \text{__________} \)

\[
\begin{array}{c}
  x \\
  \quad \downarrow \\
  y \quad T_3 \\
  \quad \downarrow \\
  T_1 \quad T_2 \\
\end{array}
\]

(b) Assume that the following tree is a binary search tree. Is it necessarily height balanced? Why or why not? \( \text{__________} \)

\[
\begin{array}{c}
  x \\
  \quad \downarrow \\
  y \quad T_4 \\
  \quad \downarrow \\
  z \quad T_3 \\
  \quad \downarrow \\
  T_1 \quad T_2 \\
\end{array}
\]
4. (10 points) Suppose a hash table has a load factor of 99/100. (a) If open addressing is being used, assuming simple uniform hashing, what is the expected number of probes in an unsuccessful search? ____________ (b) If hashing with chaining is being used, assuming simple uniform hashing, what is the expected time for an unsuccessful search? ____________

5. (10 points) (a) How fast can one sort \( n \) elements in the range \( \{1, 2, \ldots, n\} \) using \( O(n) \) storage? Give an asymptotic bound and a brief justification. ____________ b) How fast can one sort \( \sqrt{n} \) elements in the range \( \{1, 2, \ldots, n\} \) using \( O(\sqrt{n}) \) storage? Give an asymptotic bound and a brief justification. ____________

6. (10 points) Suppose there are \( n \) binary numbers, each \( m \) bits in length at most. Suppose one sorts these numbers using radix sort, treating each bit as a separate digit. What is the asymptotic time required to do this? ____________

7. (10 points) Consider the following set of elements, where \( a : b \) denotes an element with key \( a \) and priority \( b \). Construct a treap from these elements.  

8. (12 points) (a) How many comparisons are needed to find the maximum of \( n \) elements? Give an asymptotic bound. ____________ (b) How many comparisons are needed to find the maximum and second largest of \( n \) elements? Give an asymptotic bound. ____________ (c) How many comparisons are needed to find the median of \( n \) elements? Give an asymptotic bound. ____________

9. (10 points) For hashing by the multiplication method, which of the following values for A is best: (a) .125 (b) 0.01 (c) .618 (d) .618033988 ____________

10. (10 points) For hashing by the division method, which of the following is the best value for the modulus \( m \)? Give a brief justification for your answer.
11. (6 points) If the hash table size is 200 and there are 120 elements in the table, what is the load factor? ________________

12. (10 points) Suppose one is using hashing by the multiplication method and A is .528 and the table size m is 20 and the key k is 15. Which bin will this key hash to? ________________

13. (10 points) Suppose one is using hashing by the division method and the table size m is 53 and the key k is 225. Which bin will this key hash to? ________________

14. (10 points) Suppose one is doing universal hashing and the table size m is 50 and there are 1000 hash functions in all in the set H. Let x and y be two keys. What is the maximum number of hash functions h in H such that h(x) = h(y), according to universal hashing? ________________

EXTRA CREDIT: (10 points) Let $n_i$ be a random variable for the number of elements that end up in the $i^{th}$ bucket when doing bucket sort of $n$ objects, assuming there are $n$ buckets and each object has an equal chance of ending up in each bucket. What is $E[n_i]$? ________________ What is $E[n_i^2]$? ________________

EXTRA CREDIT: (10 points) Suppose $n$ elements hash randomly to $n$ slots. Let $n_i$ be the number of elements that hash to slot $i$. What is the expected value $E[\sum_i n_i^2]$? ________________

EXTRA CREDIT: (10 points) (a) How many comparisons are needed to find the maximum and minimum elements of a set of six elements? ________________ 
(b) How many comparisons are needed to find the second largest element of a set of eight elements? ________________ Justify your answers briefly.