COMP 633 - Parallel Computing

Lecture 18
November 2, 2021

BSP (3)
Parallel Sorting in the BSP model (contd)

- **Reading**
  - Skillicorn, Hill, McColl
    - » Questions and Answers about BSP (pp 1-25)
- **Programming assignment pa2**
  - online

Recall PRAM radix sort (02-pram1, Aug 13, slide 31)

```
Input:     A[1:n] with b-bit integer elements
Output:    A[1:n] sorted
Auxiliary: FL[1:n], FH[1:n], BL[1:n], BH[1:n]

for h := 0 to b-1 do
     forall i in 1:n do
          FL[i] := (A[i] bit h) == 0
          FH[i] := (A[i] bit h) != 0
     enddo
     BL := PACK(A,FL)
     BH := PACK(A,FH)
     m  := #BL
     forall i in 1:n do
            A[i] := if (i ≤ m) then BL[i] else BH[i-m]endif
     enddo
enddo
```

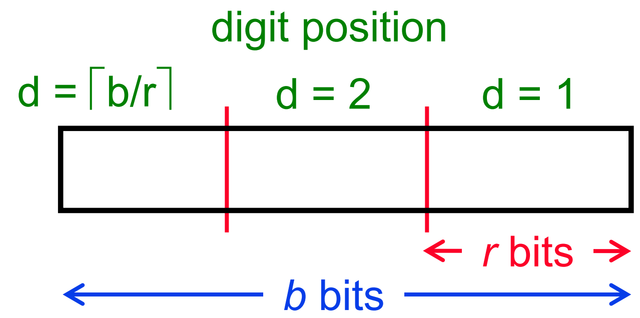
S(n) = O(b lg n)

W(n) = O(bn)

Sequential radix sort

Input: A[0 : N-1], with b-bit elements
Radix $s = 2^r$, $r \geq 1$

Result: A[0 : N-1] in sorted order




```
for d := 1 to ⌈b/r⌉ do

    -- construct histogram T[0 : s-1] of digit values in digit position d of A[0 : N-1]
    T[0:s-1] := 0
    for j := 0 to N-1 do   T[digit in position d of A[j] ]++   end do

    -- cumulative histogram W[0 : s-1]
    W[0:s-1] := exclusive_scan(T[0:s-1], +)

    -- construct permutation H[0 : N-1] that sorts A[0 : N-1] into increasing order in digit position d
    for j := 0 to N-1 do   H[j] := W[digit in position d of A[j] ]++   end do

    -- permute A[0 : N-1]
    A[ H[0:N-1] ] := A[0:N-1]

end do
```

Complexity : $T_s(N) = \left\lceil \frac{b}{r} \right\rceil \left(O(2^r) + O(N) \right)$

Parallel radix sort

- **For each digit position *d* from least to most significant**
  - use sequential algorithm to compute local histogram $T^{(j)}[0:s-1]$ for digit position *d* at each processor $1 \le j \le p$
  - construct cumulative histogram defined as $W^{(j)}[i] = \left(\sum_{i'=0}^{i-1} \sum_{j'=1}^{p} T^{(j')}[i'] \right) + \sum_{j'=1}^{j-1} T^{(j')}[i]$
  - use $W^{(j)}$ to determine the local portion of permutation H at each processor $1 \le j \le p$ and apply permutation in parallel to rearrange A

- **Example**
  - N = 20, p = 4, N/p = 5, b = 2, r = 2, $s = 2^r = 4$
  - A = [3, 1, 0, 1, 3, | 2, 0, 0, 2, 0, | 2, 2, 2, 2, 2, | 0, 1, 3, 2, 2]
  - H = [17,5,0,6,18, | 8,1,2,9,3, | 10,11,12,13,14, | 4,7,19,15,16]

processor $1 \le j \le p$

digit $0 \le i < s$

$T^{(j)}[i]$	1	2	3	4
00	1	3	0	1
01	2	0	0	1
10	0	2	5	2
11	2	0	0	1

local histogram

$W^{(j)}[i]$	1	2	3	4
00	0	1	4	4
01	5	7	7	7
10	8	8	10	15
11	17	19	19	19

cumulative histogram

Parallel computation of cumulative histogram

- **Two alternatives**

  (1) Perform exclusive parallel prefix-sums across the s successive rows of T

  - Each sum starts with the ending value on the previous row

    » total BSP cost: $s\,(2 \lg p)\,(1 + g + L)$

  (2) Multiscan method

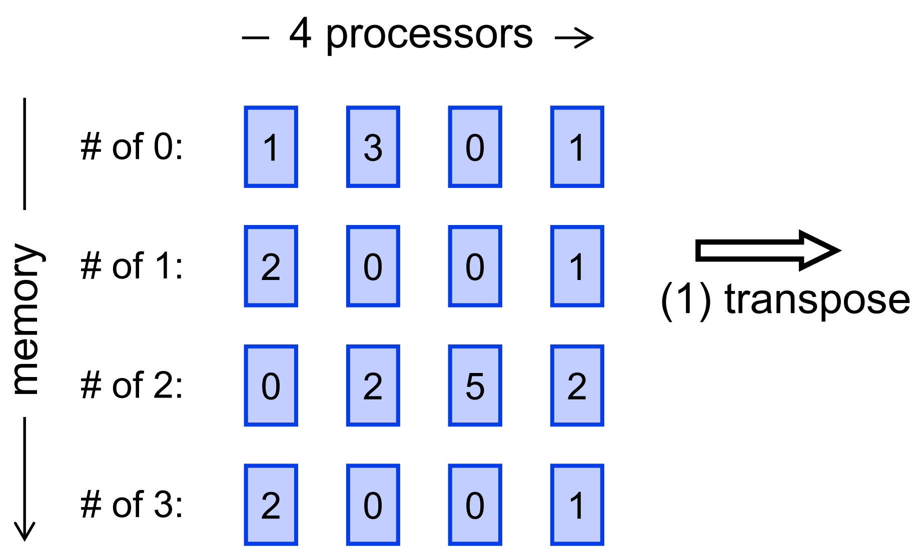
  - partition digits 0..s-1 into p contiguous intervals of size k= s/p and construct $T^{(j)}[ik: (i+1)k - 1]$ for i in 0..p-1
  - transpose T across processors
    - BSP cost: $s(1 + g + L)$
  - compute local sum, one parallel prefix sum across processors, followed by a local prefix sum
    - BSP cost: $2s + (\lg p)(1 + g + L)$
  - transpose result to yield $W^{(j)}[ik: (i+1)k - 1]$ for i in 0..p-1
    - BSP cost: $s(1 + g + L)$

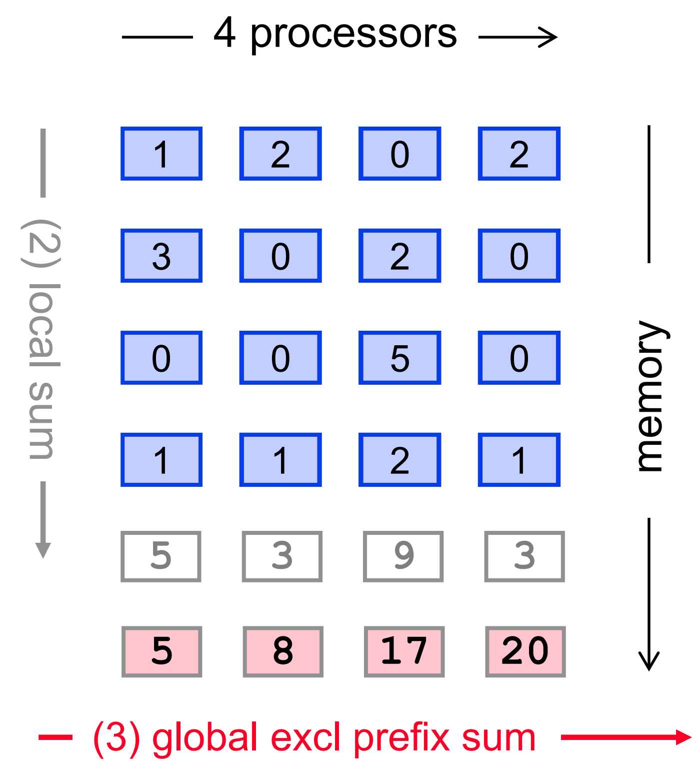
    » total BSP cost: $(4s + \lg p)\,(1 + g + L)$ superior when $p > 2$

Example of multiscan

- **Problem parameters**
  - N = 20, p = 4, N/p = 5 and r = 2, s = 2^r = 4
  - A = [`3, 1, 0, 1, 3,` `2, 0, 0, 2, 0,` `2, 2, 2, 2, 2,` `0, 1, 3, 2, 2`]

← N/p →

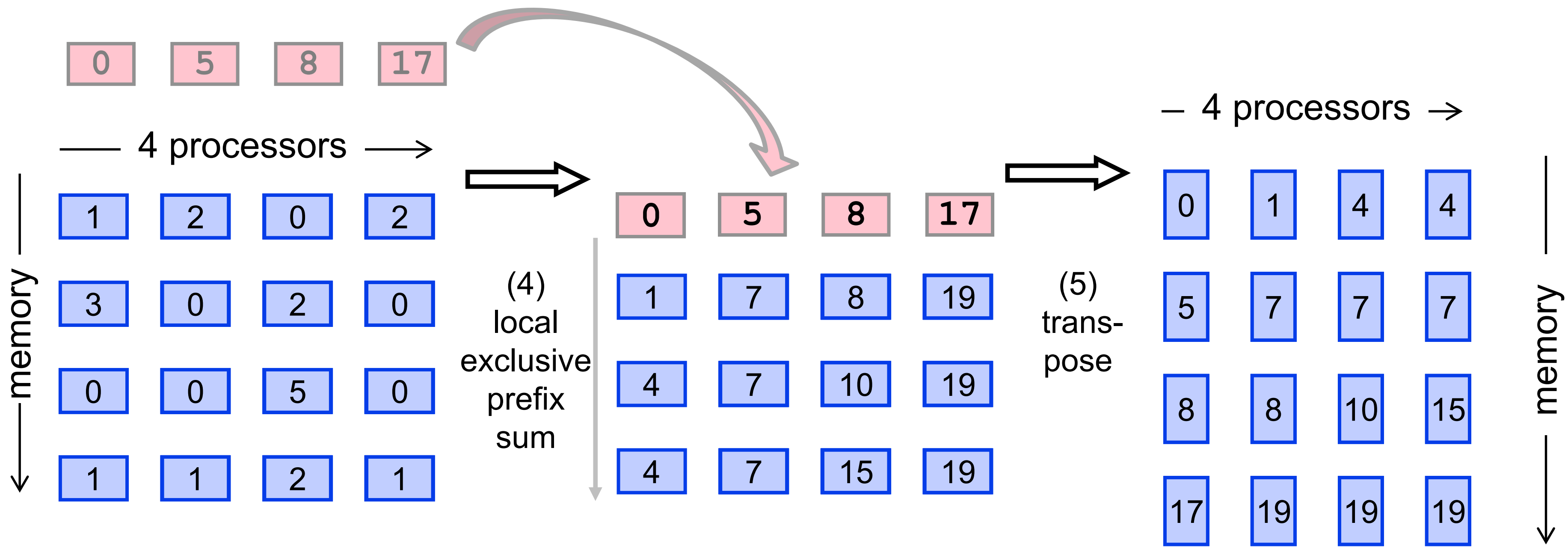




input: local histogram – count of occurrences of each “digit” 0 .. 3 within each processor’s local section of A (N/p = 5)

Multiscan example (contd)

- **Problem parameters**
  - N = 20, p = 4, N/p = 5 and r = 2, $s = 2^r = 4$
  - A = [3, 1, 0, 1, 3, | 2, 0, 0, 2, 0, | 2, 2, 2, 2, 2, | 0, 1, 3, 2, 2]



Cumulative histogram $W^{(j)}[i]$ – the destination index of the first occurrence of digit *i* held within processor *j*

Parallel radix sort - Analysis

- **Algorithm**
  - $\lceil b/r \rceil$ iterations
  - each iteration ($s = 2^r$) (using multiscan)
    - » construct histogram — O(N/p + s)
    - » transpose histograms — s · g+L
    - » local sum — O(s)
    - » global prefix sum — (lg p)(1 + g + L)
    - » local prefix sum — O(s)
    - » transpose cumulative histogram — s · g + L
    - » compute destinations — O(N/p)
    - » permute values — O((N/p)(b/64))g + L
  - BSP cost (b = 64)

$$C^{\text{RADIX}}(N,p,r) = \left\lceil \frac{b}{r} \right\rceil \Theta\left(\frac{N}{p} + 2^r\right) + \left\lceil \frac{b}{r} \right\rceil \left(\frac{N}{p} + 2^r\right) \cdot g + \left\lceil \frac{b}{r} \right\rceil (\lg p) \cdot L$$

  - How to find optimum choice of radix r?
    - » r small means N/p dominates 2^r
    - » r large means b/r is small

Predicted and measured times for radix sort

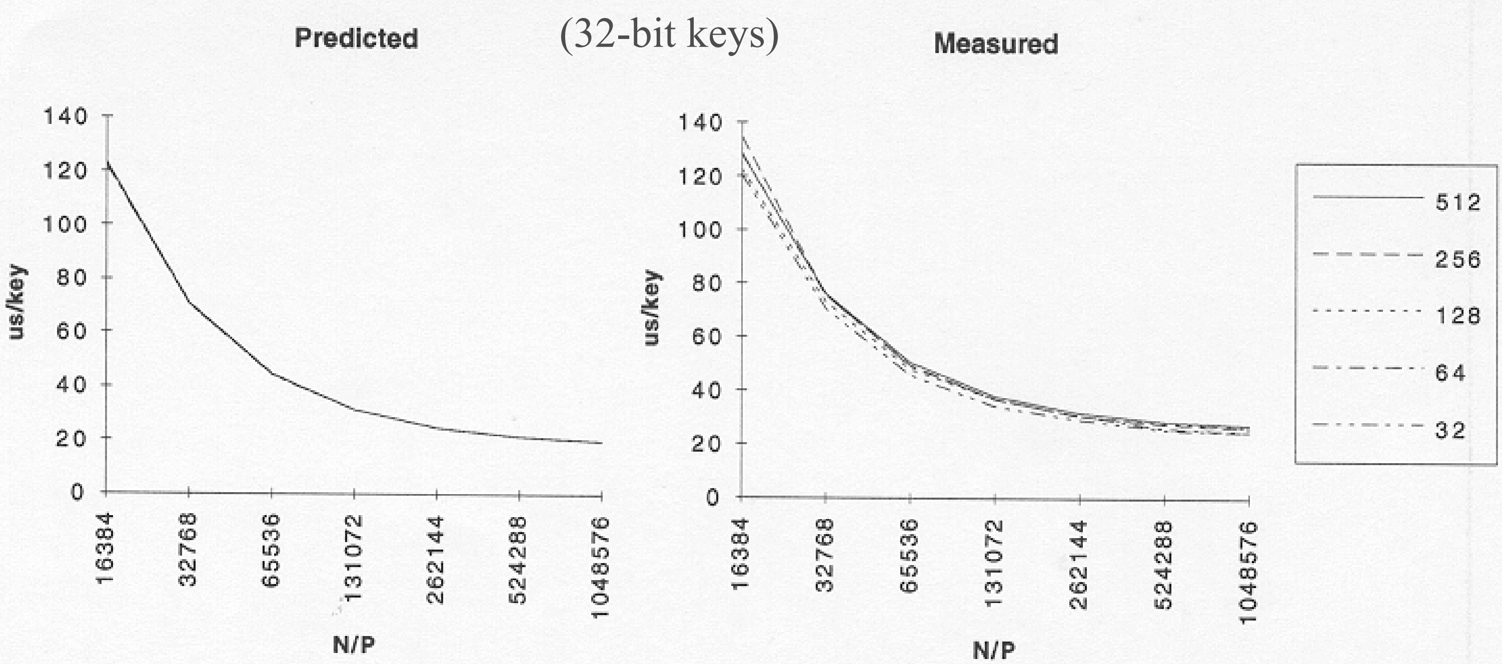


Figure 1.10 *Predicted and measured execution time per key of radix sort on the CM-5.*

Breakdown of radix sort running times

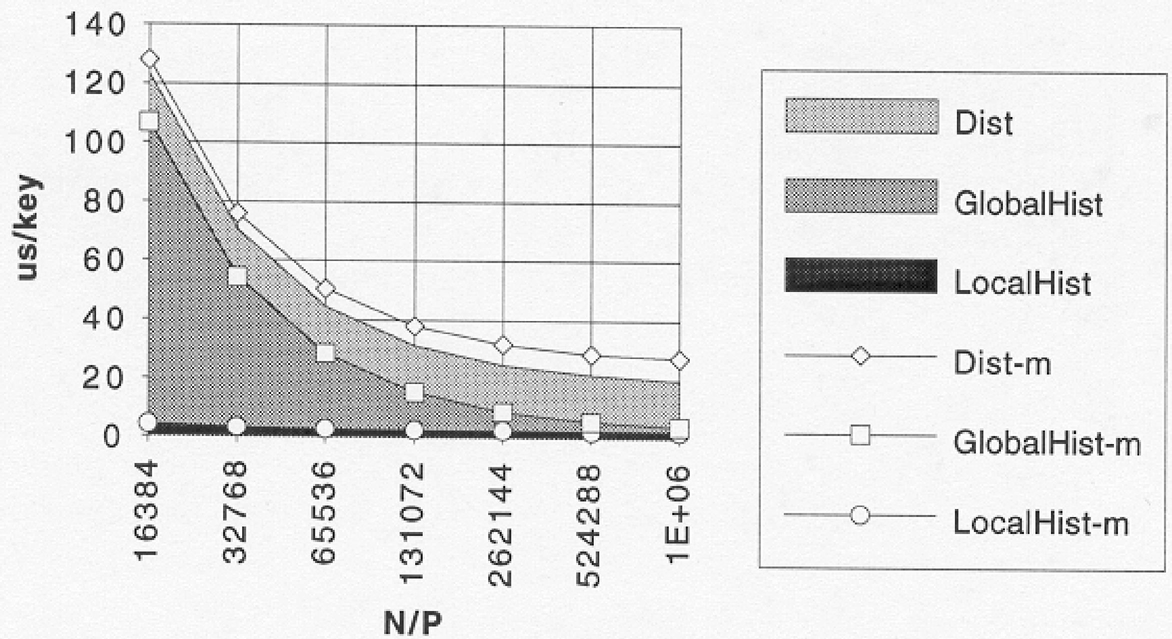


Figure 1.11 *Predicted and measured execution times per key of various phases in radix sort on 512 processors.*

Probabilistic sorting algorithms

- **Definitions**
  - An unordered collection H with N disjoint values is partitioned by splitters $S = S_1 < \ldots < S_{p-1}$ into p disjoint subsets $H_1 \ldots H_p$ such that

$$H_i = \{h \mid h \in H \text{ and } S_{i-1} \le h < S_i\} \qquad (\text{define } S_0 = -\infty, \text{ and } S_p = +\infty)$$

  - The skew $W(S)$ of a partition S is the ratio of the maximum partition size to the optimal partition size (N/p)

$$W(S) = \max_{1 \le i \le p} \left(\frac{|H_i|}{N/p} \right)$$

Determining good splitters through sampling

- **Determining a set of splitters through sampling**
  - sample $k{\cdot}p$ elements at random from H
    - » $k \geq 1$ is the oversampling ratio
  - sort this sample into order $b_1 < b_2 < \ldots < b_{k\cdot p}$ and choose $S_i = b_{k\cdot i}$

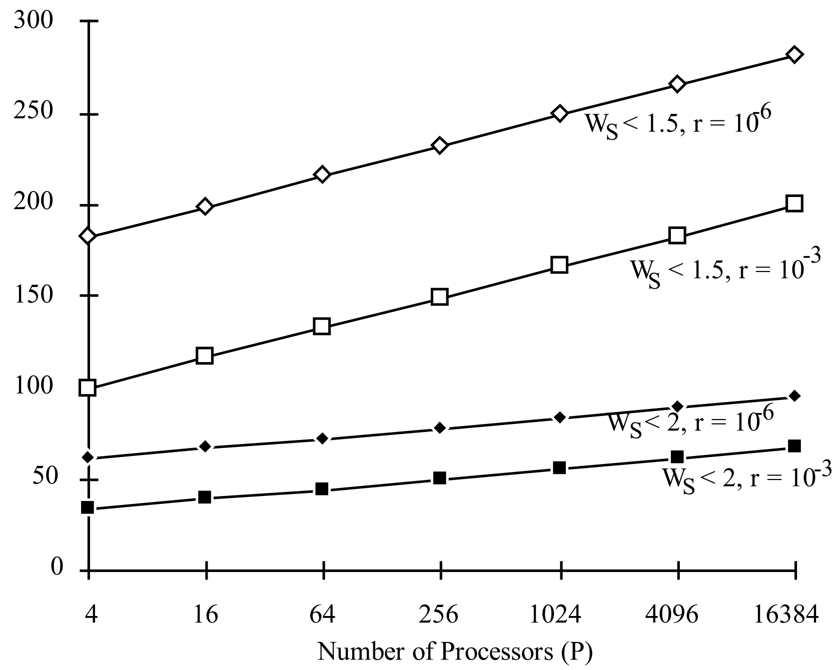
- **Probabilistic bounds on W(S) of a sampled set of splitters S**
  - given some maximum skew W and a failure probability 0 < r < 1

$$\Pr\big(W(S) > W\big) \leq r \quad \text{when} \quad k \geq \frac{2\ln(p/r)}{(1-1/W)^2 W} \quad (\text{provided } p > 1, \quad W > 1.3)$$

  - if we oversample sufficiently in choosing a set of splitters, the chance of a large skew can be made arbitrarily small

Oversampling ratio *k* as a function of *p*

- **Example**
  - for p = 100 processors, we need to sample k = 4 ln (p/r) = 74 values per processor to bound the skew W(S) < 2 with failure probability *r* = 10^{-6}



Parallel samplesort

- **Algorithm**
  1. sample k values at random in each processor to limit skew to W w.h.p.

     O(k)
  2. sort kp sampled keys, extract p-1 splitters, and broadcast to all processors
     a) by sending all samples to one processor and performing a local sort

        $O(kp) + (k+2)p \cdot g + 2 \cdot L$

     a) by performing a bitonic sort with k values per processor

        $O(k \lg^2 p) + k(1+2 \lg p) \cdot g + (1+\lg p) \cdot L$
  3. compute destination processor for each value by binary search in splitter set

     O(N/p lg p)
  4. permute values

     $WN/p \cdot g + L$
  5. perform local sort of values in each processor

     O(Ts(WN/p))

- **BSP cost**

$$C^{\text{SAMPLE}}(N,p,W) = \Theta\left(W + \lg p\right)\left(\frac{N}{p}\right) + W\left(\frac{N}{p}\right) \cdot g + \left(\lg p\right) \cdot L + O\left(k \lg p\right)\left(\lg p \cdot g + L\right)$$

Samplesort: predicted and measured times

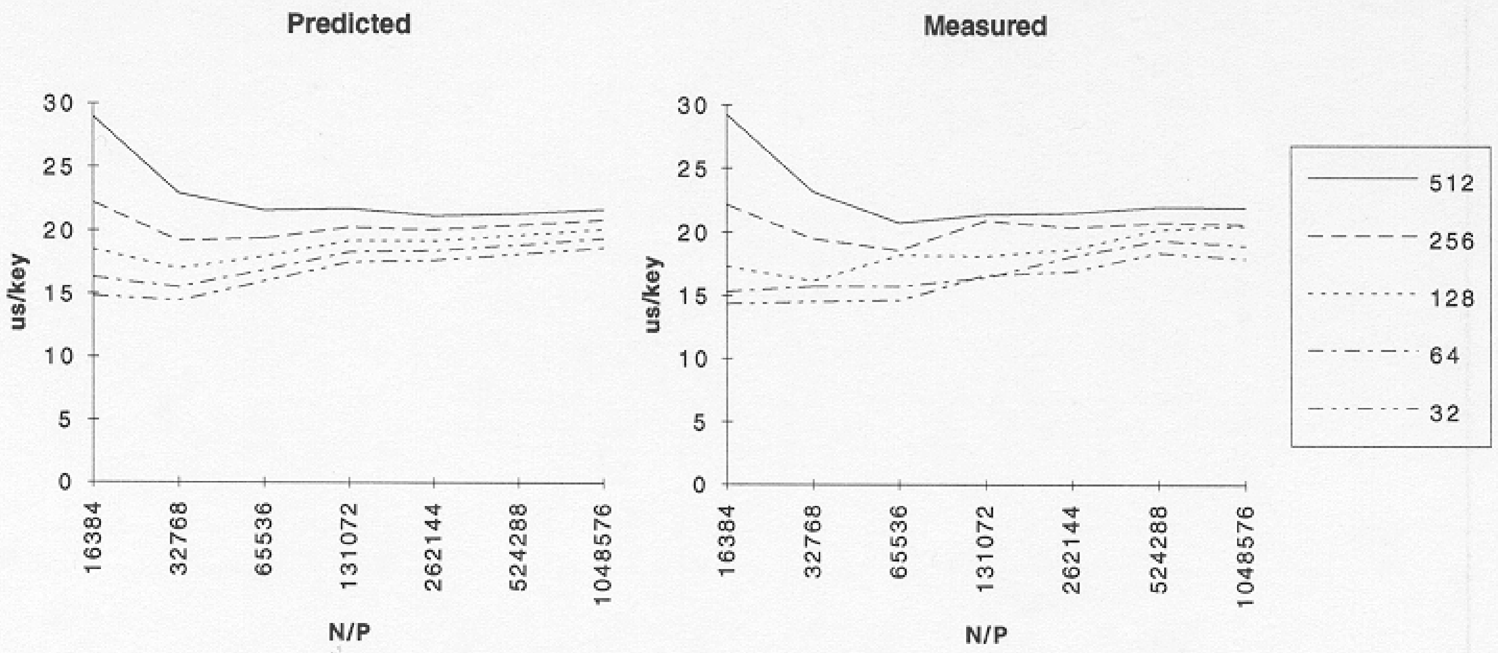


Figure 1.12 *Estimated and measured execution time of parallel sample sort on the CM-5.*

Samplesort: breakdown of execution time

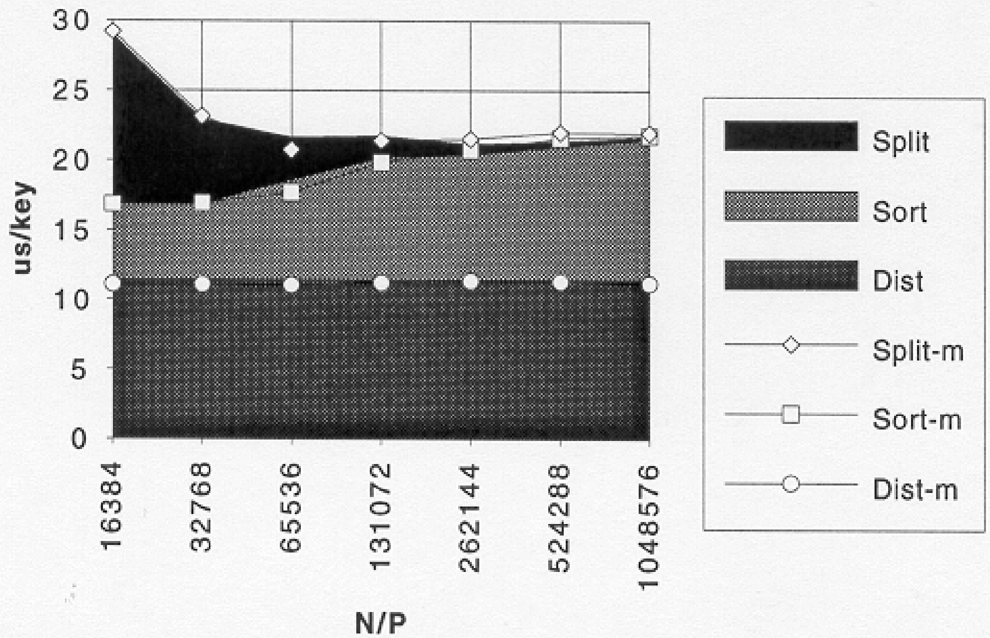


Figure 1.13 *Estimated and measured execution times of various phase of parallel sample sort on 512 processors.*

Parallel sorting: performance summary

- **32 bit values**
  - for small N/p (not shown), bitonic sort is superior

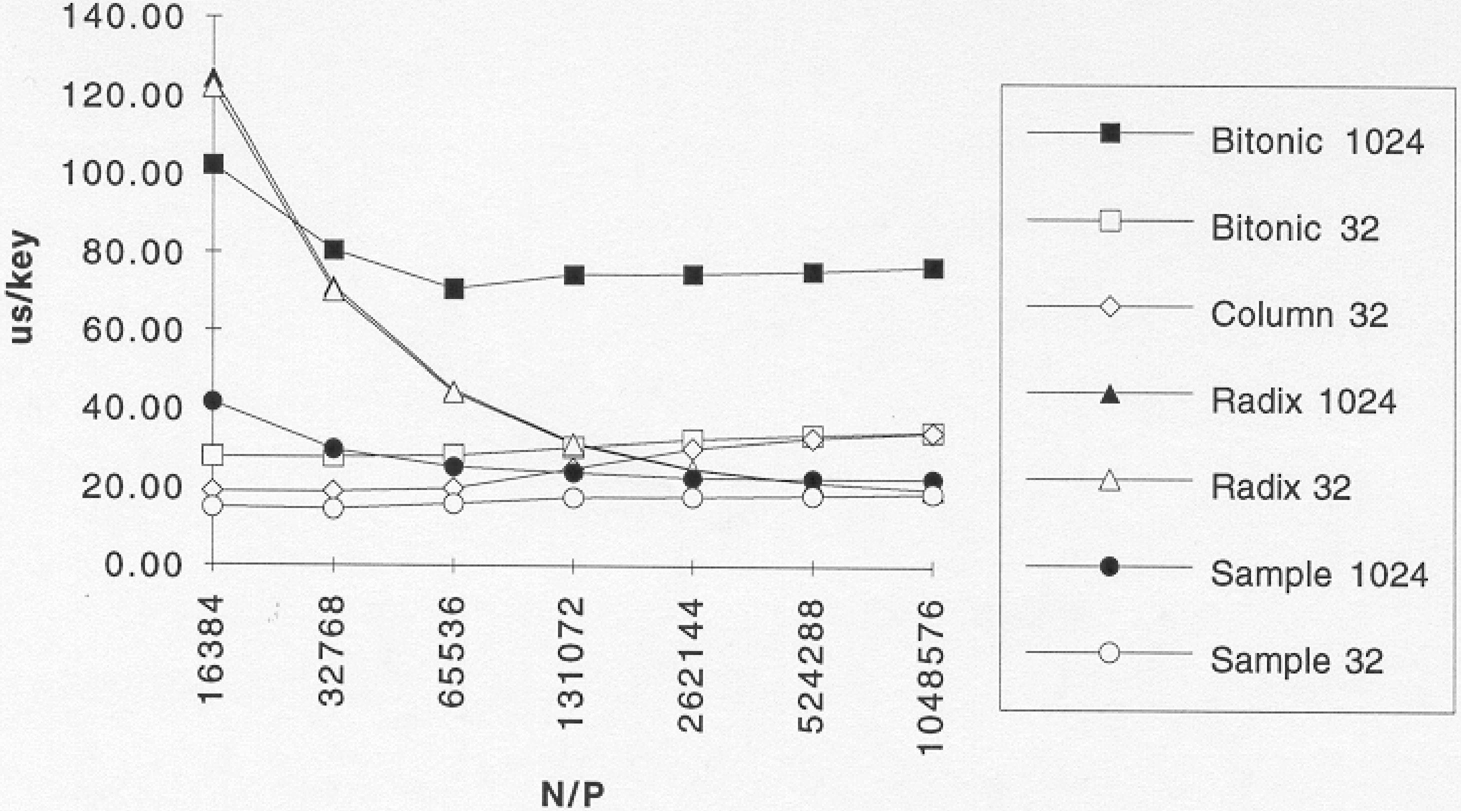
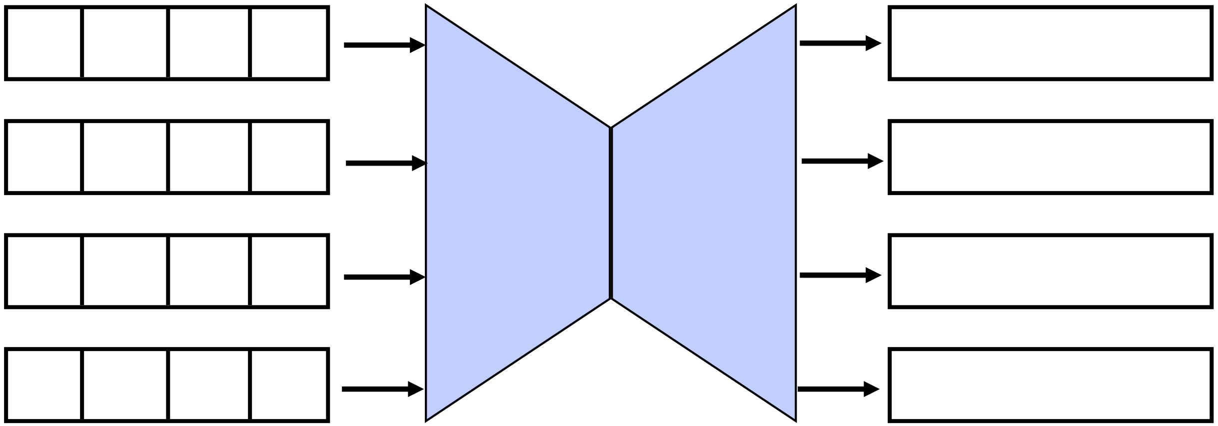


Figure 1.14 *Estimated execution time of four parallel sorting algorithms under LogP with the performance characteristics of the CM-5.*

Samplesort issues

- **Implementing the permutation**
  - What is the destination address of a given value? Two strategies:
    - » Send-to-queue operation
      - don't care, maintain queue at destination
    - » Compute unique destination for each value
      - planning cost: O(p) + 2pg + 2L
  - In what order should the values be sent?
    - » Global rearrangement defines a permutation, but piecewise implementation may yield poor performance



Samplesort issues

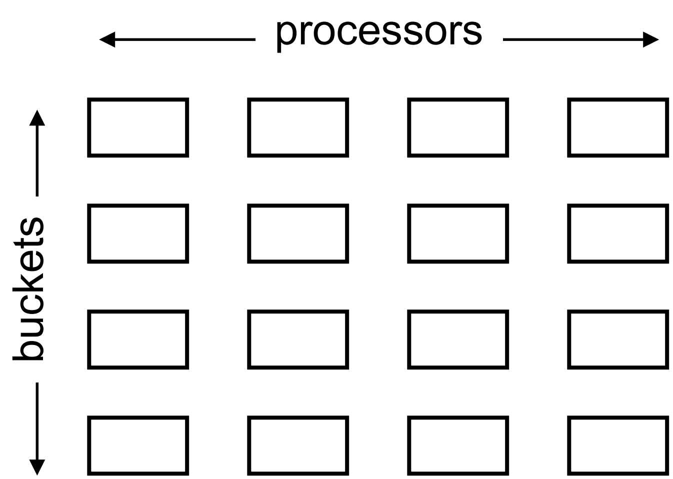
- **How to handle duplicate keys**
  - make each key unique
    - » (key, original index)
      - increases comparison cost and network traffic
  - random choice of possible destinations
    - » suppose p = 5 and splitters are

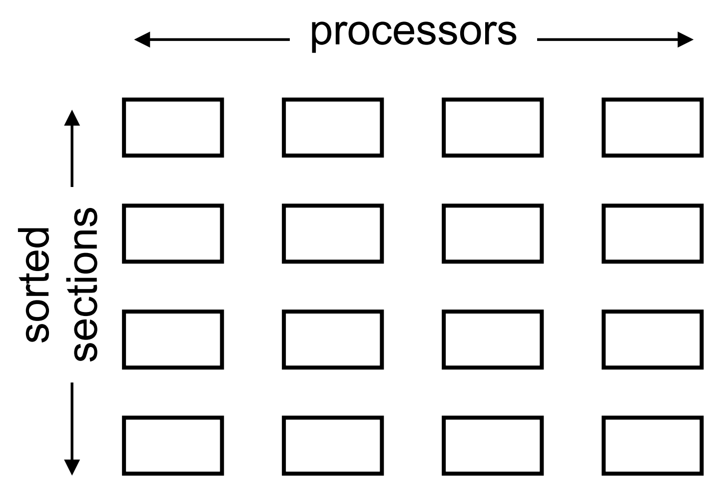
      10, 20, 20, 30

      where should we send key 20?
- **What about restoring load balance?**
  - Worst-case communication cost?

Two-phase sample sort

- **Objectives**
  - scramble input data to create a random permutation
  - highly supersample input to minimize skew





» Randomly distribute keys into p buckets

» Transpose buckets and processors
  - expected bucket size N/p^2

» Local sort

» Proc 1 selects and broadcasts splitters
  - oversampling ratio $k = N/p^2$

» Partition local keys into sorted sections according to splitters
  - expected bucket size N/p^2

» Transpose sorted sections and processors

» Local p-way merge

Two-phase samplesort

1. Randomly distribute local keys into p local buckets
2. Transpose buckets and processors

---

3. Local sort
4. Processor 1 selects (p-1) splitters
5. Broadcast splitters

---

6. Local partitioning of values into p sorted sections
7. Transpose sorted sections and processors

---

8. Local p-way merge of sorted sections

$$C^{2\text{ph}}(N,p) = O\left(\frac{N}{p}\lg N\right) + 2\left(\frac{N}{p}\right)\cdot g + L$$
$$+ O\left(p\lg\left(\frac{N}{p}\right)\right) + 2p\cdot g + 3\cdot L$$