Drainage Queries in TINs:

From local to global and back again*

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Abstract

This paper considers the cost of preprocessing a digital terrain model (DTM) represented as a triangulated irregular network (TIN) so that drainage queries—e.g., what is the watershed of a query point, or how much water passes through a point given that rain is falling at a known rate—can be answered by simply evaluating a summary function. Although the worst-case storage and preprocessing costs are high, the experimentally-observed costs are reasonable.

In order to compute a compact and consistent summary function, the drainage network needs a rigorous definition. This paper, therefore, also surveys some of the previous definitions, extends them, and establishes a number of properties of drainage networks with a focus on TINs.

1 Introduction

Terrain drainage characteristics provide important information on water resources, possible flood areas, erosion and other natural processes. In natural resource management, for example, the basic management unit is the *watershed*, the area around a stream that drains into the stream. Road building, logging, or other activities carried out in a watershed all have the potential to affect the defining stream. Manual quantification of terrain drainage characteristics is a tedious and time consuming job. Thus, many researchers have developed spatial analysis algorithms to compute drainage characteristics from digital terrain models (DTMs).

We began this research intending to go one step further. Many of the spatial analysis algorithms on terrains are batch processes whose product is a map depicting drainage characteristics; our aim is to provide this map, but also provide richer structures that could answer interactive queries for watershed areas, regions, or flow rates. We had to start, however, by taking a step back to reconsider definitions of the drainage network and its components, especially for triangulated irregular networks (TINs). Thus, Section 2 begins with a brief critique of definitions and algorithms for drainage in various terrain models.

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Frank et al. [6] point out that formal definitions should be used to define terrain-specific features so that properties of the structure can be established mathematically and contradictory definitions (or at least those in disagreement with other related research) can be avoided. We concur, and propose definitions in Section 2.3 that capture the global behavior of non-inertial flow on a surface. Our definition extends the local definitions of Frank et al. [6] to include accumulation of water. The focus of the definitions and methods is on triangulated irregular networks (TINs).

Using a global definition for the drainage network (with accumulation) is natural but may make it more difficult and time consuming to extract additional drainage information. Section 3 describes how to augment the drainage network with quantitative information like areas of watersheds and flow rates through the network. This information is represented in a compact and consistent manner by summary functions, which are computed for all parts of the drainage network. The summary functions turn global drainage queries into local ones.

Several useful applications of the summary functions cannot be obtained easily from local drainage models. Firstly, user-defined questions on flow rate and basin areas can easily be answered, since these are quantified by the accumulation in the model. Secondly, by thresholding on the flow rate in the drainage network we mmediately obtain a multi-scale drainage network, useful for map generalization. Thirdly, by thresholding on basin areas (or volumes), local depressions that could be artifacts of data acquisition or modelling can be removed in a uniform way.

For an n-point TIN, a fiendish (and improbable) worst-case example shows that the size of this structure can be cubic, but preliminary experimental results indicate that the size is more likely to be proportional to n. We then turn to the delineation of the watershed of a point, or basins for whole parts of the drainage network. Section 4 relaxes some of our initial assumptions, indicates possible extensions to the model and at the same time, discusses ideas for further research.

2 Definitions and assumptions for drainage

A terrain is the graph in R^3 of a continuous function $z = f(x, y)$ defined on a closed, compact, connected subset of the xy plane. A TIN is a terrain that consists of triangles.

2.1 Assumptions

When one looks at a terrain, one can intuitively identify a network of watercourses and ridges based on the notions that water flows downhill, tributaries join larger streams, rivers run to the sea, and ridges divide rivers. Let us be more precise about the first three of these assumptions; we consider the fourth in Section 3.2.

A1 At any point, water follows the steepest descent,

A2 Watercourses can merge, so that the drainage network has the topology of a set of trees, and

A3 Watercourses end only at a local minimum of the terrain.

Precise definitions of watercourse and ridge networks that satisfy these assumptions are rare in the literature, especially when these definitions are applied to a terrain model that is stored in the computer. In hindsight, this should not have been a surprise; from a continuous, two-dimensional terrain, represented in the computer by a discrete set of bits, one wishes to extract connected networks that are one-dimensional but have a discrete (graph) topology. Thus, if the reader will permit some hasty generalizations, the mathematical literature is "too continuous," being concerned with the differential geometry of flows on smooth surfaces, the GIS literature is too focused on computation, often defining the drainage network as the output of some algorithm, and the hydrology literature is willing to introduce uncomputable "infinite regress" to have duality properties for interlocking ridge and channel networks. The issue of interlocking networks is discussed in Section 3.2.

2.2 Earlier definitions in various terrain models

In addition to drainage definitions for smooth mathematical surfaces, there are different definitions for the three main forms of digital terrain models (DTMs): gridded DEM (digital elevation model storing elevations at points of a regular grid), digital contours, and TINs (triangulated irregular networks). Drainage has also been defined in terms of other surface networks.

Smooth mathematical surfaces The study of drainage on *smooth* mathematical surfaces has a long history, including mathematicians such as Cayley [3] and Maxwell [16]. Koenderink and van Doorn [11, 12] credit Rothe [27] with the first characterization of ridges and channels in 1915; Koenderink and van Doorn's work expresses this solution in the terminology of modern differential geometry and includes examples of why other attempts at characterizing drainage on smooth surfaces (especially using those using local criteria) are inadequate.

Unfortunately, in smooth surfaces, watercourses tend not to join except in the limit, violating A2.A3 can also be violated; watercourses can end at points that are not local minima. A further complication is that most smooth surfaces and their flows can only be approximated in the computer. Thus, it is debatable whether smooth surfaces are appropriate models of physical terrain.

Contours Some researchers [8, 13, 19, 32] define the drainage networks in terms of contour lines: A points is on the the watercourse or ridge network if it is a local minimum or maximum on its contour line. Koenderink and van Doorn [11] have shown that this definition can violate A1—the computed network does not necessarily travel in the direction of steepest descent. This definition also implies that the terrain should be twice differentiable, which again hinders streams from joining (A2). Three of the above-mentioned papers [8, 13, 32] continue the computation of the drainage network on gridded models.

Grid or raster-based DEM There is no accepted definition of the drainage network in a grid. Frequently the network is defined as the output of an algorithmic procedure, which may not be completely specified outside of the computer code. We can, as other researchers have done, classify these computations by their locality and by whether they have an explicit surface model.

A number of methods [5, 9, 10, 13, 17, 21, 23, 28, 31] are based on extracting features from a raster DEM by local operations, using techniques that are common in raster image processing. Local filters are are used to detect potential pit, peak, channel, and ridge cells, then the channel and ridge cells are linked according to some heuristic to form the drainage network. Basin boundaries are formed in a similar fashion [9, 17]. Assumptions A1 and A3 are frequently violated, since the global nature of the terrain's drainage network is not adequately reflected. As with image processing, there are inherent ambiguities when combining information from local filters—one research team developed an expert system for this task [26]. Different heuristics will result in different sets of extracted terrain features.

Mark [15] proposed a global computation: sort all the cells in order of decreasing elevation, and initially assign each cell one unit of fluid. Then each cell in order adds its fluid to the lowest of its eight neighbors. In the end, cells receiving more than a threshhold quantity of fluid are declared to be on the drainage network. This method has an implicit surface and approximates a steepest descent; at lower elevations the network may become two-dimensional.

A grid of points with elevations does not give a terrain unless a continuous surface is defined. (One early approach to drainage [23] has no concept of surface.) Approaches that define flow based on 4 or 8 neighbors in the grid [2, 9, 15, 21, 24, 28] define an implicit surface, which may not have geometric interpretation in which water flows according to steepest descent. If an interpolation method is specified, then the grid can be considered as an explicit surface on which flow can be properly defined. Douglas [5] inserts a virtual grid point at the center of each square for a piecewise linear interpolation; Haralick [7] uses bicubic polynomial surface patches for a continuous and smooth surface. Afterwards, however, flow is still approximated in raster pieces.

We note that both the surface and the flow are discretized in many grid algorithms. The discontinuities in both make it difficult to compute summary functions and lead us to prefer TINs.

TINs Triangulated irregular networks, or TINs, define a continuous terrain that is not differentiable at the edges and vertices. Flow along triangles and edges are easy: the steepest descents in a triangle lie on parallel lines; an edge becomes a channel if flow comes from both adjacent triangles.

In the literature, the continuous nature of the terrain is often ignored; this happens in one of two ways. First, the flow may be discretized and passed triangle to triangle based on neighbor relations in the direction of steepest slope [22]. This approximates steepest descent (A1). It also calls for small elements, which works against the adaptive nature of TINs. Second, a subset of edges that catch flow (the *cofluent* edges or local channels as defined in the next section) may be considered important features and cross-triangle flows may be ignored [6, 20, 30, 33]. Watercourses then stop at places that are not local minima (A3).

Surface networks Related to drainage networks are the so-called surface networks by Mark [14], Pfaltz [25], Warntz [35], Shinagawa and Kunii [29], and Wolf [39, 40]. Also worth mentioning as related work is the computation of terrain data from hydrological information [19, 32]; hydrologists use complex models that differentiate between surface flow and subsurface flow, and therefore are also dependent on permeability of the ground, rainfall, and other quantities [18, 30, 34].

2.3 Definitions for global drainage in TINs

We make one more assumption in order to simplify our definitions:

A4 At any point, there is a unique direction of steepest descent.

This assumption rules out flat faces and enforces a canonical choice at edges and vertices where there may be more than one descent. It also means that watercourses in our model do not bifurcate. We modify and relax this assumption in Section 4.

General definitions The following definitions capture infinitesimal, viscous flow on a surface.

- The *trickle path* $\tau(p)$ of a point p on a surface S is the path that begins at p and follows the steepest descent until it reaches a local minimum or the boundary of S.
- The *watershed* of a point p is the set of all points whose trickle paths contain p.
- The *watercourse network* (or *drainage network*) consists of all points whose watersheds have non-zero area (or more mathematically: whose watersheds have two-dimensional Lebesgue measure).

A key observation is that trickle paths cannot cross—if two trickle paths join at some point, then by our assumptions they must continue together along the steepest descent path.

As we shall see next, these definitions capture the local definitions in a TIN by Frank et al. [6, 33] and extend them to satisfy our assumptions on global behavior of a drainage network.

In smooth terrains these definitions omit some of the special slope lines that satisfy Rothe's differential equations [11, 12] because trickle paths in smooth terrains may merge only in the limit. The expected watercourse at the bottom of a smooth gutter, for example, does not materialize. Rothe's equations do not apply in non-differentiable terrains such as TINs, however.

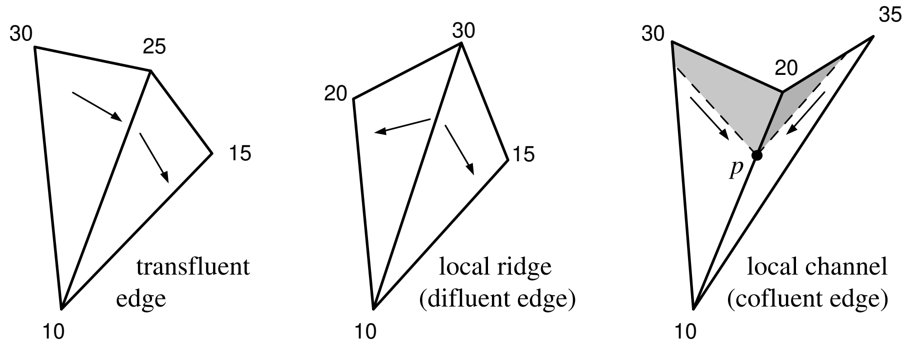


Figure 1: Definitions in a TIN: Any point p on a local channel has a watershed with non-zero area

The relation to TIN-based definitions Figure 1 illustrates definitions from Frank et al. [6] that are specific to edges of a TIN. We say that an edge e of a triangle *receives flow*, *sends flow*, or does neither, depending on whether the steepest descent direction for the triangle moves flow into, out of, or parallel to the edge. By convention, edges do not include their endpoints.

- An edge e of the triangulation is *transfluent* if it is receives flow from one adjacent triangle and sends flow to the other.
- An edge e of the triangulation is a *local channel* (or is *cofluent*) if it receives flow from an adjacent triangle and is not transfluent.
- An edge e of the triangulation is a *local ridge* (or is *difluent*) if it sends flow to an adjacent triangle and is not transfluent.

The following definitions apply to general surfaces.

- A *peak* is a local maximum of the surface. In a TIN, a peak is a vertex with higher elevation than its incident edges and triangles.
- A *pit* is a local minimum of the surface. In a TIN, a pit is a vertex with lower elevation than its incident edges and triangles.
- A *basin* is the watershed of a pit.

These definitions imply that all local channels are on the watercourse network and that none of the local ridges are: A point p on a local channel collects fluid from the region bounded by steepest ascents in the adjacent triangles, as shown shaded in Figure 1. A point q on a ridge is on the trickle path for itself. In fact, the trickle paths from local channels determine the watershed network.

Lemma 2.1 *The watercourse network in a TIN is collection of disjoint (graph-theoretic) trees rooted at pits whose leaves are local channels.*

Proof: The trickle path from any point on the watercourse network is contained in the network. Since a trickle path can only end at a pit, the watercourse network is, what is called in graph theory, a forest of rooted trees. These trees are disjoint because the trickle paths cannot bifurcate.

We have already argued that local channels are contained in the watercourse network, and one can check that points on triangles, transfluent edges, or vertices are on the network only if some point above is also on the network. Therefore the watercourse network begins at local channels. ∎

Knowing the topology of the watercourse network, the following definitions are natural. We'll use them throughout the paper.

- A *(watercourse) junction* is a point of the watercourse network that has degree at least 3 in graph terminology, that is, there are at least two separate incoming parts of the watercourse network.

- A *watercourse* segment is a maximal portion of the watercourse network that doesn't contain junctions, leaves, or pits of the watercourse network.
- A *catchment area* of a watercourse segment is the part of the terrain that drains directly into the watercourse segment; it does not enter another watercourse segment first.

Junctions occur, for example, where two local channels have the same downstream vertex or where cross-triangle flow reaches a local channel.

3 Processing a TIN for drainage queries

The definitions from the previous section describe global flow on a TIN surface. There are now a number of queries or operations that we would like to support, including:

1. What is the area of the watershed of point q?
2. What is the area of the projection of the watershed of point q on the xy plane?
3. What is the flow rate through q if water falls at a uniform rate r?
4. Construct the watershed of point q.
5. What is the catchment area (projected area, flow) of a given watercourse segment?
6. What points have watershed areas of size (at most, at least, exactly) A?

With additional processing (after a few more definitions) we can partition a TIN surface into pieces for which simple functions summarize the drainage behavior—in essense turning drainage computations back into local computations.

3.1 Partitioning a TIN based on drainage

To compute the summary functions needed to solve the drainage queries, we need a few more definitions that are tailored for this purpose. The *drain of a point* p, denoted $drain(p)$, is the point where the trickle path $\tau(p)$ first meets the watercourse network. If p is on the network, then $drain(p) = p$.

The set of basins, one for each pit, form a coarse level partition of a TIN. The set of catchment areas for the watercourse segments refines this partition. For a watercourse segment s, its catchment area consists of all points p for which $drain(p)$ is on s.

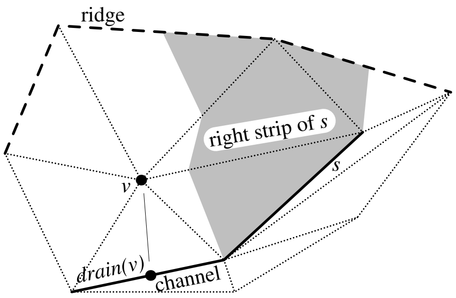


Figure 2: Right strip for segment s

Define *strips* as a further refinement: For every triangle vertex v of the TIN, introduce a node at $drain(v)$. These nodes cut the watercourse segments into yet smaller segments that we call *base segments*. One can easily verify that the nodes can only lie on local channels. Figure 2 gives an illustration. For a base segment s of a local channel, define the *right strip* to be the set of points p such that $drain(p)$ lies on s and the trickle path $\tau(p)$ comes from the right, assuming base segments are directed consistent with the flow. Define the *left strip* similarly. Strips, whether left or right, have a simple structure.

Lemma 3.1 *A strip is a region bounded by a base segment from a local channel, a segment of a local ridge, and two steepest descent paths. No vertices of the TIN lie inside a strip.*

Proof: We can establish this lemma for a right strip by induction on the number of triangles that the strip intersects.

Consider the triangle T to the right of the base segment s. The points of T in the strip are contained in a trapezoid (or triangle) formed by extending the endpoints of s in the direction of steepest ascent until they hit a triangle edge. Let s' denote the segment of this edge that forms the upper boundary of the trapezoid. This really is a trapezoid (or triangle, if an endpoint of s is a vertex of T): only one of the other two edges of T can be hit, since the relative interior of s does not contain $drain(v)$ for any triangle vertex v.

Now, if s' lies on a ridge, the trapezoid is the strip. Otherwise, the strip is the trapezoid together with the strip of s'. Mathematical induction completes the proof. ∎

This lemmas has immediate corollaries about the structure of coarser partitions and about summarizing area and flow.

Corollary 3.2 *Basins, watersheds, and catchment areas are bounded by local ridges and steepest descent paths.*

Proof: All three are composed of left and right strips for watercourse segments. ∎

Corollary 3.3 *Watershed area, projected area, or steady-state flow rate under uniform rainfall assumptions can be summarized by piecewise quadratic functions on segments of the refined watercourse network.*

Proof: For a query point q on the watercourse network, the base segment s determines left and right strips that are partially inside the watershed of q. Every other strip is either entirely outside of the watershed of q or entirely inside and contributes a constant amount of area, projected area, or steady-state flow.

The contribution from the left and right strips with base s can be summarized as a quadratic function $at^2 + bt + c$, where t is a parameter indicating the position of q on s. ∎

3.2 Ridge networks and duality

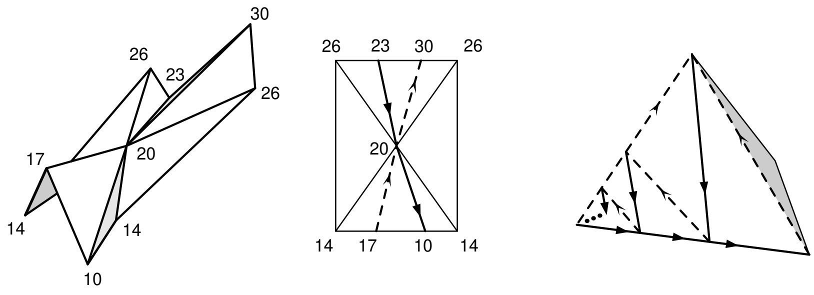
The smallest natural units in the watercourse network are the watercourse segments. To partition the terrain into hydrological units a natural choice, therefore, is to delineate the catchment areas of the watercourse segments. We have seen that these are bounded by local ridges and steepest descent paths, and they are the union of sequences of strips. The efficient computation of watershed delineation is based on tracing the trickle paths of junctions *back upward* to local ridges.

In the full paper, we investigate the structure and computation of ridge networks in more detail: The important role of *passes,* also called *saddle points,* arises naturally from our definitions. For this extended abstract, we confine ourselves to some comments on the dual relation of rigdges and watercourses. Two types of duality are referred to in work defining ridges and courses; both have limited validity.

The first is that ridges and watercourses change roles when a terrain is inverted (or when all the elevations are negated). This is true for the local features: Local ridges and local channels, pits and peaks are duals in this sense. The global assumption A3, that water flows to a pit, breaks this duality at vertices, however. We cannot assume that ridges continue to a peak along the lines of steepest ascent without allowing watercourses to flow across ridges. Figure 3(a) shows such an example.

A second is that, in a given terrain, the ridge and watercourse networks interlock—that there is a ridge between two adjacent courses and a course between adjacent ridges. Werner [38] shows that in such a case the ridges and courses can be embedded in dual graphs and that specific properties of one network determine specific properties of the other. Here one must be careful with the meaning of "between." Werner's definition of "between" essentially guarantees that an embedding is possible, and thus his theorems are about graphs and need not reflect a physical situation.

Frank et al. [6] pointed out that this assumption can lead to infinite regress, a fact that was known to Warntz [36], even in a single triangle. Consider the side of a pyramid as having two ridges and a watercourse at the base, as in Figure 3(c). To separate the ridges, one must introduce a course splitting the triangle; one of the pieces now needs a ridge to separate two courses, ad infinitem. Although Frank et al. attributed this to the discrete nature of a TIN, continuous examples can also be given. Dawes and Short [4] argue that the problem occurs when one insists that the ridge and course networks are joined; if they are not joined,



(a) steepest ascent can cross steepest descent (b) top view of (a) (c) inifinite regress

Figure 3: Ridges and watercourses are not strict duals

however, then watershed boundaries are hard to determine and the word “between” must again be carefully defined.

In our global approach, every division of the watercourse network into segments induces a ridge network so that the two networks, taken together, form the catchment areas for each side of each watercourse segment.

3.3 Answering queries

Using the summary functions or ridge networks described above, we can answer the six queries from the beginning of this section.

1–3. What is the watershed area, projected area, or flow rate through point q?

These queries can be answered from the summary functions attached to the watercourse network at q. (If q is not on the watercourse network, then the answer is “no measurable amount.”

4. Construct the watershed of point q.

Construction queries cannot be summarized, but can be answered in time proportional to the number times the output intersects triangles. Simply extend all strips through the triangles.

5. What is the catchment area (projected area, flow) of a given watercourse segment?

The area can be determined by subtracting the watershed areas for the nodes at the top from that for the node at the bottom of the watercourse segment. The other quantities can be determined analogously.

6. What points have watershed areas of size (at most, at least, exactly) A?

The summary functions can be evaluated to determine where a threshhold is reached or exceeded. “At least” or “at most” queries can be answered in time proportional the output size (plus the number of pits in the “at least” case) by starting with each pit or leaf, respectively, and growing the answer until the threshhold is reached.

3.4 Data structure size and computation

Of course, one must pay a price for considering a global definition of drainage and again for computing and storing local summaries. We briefly investigate the memory size and computation time requirements for the watercourse network and for augmenting it with local summaries.

In a TIN with n vertices, the worst-case size of the network alone can be horrible. In conjunction with a number of computational geometers [1], we have shown that it can require on the order of n^3 memory cells. The upper bound can be proved by showing that a trickle path croses any edge at most n times; the lower bound is by a construction depicted in figure 4, in which roughly n rivers cross n triangulation edges n times each.

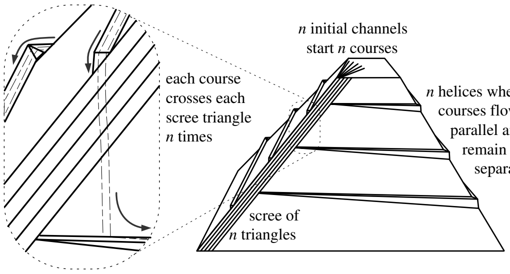


Figure 4: A watercourse network with cubic size

As you can see, this construction is unlikely to occur in physical geography, with the possible exception of when the Egyptians were building the pyramids. In our preliminary experiments on TINs derived from sampling data from St. Mary's Lake, B.C., the network has slightly fewer segments than there are vertices. (For the final paper we will have more extensive tests on data provided by Facet Decision Systems.)

No. vertices	450	1800	4956	11125
Network size	353	1427	4134	10579

The network can be computed in time proportional to its size by simply tracing down from every local channel, marking which local channels we have visited so that we can stop as soon as we encounter a marked edge.

The memory required for local summaries is easy to analyze. For each vertex v of the TIN, $drain(v)$ is a node of the augmented network, which requires a position and three coefficients of a quadratic polynomial.

We compute $drain(v)$ for each vertex v by simply tracing the trickle path $\tau(v)$ through the triangles to $drain(v)$. Several clever data structuring techniques could have been used to trace a several points at once and reduce the worst-case construction time, but our preliminary experiments indicated that this would not be a significant savings over the practical cases that we were attempting. The summary functions for area, projected area, or flow could be computed from areas of trapezoids in strips; we measure the quantities at three points in a watercourse segment and fit a quadratic polynomial to the measurements.

4 Extensions and future research

Our drainage model can be extended to take into account the absorption of water by different ground types. It can also be extended to take into account the fact that water has volume and can collect—one clearly should do so in terrains because errors in data or in sampling can create spurious local minima that are shallow pits. Any significant volume of water would collect to fill these and then spill over the lowest pass to rejoin the main network. Also, rivers whose flow is less than a certain volume can be pruned from the network. This is important for generalization of terrains [37]. By choosing the appropriate threshold value generalization to various scales is possible.

In our processing, we may desire that local depressions be filled to form lakes and that their outlet points be identified, so that water is assumed to pass through the depression and flow down from the outlet to find its final destination. Again we may use a threshold on the basin area to determine which local depressions should be filled, and smoothing operations can be performed to remove them. O'Callaghan and Mark [21] reported 90% removal rate for one pass of a simple binomial smoothing operator. For larger one, explicit methods have to be used to fill them.

We can also incorporate water absorption into the flow model. By employing precipitation models and water absorption models to different areas of the terrain (based on the surface type and vegetation, for instance), we can either fill the small pits into lakes if the influx is greater then the absorption, or leave it as a real pit (sink), if otherwise.

Finally, it is also possible to incorporate bifurcation of streams, and to allow horizontal triangles. Whether and how to do this will depend on the specific applications and desired drainage characteristics, and therefore we have chosen not to make any particular choices in our model.

Acknowledgment

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