Cross-ratios and angles determine a polygon

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Abstract

We prove that a unique simple polygon is determined, up to similarity, by the interior angles at its vertices and the cross-ratios of diagonals of any given triangulation. (The cross-ratio of a diagonal is the product of the ratio of edge lengths for the two adjacent triangles.) This establishes a conjecture of Driscoll and Vavasis, and shows the correctness of a key step of their algorithm for computing Schwarz-Christoffel transformations mapping a disk to a polygon.

1 Introduction

Computing meshes for numerical solution of problems on domains with irregular geometry poses interesting challenges because it interweaves continuous mathematics with discrete geometry and then seeks an implementation on an even more discrete computer. One excellent example is a new algorithm by Driscoll and Vavasis for computing a conformal mapping of a unit disk to a simple plane polygon [2]. We describe this algorithm in some detail in Section 2. The correctness of one of its steps depends on a purely geometric conjecture [2, Conjecture 1]:

> Let P be an n-vertex, triangulated, simple polygon. Then P is uniquely determined up to similarity transform by the following data: the sequence of all interior angles of P at its vertices, and the list of $n-3$ absolute values of cross-ratios of the quadrilaterals determined by the triangulation of P.

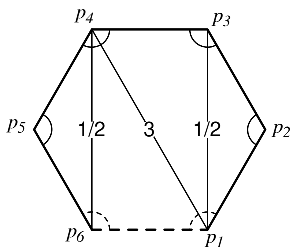

Figure 1: Angles and cross-ratios

Pairs of adjacent triangles, which share a *diagonal edge* of the triangulation, form the above-mentioned quadrilaterals. The *cross-ratio* of a quadrilateral is the product of the lengths of two sides clockwise from the endpoints of the diagonal divided by the product of the lengths of the two sides counterclockwise. In the triangulated hexagon of Figure 1, diagonal edges are labeled with their cross-ratios.

In this paper, we establish their conjecture by establishing properties of *partially-specified polygons*, which we define in Section 4. We employ some observations on the ratios of triangles that we make in Section 3.

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2 The Schwarz-Christoffel transform and the CRDT algorithm

If we think of a simple polygon $P = \{p_1, p_2, \ldots, p_n\}$ as a simply-connected, open subset of the complex plane and let D be the unit disk, then the Riemann Mapping Theorem [3] says that there is an analytic function $f\colon D \to P$ mapping the unit disk to the polygon. If the point $f(0)$ and direction of $f'(0)$ are specified, then f is unique. With this mapping, some problems on P (e.g., Laplace's equation) can be solved on the simpler domain D. Other problems that require a mesh on P can begin with a mesh on D; conformal mapping preserves angles in the mesh.

All mappings $f\colon D \to P$, from the unit disk D to an n-gon P, can be written using the Schwarz-Christoffel formula

$$f(w) = p + q \int_0^w \prod_{1 \le j \le n} \left(1 - \frac{\omega}{w_j}\right)^{\beta_j} d\omega.$$

In this formula, the complex numbers p and q translate, rotate, and scale the image, the scalars $\beta_j = \psi_j/\pi - 1$, where ψ_j is the interior angle at polygon vertex p_j, and the unit-length complex numbers w_j are the *prevertices*, the points on the boundary of the disk that map to the vertices of the polygon. The integration is a complex contour integral in the disk. The S-C formula has $n+4$ parameters. Eliminating similarity transformations of the polygon and the freedom to choose $f(0)$ and the direction of its derivative leaves $n-3$ parameters unspecified. These are determined by numerically solving a difficult nonlinear system [1, 2, 4, 5].

This direct approach suffers from *crowding*—skinny regions of P cause the angle between the prevertices to decrease exponentially in the aspect ratio. This causes problems for numerical techniques; many bits of precision can be lost when determining the angle between two prevertices.

Driscoll and Vavasis [2] have developed an clever combination of numerical and geometric ideas to minimize the impact of crowding. They use a constrained Delaunay triangulation of P, with new vertices introduced on the boundary so that triangles in P have good aspect ratio. There is a corresponding triangulation of prevertices in D (join prevertices iff an edge joins their vertices in P); Driscoll and Vavasis reformulate the nonlinear system to compute the cross-ratios of this triangulation. These $n-3$ cross-ratios are three parameters too few to specify the n prevertices completely. They note that when an application is concerned with accuracy of computations inside a particular triangle, it can specify the three prevertices of this triangle to be spread out around the disk, then determine the rest of the prevertices from the cross-ratios in linear time. Crowding increases further from the specified triangle, but usually the importance of a vertex decreases with distance as well.

As an illustration of their technique, they produced the right side of Figure 2, which shows a mapping from a rectangle to a "maze" polygon. Solid lines in the rectangle at $i/5$, for $1 \le i \le 4$, and dotted lines at 10^{-2}, 10^{-4}, $\ldots$, 10^{-16} map to their images at right. Driscoll and Vavasis [2] report that "all computations were performed in double precision."

While the experimental evidence supports the correctness and efficiency of their approach, it does depend on some conjectures. The first, cited in Section 1 and established in this paper, relates to their iteration to find the cross-ratios of the triangulation of prevertices. Their iteration uses the S-C transform to determine cross-ratios of P from the guessed cross-ratios of prevertices; the question is whether, having determined the cross-ratios of polygon P, the algorithm has determined P. There

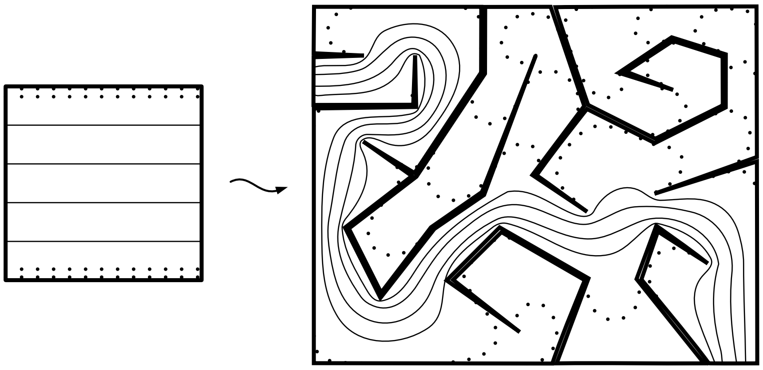

Figure 2: Conformal mapping from rectangle to maze. (From Driscoll and Vavasis [2])

are other interesting conjectures as to the relationship between the magnitudes of cross-ratios in P and in D, which we cannot yet address.

3 Observations on the ratios of triangles

To begin, we establish several sufficient conditions on sides and angles of a triangle for its ratio to increase. We use the notation illustrated in Figure 3: Points p and q are fixed so that the segment $\overline{pq}$ is horizontal from left to right. Points r_1 and r_2 are chosen above the line $\overleftrightarrow{pq}$, and the angles of triangle pqr_i measure α_i, β_i, and γ_i at p, q, and r_i, respectively. Let a_i and b_i denote the lengths of qr_i and pr_i, respectively. The *ratio* of the triangle pqr_i, will thus be a_i/b_i.

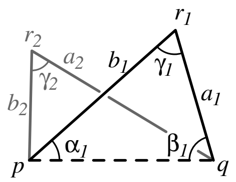

Figure 3: Triangle notation

If we fix $\gamma_1 = \gamma_2$, then the triangle is determined when α_i is chosen. Thus, we establish the next condition by a continuous argument and drop the subscripts.

Lemma 1 *Using the above notation, when angle γ is fixed, the ratio $r(\alpha) = a/b$ is a strictly-increasing function of α, for $\alpha \in [0, \pi - \gamma]$.*

Proof: By the law of sines, the ratio

$$r(\alpha) = \frac{a}{b} = \frac{\sin\alpha}{\sin(\alpha+\gamma)} = \cos\gamma - \sin\gamma \cdot \cot(\alpha+\gamma).$$

This is an invertible function for $0 < \alpha \leq \pi - \gamma$, and its derivative $\sin\gamma/\sin^2(\alpha+\gamma)$ is positive for $0 < \gamma < \pi$, which we must have for pqr to be a triangle. ∎

By the way, all of our conditions are established by algebraic arguments, which is a good way to build character. To develop intuition, it is sometimes more pleasant to observe the conditions geometrically by considering the interaction of four families of constraint curves that cover the plane, which are illustrated in Figure 4. We use these to illustrate the algebraic proofs.

Observation 2 *Let pqr be a triangle with p and q fixed and r to the left of $\overline{pq}$. The following families are the loci of points r with the corresponding parameter of triangle pqr fixed:*

A. points with fixed angle $\alpha = \angle rpq$ lie on the lines through p,

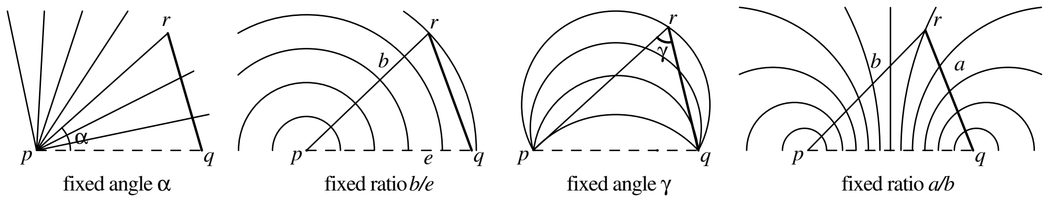

Figure 4: Four families of constraint curves for triangle endpoint r.

B. *points with fixed ratio $b/e = rp/pq$ lie on the circles centered at p,*

C. *points with fixed angle $\gamma = \angle qrp$ lie on circles through p and q, and*

D. *points with fixed ratio $a/b = qr/rp$ lie on circles orthogonal to those of C.*

These curves intersect nicely; a pair of curves chosen from A and B, A and C, B and D, or from C and D will have only one intersection above the x axis. Families A and B and families C and D are orthogonal—any two of their curves intersect at right angles.

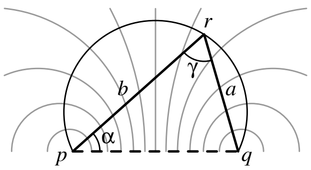

Figure 5: Fixed γ

A geometric argument for Lemma 1 observes that with angle γ fixed, the point r moves along a circular arc of family C as α_i increases, see Figure 5. As r moves, it crosses each curve of family D, and the ratio increases over $(0, \infty)$.

Lemma 3 *Using the triangle notation of Figure 3, the ratios $a_1/b_1 < a_2/b_2$ if $r_1 \neq r_2$ and any of the four following conditions hold:*

(1). if $a_1 \leq a_2$ and $b_1 \geq b_2$,

(2). if $b_1 \geq b_2$, $\alpha_1 \leq \alpha_2$, and $\gamma_1 \geq \gamma_2$,

(3). if $a_1 \leq a_2$, $\beta_1 \geq \beta_2$, and $\gamma_1 \leq \gamma_2$, or

(4). if $a_1 \geq a_2$, $b_1 \leq b_2$, $\alpha_1 \leq \alpha_2$, and $\beta_1 \geq \beta_2$.

Proof: Condition (1) is the easiest. With the assumption that $r_1 \neq r_2$, one of the inequalities $a_1 \leq a_2$ or $b_1 \geq b_2$ must be strict, so when condition (1) holds, $a_1/b_1 < a_2/b_2$.

For the remaining conditions, it helps to express ratios of sides using the law of sines and trigonometric identities. For example,

$$\frac{a_i}{b_i} = \frac{\sin \alpha_i}{\sin \beta_i} = \frac{\sin(\gamma_i + \beta_i)}{\sin \beta_i} = \cos \gamma_i + \cot \beta_i \sin \gamma_i = \cos \gamma_i + \cos \beta_i \left(\frac{e}{b_i}\right).$$

Note that all angles will be in $(0, \pi)$, so sines are positive. Cosine and cotangent are decreasing functions over this angle range, crossing zero at angle $\pi/2$.

When condition (2) holds we consider two cases, depending on whether angle β_1 is obtuse.
(2), β_1 obtuse: Assume, for the sake of deriving a contradiction, that $a_1/b_1 \geq a_2/b_2$, or equivalently

$$\cos \gamma_1 + \cot \beta_1 \sin \gamma_1 \geq \cos \gamma_2 + \cot \beta_2 \sin \gamma_2.$$

Since $\beta_1 > \pi/2$, the value $-\cot\beta_1$ is positive. Also, $\pi/2 > \gamma_1 \geq \gamma_2$, so $\cos\gamma_1 \leq \cos\gamma_2$ and $\sin\gamma_1 \geq \sin\gamma_2$. Combining these inequalities, we observe that $-\cot\beta_1 \leq -\cot\beta_2$; We conclude that $\pi_2 < \beta_1 \leq \beta_2$ and $\alpha_1 \leq \alpha_2 < \pi/2$.

But then, $\sin\alpha_1 \leq \sin\alpha_2$ and $\sin\beta_1 \geq \sin\beta_2$, and, if $r_1 \neq r_2$, one of these inequalities is strict. Thus we derive a contradiction,

$$\frac{a_1}{b_1} = \frac{\sin\alpha_1}{\sin\beta_1} < \frac{\sin\alpha_2}{\sin\beta_2} = \frac{a_2}{b_2}.$$

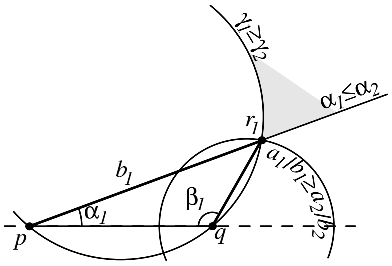

Figure 6: (2), β_1 obtuse

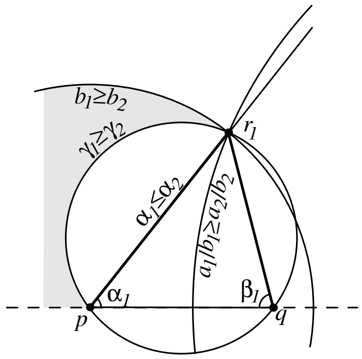

Figure 7: (2), β_1 not obtuse

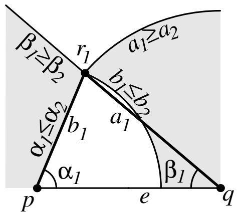

Figure 8: (4), angle & distance constraints

(2), β_1 not obtuse: Assume, for the sake of deriving a contradiction, that $a_1/b_1 \geq a_2/b_2$, The condition $\alpha_1 \leq \alpha_2$ implies $\cos\alpha_1 \geq \cos\alpha_2$. By the condition $b_1 \geq b_2$, we know

$$\cos\alpha_1 + \cos\beta_1\left(\frac{a_1}{b_1}\right) = \frac{e}{b_1} \leq \frac{e}{b_2} = \cos\alpha_2 + \cos\beta_2\left(\frac{a_2}{b_2}\right).$$

Since $\cos\beta_1$ is non-negative, we can combine the above inequalities to derive that $0 \leq \cos\beta_1 \leq \cos\beta_2$. But since $\cos\gamma_1 \leq \cos\gamma_2$ and $e/b_1 \leq e/b_2$, with one of these inequalities strict if $r_1 \neq r_2$, we again derive a contradiction:

$$\frac{a_1}{b_1} = \cos\gamma_1 + \cos\beta_1\left(\frac{e}{b_1}\right) < \cos\gamma_2 + \cos\beta_2\left(\frac{e}{b_2}\right) = \frac{a_2}{b_2}.$$

(3): We reduce condition (3) to condition (2). Reflect the triangle about a vertical line, which exchanges 'a's with 'b's, and 'α's with 'β's. Then exchange '1's with '2's, and apply the proof for condition (2). When we return the conclusion to the original notation, we obtain that $b_2/a_2 < b_1/a_1$. Taking reciprocals gives the desired result.

(4): The final part of the lemma is vacuously true—condition (4) can never be satisfied.

By the law of sines $b_i = e\sin\beta_i/\sin\gamma_i = e\cos\alpha_i + a_i\cos\gamma_i$. Since $\cos\alpha_1 \geq \cos\alpha_2$ and $a_1 \geq a_2$, in order to have $b_1 \leq b_2$ (and $r_1 \neq r_2$), we must either have $0 \leq \cos\gamma_1 < \cos\gamma_2$ or $\cos\gamma_1 < 0$. Similarly, $a_i = e\cos\beta_i + b_i\cos\gamma_i$, $\cos\beta_1 \leq \cos\beta_2$, and $b_1 \leq b_2$. In order to have $a_1 \geq a_2$ (and $r_1 \neq r_2$), we must either have $0 \leq \cos\gamma_2 < \cos\gamma_1$ or $\cos\gamma_2 < 0$. Because the inequalities with non-negative cosines are incompatable, we must have that both γ_1 and γ_2 are obtuse angles.

We can now observe a contradiction: angles α_i and β_i are not obtuse, so $\sin\alpha_1 \le \sin\alpha_2$ and $\sin\beta_1 \ge \sin\beta_2$. By the law of sines, $a_i/b_i = \sin\alpha_i/\sin\beta_i$, but

$$\frac{a_1}{b_1} \le \frac{a_2}{b_2} \qquad \text{and} \qquad \frac{\sin\alpha_1}{\sin\beta_1} \ge \frac{\sin\alpha_2}{\sin\beta_2}.$$

Only if $r_1 = r_2$ can these inequalities be satisfied simultaneously. ∎

4 A proof using partial polygons

The remainder of this paper proves that the interior angles and cross-ratios of a triangulated polygon P are sufficient to determine P up to similarity. The proof uses *partial polygons,* for which all cross-ratios and all angles except the first and last are specified. In notation, let $P = p_1p_2\dots p_n$ be a simple polygon in the Euclidean plane. Assume that we are given:

- $n-2$ angles $\psi_2, \psi_3, \dots, \psi_{n-1}$, where $0 < \psi_i \le 2\pi$ measures the internal angle at p_i.
- $n-3$ cross-ratios for the quadrilaterals formed by the pairs of triangles adjacent to the $n-3$ diagonals in the triangulation of P.

Note that angles ψ_1 and ψ_n at vertices p_1 and p_n are not specified. We call ψ_n the *variable angle* of partial polygon P. If we set $\psi_n = \theta$, we can determine ψ_1 since the sum of interior angles $\sum_{1\le i\le n} \psi_i = (n-2)\pi$.

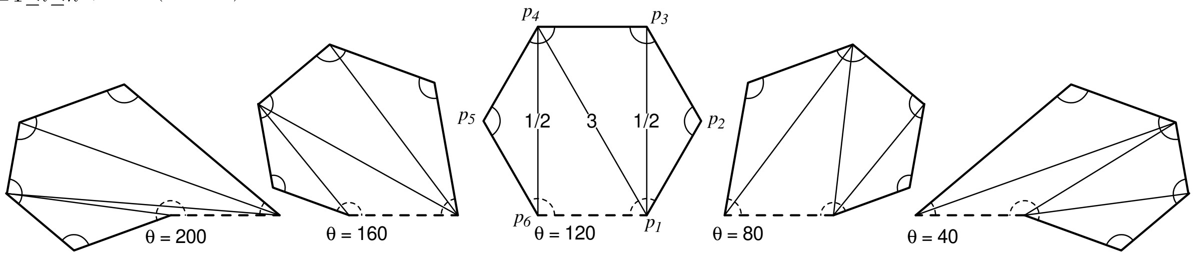

Figure 9: Realizations of a partial polygon P for five different angles θ. In each, the angles at p_2 through p_5 are 120°, and cross-ratios are 1/2, 3, and 1/2 as labeled in the middle hexagon.

A *realization of partial polygon P* is an embedding that respects the specified angles and cross-ratios and has positive, finite area for all triangles; Figure 9 shows five realizations of a triangulated hexagon for different values of θ. No global monotonicity properties for realizations of P seem to be evident as θ varies—e.g., the area and sides of triangle $p_1p_2p_3$ shrink and grow again as θ decreases in Figure 9.

We call $\overline{p_np_1}$ the *unspecified edge* of partial polygon P. It is one side of a triangle in the triangulation of P; let a and b be the lengths of the other two sides in counter-clockwise order, and define the *ratio of* P to be a/b. We prove by induction that the ratio of P is a partial function of θ that is strictly-increasing over its domain.

Theorem 4 *Let P be a partial polygon with variable angle θ. For each θ, the realization of P, if it exists, is unique, and the ratio of P is a strictly-increasing partial function $r(\theta)$.*

Proof: We prove this by induction on the number of triangles in P.

Base: In the base case, partial polygon P is a single triangle $p_1p_2p_3$. We can choose the positions of p_1 and p_3 as in Figure 10 to fix the similarity transformation. Since angle ψ_2 at p_2 is specified, Lemma 1 applies to show that the ratio is a strictly-increasing function of the variable angle θ at p_1.

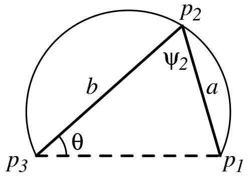

Figure 10: Base case

In the induction step, we are given a partial polygon P, having $n > 3$ vertices, $n-2$ triangles, and variable angle θ and unspecified edge e, which we fix as horizontal. Removing the triangle incident on the unspecified edge leaves a pair of partial polygons or, when this triangle uses vertex p_{n-1} or p_2, a single partial polygon. We handle these cases separately.

Single using p_{n-1}: When triangle $p_1p_{n-1}p_n$ is adjacent to the unspecified edge e of P, then removing this triangle leaves a partial polygon P' from p_1 to p_{n-1} to which we may apply the induction hypothesis.

We define notation as in Figure 11: Let ϕ be the variable angle for P', and let a be the length of the unspecified edge of P'. Let $\gamma = \psi_{n-1} - \phi$. Let b be the length of $\overline{p_{n-1}p_n}$, which is the third side of the removed triangle, and let β be the angle opposite the side of length b. Let κ be the cross-ratio specified for diagonal $\overline{p_{n-1}p_2}$. Define the *cross-ratio function* $s(\phi)$ to be the ratio for P' divided by κ. By the induction hypothesis, $s(\phi)$ is a strictly-increasing partial function.

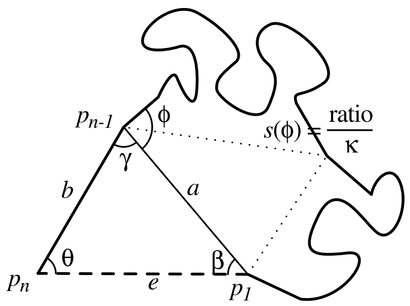

Figure 11: Single partial polygon

We want to prove that the ratio $r(\theta) = a/b$ is a strictly-increasing partial function. First, we note that a given angle θ determines at most one realization of P. As ϕ varies from 0 to ψ_{n-1}, the function $s(\phi)$ increases. At the same time, the angle $\gamma = \psi_{n-1} - \phi$ decreases; for a fixed θ, Lemma 1 says that the ratio e/b also decreases. Because any realization of P has $e/b = s(\phi)$, the realization for any fixed θ is unique.

Now, to show that $r(\theta) = a/b$ is strictly increasing, we compare the ratios for two realizations of P with two different angles, $\theta_1 < \theta_2$. We consider two subcases, depending on whether the corresponding angles $\gamma_1 \geq \gamma_2$ or $\gamma_1 < \gamma_2$. (As with the lemmas of the previous section, we can assume that e remains fixed, and subscript other lengths and angles in the two realizations.)

If $\gamma_1 \geq \gamma_2$, then $\phi_1 \leq \phi_2$, so the ratio $e/b_1 = s(\phi_1) \leq s(\phi_2) = e/b_2$. Since angles $\theta_1 < \theta_2$ correspond to the αs Lemma 3, we have the conditions of part (2) and can conclude that the ratio increases: $r(\theta_1) = a_1/b_1 < a_2/b_2 = r(\theta_2)$.

On the other hand, if $\gamma_1 \leq \gamma_2$, then $\phi_1 \geq \phi_2$, so the ratio $e/b_1 = s(\phi_1) \geq s(\phi_2) = e/b_2$. Since the angles of the triangle sum to π, we know that $\beta_1 \geq \beta_2$. Thus, when $a_1 \leq a_2$ then we apply Lemma 3(3), and when $a_1 \geq a_2$ we apply Lemma 3(4); in either case the ratio increases: $r(\theta_1) = a_1/b_1 < a_2/b_2 = r(\theta_2)$.

This completes the argument for the single case using p_{n-1}.

Single using p_2: When triangle $p_1p_2p_n$ is adjacent to the unspecified edge of P, we can reduce this case to the previous. Reflect P about a line, taking the reciprocals of all specified cross-ratios, to obtain P_R. Apply the argument of the previous case to show that there is a strictly-increasing function of angle θ_R for the ratio $r_R(\theta_R)$ of P_R. Then there is a strictly-increasing function of angle θ for the ratio $r(\theta)$ of P, namely

$$r(\theta) = \frac{1}{r_R\Big((n-2)\pi - (\sum_{2\le i\le n-1}\psi_i) - \theta\Big)}.$$

Uniqueness is preserved under reflection.

Pair using p_j: When triangle $p_1p_jp_n$ is adjacent to the unspecified edge e of P, for some $2 < j < n-1$, we can label the sides $a = \overline{p_1p_j}$ and $b = \overline{p_jp_n}$, and angles α, β, and $\gamma = \pi - \alpha - \beta$ opposite the sides a, b, and e. We will fix the similarity by choosing edge e to be horizontal as in Figure 12.

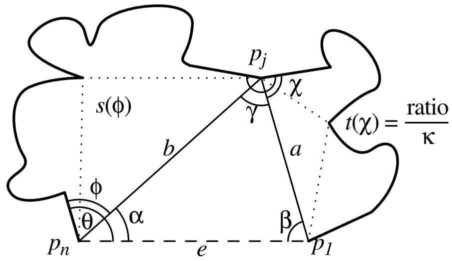

Figure 12: Pair of partial polygons

Removing triangle abe leaves two partial polygons: one from p_j to p_n with variable angle $\phi = \theta - \alpha$ and cross-ratio function $s(\phi)$, and another from p_1 to p_j with variable angle χ and cross-ratio function $t(\chi)$. (Because the angle at p_j is specified as ψ_j, and the angles of a k-gon sum to $(k-2)\pi$, we know that $\chi + \gamma - \phi$ is a constant, namely $(n-j-1)\pi - \sum_{j\le k<n}\psi_k$. Similarly, $\theta + \beta - \chi$ is a constant.) We can apply the induction hypothesis to both partial polygons, so the cross-ratio functions $s(\phi)$ and $t(\chi)$ are strictly-increasing partial functions of their angles.

We begin with a different decomposition of P, however, to show that the realization of P for a given θ is unique. Break P into two partial polygons at edge a. If we fix θ, then the induction hypothesis applies to the polygon from $p_j, \dots, p_n, p_1$: the ratio b/e is a strictly-increasing partial function of variable angle β, and its reciprocal e/b is decreasing. With $\theta + \beta - \chi$ a constant and θ fixed, however, angle χ and ratio function $t(\chi)$ are also strictly-increasing (partial) functions of β, and $t(\chi) = e/b$. Equality holds at most once for any specified θ.

Now, to show that $r(\theta) = a/b$ is strictly increasing, we compare the ratios for two realizations of P with two different angles, $\theta_1 < \theta_2$, and consider four subcases depending on the comparisons of corresponding angles ϕ_1 with ϕ_2 and χ_1 with χ_2. (As before, we assume that e remains fixed, and subscript other lengths and angles in the two realizations.)

Pair, $\phi_1 < \phi_2$ and $\chi_1 \le \chi_2$: Since $s(\phi_1) = a_1/e < s(\phi_2) = a_2/e$ and $t(\chi_1) = e/b_1 \le t(\chi_2) = e/b_2$, the ratio $r(\theta_1) = a_1/b_1 < a_2/b_2 = r(\theta_2)$ increases in this case. Note that this is Lemma 3(1), but it is easier to observe directly than by paging back to the lemma.

Pair, $\phi_1 \ge \phi_2$ and $\chi_1 \le \chi_2$: Here, $t(\chi_1) = e/b_1 \le t(\chi_2) = e/b_2$, so $b_1 \ge b_2$. Since $\alpha_i = \theta_i - \phi_i$, angles $\alpha_1 < \alpha_2$, and since $\gamma_i + \chi_i - \phi_i$ is a constant, $\gamma_1 \ge \gamma_2$. This satisfies the conditions for Lemma 3(2), so $r(\theta_1) = a_1/b_1 < a_2/b_2 = r(\theta_2)$ in this case, also.

Pair, $\phi_1 < \phi_2$ and $\chi_1 > \chi_2$: Here, $s(\phi_1) = a_1/e < s(\phi_2) = a_2/e$. The angles $\gamma_1 < \gamma_2$, since $\gamma_i + \chi_i - \phi_i$ is a constant, and $\beta_1 > \beta_2$, since $\beta_i + \theta_i - \chi_i$ is a constant. This satisfies the conditions for Lemma 3(3), so $r(\theta_1) = a_1/b_1 < a_2/b_2 = r(\theta_2)$.

Pair, $\phi_1 \geq \phi_2$ and $\chi_1 > \chi_2$: Now, $s(\phi_1) = a_1/e \geq s(\phi_2) = a_2/e$ and $t(\chi_1) = e/b_1 \geq t(\chi_2) = e/b_2$. Angles $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$, since $\alpha_i = \theta_i - \phi_i$ and $\beta_i + \theta_i - \chi_i$ is a constant. This satisfies the conditions for Lemma 3(4), so $r(\theta_1) = a_1/b_1 < a_2/b_2 = r(\theta_2)$—vacuously, in fact.

This completes the analysis of the pair case. Therefore, by mathematical induction, the theorem is established. ∎

Because any fully-specified polygon can be made into a partial polygon by omitting the first and last internal angles, we have the following corollary.

Corollary 5 *A triangulated n-gon P is uniquely determined, up to similarity, by its sequence of internal angles and the cross-ratios of its diagonals.*

5 Conclusions

We have proved by induction that angles and cross-ratios determine a unique simple polygon. It might be nice to extract from the proof an algorithm to compute the realization, but because the proof reveals that the polygon's vertex coordinates are related by compositions of trigonometric functions, an exact implementation would require expensive, high-precision arithmetic. The CRDT algorithm of Driscoll and Vavasis [2] performs an iterative computation to find a Schwarz-Christoffel mapping of a disk to a polygon; our theorem shows that their algorithm does produce a mapping to the correct polygon.

Many interesting questions about the CRDT algorithm remain, including to analyze its rate of convergence and quantify the worst-case crowding. It would be good to consider other interesting combinations of continuous and combinatorial geometry that address precision questions.

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