# Erased Arrangements of Lines and Convex Decompositions of Polyhedra

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**Abstract**

In 1984, B. Chazelle [*SIAM J. Comp.,* 13 (1984), pp. 488–507] proposed a notch-cutting procedure for decomposing a non-convex polyhedron into convex pieces. This paper shows that notch-cutting, when applied to a polyhedron with $n$ faces and $r$ reflex dihedral angles, gives a convex decomposition with $\Theta(nr + r^{7/3})$ worst-case complexity. The upper and lower bounds are obtained by studying the complexity of the horizon of a segment in an *incrementally-constructed erased arrangement* of $n$ lines. Efficient deterministic algorithms to compute this decomposition are also described.

# 1 Introduction

Some tasks of spatial solid modeling and computer-aided design (CAD) can be performed more easily on convex polyhedra than on non-convex ones. Examples include intersection, mesh generation, checking separation, and construction from primitives. Thus, a procedure that decomposes a polyhedron into convex pieces can be an important first step in these tasks.

Define the *complexity* of a set of polygons or polyhedra to be the total number of their vertices. When decomposing a polyhedron into convex pieces, one would like to keep the complexity small because it has a direct impact on the difficulty of solving convex subproblems. Finding the decomposition with minimum complexity is NP-hard [21], but in 1984 Chazelle [5] gave a procedure for which the complexity of the decomposition of a polyhedron depends on the number of its faces, $n$, and the number of its *reflex edges* or *notches,* $r$—an edge is reflex if its dihedral angle is greater than $180°$. (By Euler's formula (discussed in any book that covers planar graphs) the number of vertices is linearly related to the numbers of edges and faces.)

## 1.1 The notch cutting procedure

Chazelle's procedure works by cutting notches until only convex pieces remain. It works in two stages. First, it arbitrarily orders the notches $e_1, e_2, \ldots, e_r$ and chooses a *notch plane* $\pi_i$ for each notch $e_i$ of $P$, such that $\pi_i$ contains $e_i$ and forms dihedral angles of at most $180°$ with the two faces incident to $e_i$. Notch plane $\pi_i$ can be a plane containing a face incident to $e_i$.

Second, it resolves each notch $e_i$ by cuts in $\pi_i$. For the first notch $e_1$, the polyhedron $P$ is cut along the component of $P \cap \pi_1$ that contains $e_1$. This may break notches into segments called *subnotches,* but no new notches are created. For each successive notch $e_i$, all subnotches of $e_i$ are resolved as above by cutting simultaneously along $\pi_i$ in the pieces of $P$ that contain pieces of $e_i$. After resolving each of the $r$ notches, only convex pieces remain.

In Section 3 we consider the problem:

**Problem 1.1** *Given a polyhedron $P$ with $n$ faces and $r$ reflex edges, what is the complexity of Chazelle's decomposition into convex pieces?*

Chazelle's analysis showed that his procedure created $O(r^2)$ pieces with total complexity $O(nr^2)$ in $O(nr^3)$ time. Dey [9] gave an implementation with running time $O(nr^2 + r^3 \log r)$. He also claimed that the complexity of the decomposition was bounded by $O(nr + r^2\alpha(r))$. Unfortunately, our example in Section 3.1 shows that his analysis was incorrect; Chazelle's algorithm can produce decompositions with complexity $\Omega(nr + r^{7/3})$. Section 3.2 establishes this as an upper bound as well. We view this as the main contribution of our paper. Section 3.3 describes an algorithm to compute the decompostion in $O(nr + r^3)$ time and space. This improves on Chazelle's and Dey's time bounds. Joe [19] has done interesting work on chosing notch planes in practical examples to obtain few or well-shaped convex pieces.

## 1.2 2-d problems in notch cutting

We prove our combinatorial bounds by looking at two planar problems that are of independent interest.

**Problem 1.2** *What is the total complexity of $m$ convex polygons with disjoint interiors whose sides lie on $n$ lines?*

**Problem 1.3** *What is the horizon complexity of a segment that intersects $m$ faces in an "arrangement" constructed in $n$ stages, where stage $i$ adds to the arrangement some of the segments given by intersecting line $\ell_i$ with the faces of the "arrangement" of stage $i-1$?*

Problem 1.2 is more general than Problem 1.3; thus, the upper bound of $O(m^{2/3}n^{2/3}+n)$ that we prove for the first problem in Section 2.2 also applies to the second. Halperin and Sharir [15] have proved a similar bound on the number of turns of $m$ monotone, disjoint, convex chains defined on $n$ lines. Our upper bounds on erased arrangements were proved independently, but can also be derived from their theorem. We give matching lower bounds for both problems in Section 2.1.

We can state both problems in terms of an *erased arrangement of $n$ lines*, which we define as a convex subdivision whose segments lie on the $n$ lines. The complexity of a face in an erased arrangement is the number of its corners. An *incrementally-constructed erased arrangement* is an erased arrangement formed by first ordering the lines $\ell_1, \ell_2, \ldots, \ell_n$ and then adding segments on line $\ell_i$ in stage $i$ such that each stage is an erased arrangement. Figure 1 shows two erased arrangements; the second is an incrementally-constructed erased arrangement. We also make one observation on erased arrangements.

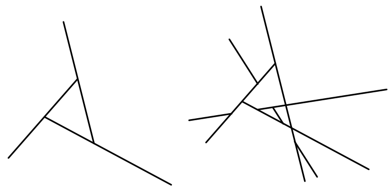

Figure 1: Erased arrangements

**Observation 1.4** *Let $L$ be a set of lines in general position. A subdivision of the plane induced by segments of $L$ is an erased arrangement iff every endpoint of a segment is contained in the interior of another segment.*

In the setting of erased arrangements, Problem 1.2 asks for complexity of $m$ faces. Clarkson et al. [7] proved an $O(m^{2/3}n^{2/3}+n)$ upper bound on the complexity of many faces in an arrangement of lines; our upper bound proof in Section 2.2 is based on their proof and our result includes theirs as a special case. Problem 1.3 asks for the complexity of the horizon of a segment in an incrementally-constructed erased arrangement. Section 2.1 proves that this is $\Theta(m^{2/3}n^{2/3}+n)$, which should be contrasted with the linear complexity of the horizon in a line arrangement [4, 11, 13].

### 1.3 Other results on decomposition

We were led to notch cutting by our desire to find small constructive solid geometry (CSG) representations of polyhedron [10]. A notch cutting decomposition of a polyhedron $P$ can use the planes of the faces to form convex pieces. Thus, one obtains a "union of intersections" formula whose variables corresponding to faces of $P$ and whose size is the complexity of the decomposition. (Notch cutting with the planes of faces also ensures that the arithmetic precision required by the decomposition is the same as that required to represent the original planes.)

Another way to generate CSG formulæ is by using binary space partitions. A BSP tree for $n$ polygons in space is defined recursively. If $n = 0$ then return a leaf node. If $n > 1$ then create a tree node storing a chosen plane $\pi$ and recursively create two BSP trees as children—one for the portions of the polygons on each side of $\pi$. Paterson and Yao [22] showed that $n$ triangles in space have a BSP tree of size $O(n^2)$ that can be computed in $O(n^3)$ time. This gives a quadratic-size formula for a polyhedron.

BSP trees are useful for establishing a back-to-front order for hidden surface calculations in graphics. They do not, however, give an explicit convex decomposition—converting a BSP tree to a decomposition can result in a decomposition of $\Omega(n^3)$ complexity.

Chazelle and Palios [6] gave a procedure that decomposes polyhedra that are homeomorphic to a ball and have $n$ faces and $r$ reflex edges into convex pieces with total complexity $O(n + r^2)$ in time $O((n+r^2)\log r)$. This complexity is best possible. (Notch-cutting techniques cannot do better than $\Omega(nr)$, since each of the $r$ notch planes can cut $\Omega(n)$ faces.) The decomposition of Chazelle and Palios does not use notch-cutting, so it does not give a CSG formula using halfspaces and representing the decomposition may take more algebraic complexity than representing the input planes. It would be interesting to extend their procedure to polyhedra that are not homeomorphic to a ball.

All of these decompositions create additional vertices. If no extra vertices are permitted, then there may be no convex decomposition; Ruppert and Seidel [23] have shown that it is NP-complete to determine whether a decomposition using the original vertices exists.

## 2 Erased arrangements

Because each face of an erased arrangement is convex, each line of $L$ can contribute at most one segment to its boundary, so a single face can have maximum complexity $n$. In the next two subsections, we consider the maximum total complexity of multiple faces in an erased arrangement. It is important to remember that the complexity of a face in an erased arrangement is the number of corners of the face, which may be much smaller than the number of segments that end on the boundary of the face. This is appropriate for the many algorithms that navigate arrangements by walking from corner to corner.

### 2.1 A lower bound for horizon complexity

In this section we examine lower bounds on the complexity of many faces and on the horizon complexity of a line in an erased arrangement. A construction of Edelsbrunner and Welzl [12] establishes the following theorem:

**Theorem 2.1 (Edelsbrunner and Welzl [12])** *The complexity of $m$ faces in an arrangement of $n$ lines can be $\Omega(m^{2/3}n^{2/3} + n)$, for $1 \leq m \leq \Theta(n^2)$.*

Because every arrangement is an erased arrangement, their result implies the same bound for erased arrangements. We briefly describe their construction and modify it so that half of the complexity appears as the horizon of a single line $\ell$. The $\Omega(m^{2/3}n^{2/3} + n)$ bound should be contrasted with the horizon complexity of a line in an arrangement of $n$ lines, which is linear in $n$ [4, 11].

**Theorem 2.2** *For any $m \leq n$, there exists an incrementally-constructed erased arrangement on $n$ lines and a line $\ell$ that intersects $2m + 1$ faces with total complexity $\Omega(m^{2/3}n^{2/3} + n)$.*

**Proof:** Edelsbrunner and Welzl's construction can be described as follows. Place $m$ discs of radius $0 < \epsilon < 1$ with their centers on the vertices of an $\sqrt{m} \times \sqrt{m}$ Cartesian grid. Construct polygons that circumscribe these disks by selecting $2n$ lines of small rational slopes that are tangent to these circles in the following manner: e.g., for slope $p/q$ (in reduced form) there are on the order of $m/\max(p,q)$ lines that each contribute $\sqrt{m}/\max(p,q)$ edges to polygons that circumscribe $\epsilon$-disks. Results from number theory on the number of divisors (specifically, Euler's $\phi$ function [16]) imply that, after adding $2n$ lines, the polygons circumscribing the $m$ disks have $\Omega(m^{2/3}n^{2/3})$ edges.

Note that the circumscribing polygons are centrally symmetric about the disk centers. By choosing $\epsilon$ small, rotating, and translating the configuration we can ensure that the circumscribing polygons are above the $x$ axis and project vertically onto disjoint intervals of the $x$ axis. We can incrementally construct an erased arrangement with channels to these polygons to prove Theorem 2.2.

To incrementally construct the erased arrangement of $2m + 1 + 2n$ lines that is depicted in figure 2, first add $2m$ vertical lines tangent to the circumscribing polygons. The pair of lines tangent to a given polygon define a *channel*; by construction, any two channels are disjoint. Second, add the $x$ axes, drawing only those segments that are not in any channel. Third, add the lines of Edelsbrunner and Welzl in any order, following three rules for each line $\ell$:

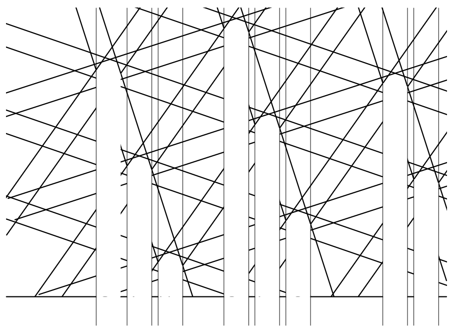

Figure 2: Channels to complex faces

1. no segment of $\ell$ is drawn below the $x$ axis,
2. no segment of $\ell$ is drawn in a channel between the $x$ axis and the circumscribing polygon whose tangents define the channel.
3. all other segments are drawn.

Construction rule 1 guarantees that the horizontal line $y = -1$ intersects only $2m + 1$ faces of the erased arrangement. Rules 2 and 3 ensure that the faces in each channel that the line $y = -1$ intersects are bounded by half of the (centrally-symmetric) circumscribing polygon that defines that channel. Thus, the total complexity is $\Omega(m^{2/3}n^{2/3} + n)$. ∎

### 2.2 The upper bound for many faces

In this section we show that the complexity of $m$ faces in an erased arrangement of $n$ lines is $O(m^{2/3}n^{2/3} + n)$. This bound also applies to the more restricted cases in which the erased arrangement is incrementally constructed or the $m$ faces lie on a common line.

We begin with a weaker upper bound of $O(mn^{1/2})$ of the type proved by Canham [3, 13]. Then we use the machinery of random sampling to divide the arrangement into pieces that involve fewer lines and invoke the Canham bound on these pieces. This technique was pioneered by Clarkson et al. [7] in their analysis of the complexity of $m$ faces in an arrangement. In our application, we must be careful to capture the face complexity that spills out of the pieces. Halperin and Sharir [15] proved their bound on the number of turns of $m$ monotone, disjoint, convex chains defined on $n$ lines by the same technique. The results of this section also follow from their theorem.

**Lemma 2.3** *The complexity of $m$ faces in an erased arrangement of $n$ lines is $O(mn^{1/2} + n)$.*

**Proof:** Form a bipartite graph $G$ whose vertex sets are the set of $m$ faces and the set of $2n$ halfplanes on the $n$ lines. Join a face $f$ to a halfplane $h$ by an edge if $h$ contains $f$ and contributes a segment to the boundary of $f$. We bound the complexity of the $m$ faces by bounding the number of edges of $G$.

Because the faces are convex, if three halfplanes $h_1$, $h_2$, and $h_3$ form part of the boundary of two faces $f_1$ and $f_2$, then $f_1$ and $f_2$ must intersect; see figure 3. Since all faces are disjoint, it follows that there is no $K_{2,3}$ in $G$. By a result of extremal graph theory [20], $G$ has at most $O(mn^{1/2} + n)$ edges. ■

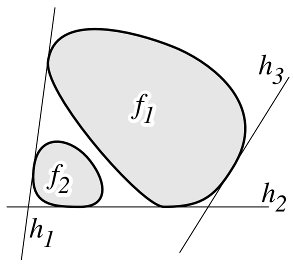


Figure 3: No $K_{2,3}$ subgraph

Canham's original proof for line arrangements [3], which does not use extremal graph theory, can also be modified to apply here. It gives a smaller constant in the big-$O$.

By probabilistic arguments [7, 8], we know that we can choose a sample of $\Theta(r)$ of the lines and decompose its arrangement into $r^2$ trapezoids, which Clarkson et al. [7] call *funnels,* such that every trapezoid is cut by $O(n/r)$ lines. We will choose $r = O(m^{1/2})$. If we merge the trapezoidation with the erased arrangement, we find that the complexity of our original $m$ faces may be spread over several trapezoids and that each trapezoid may intersect several faces. We will account for the complexity on a per-trapezoid basis.

We distinguish eight types of trapezoid/face interactions; see figure 4. We label each pair $(t, f)$, where $t$ is a trapezoid and $f$ is a face, with one of the following:

type $X$: The trapezoid $t$ does not intersect the interior of the face $f$ (i.e., $f \cap t = \emptyset$).

type $I$: The trapezoid $t$ is completely contained in the face $f$.

type $C$: One or more corners of trapezoid $t$ lie inside the face $f$, but $(t, f)$ is not type $I$.

type 0—type 4: The corners of the trapezoid $t$ lie on or outside the face $f$, the intersection $f \cap t \neq \emptyset$, and 0, 1, 2, 3, or 4 edges of $t$ intersect the interior of $f$.

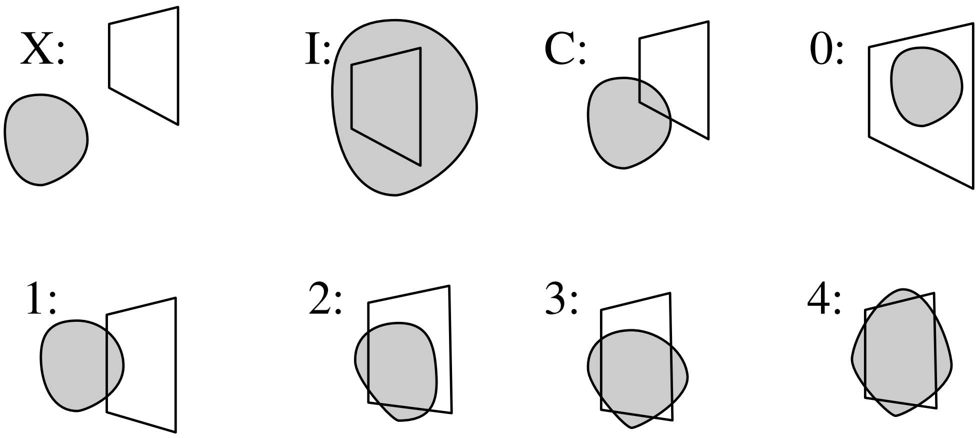


Figure 4: The eight trapezoid/face intersection types

In the next paragraphs, we account for the number of line segments contributed to the total by each type of interaction. Lemma 2.4 shows that the contributions are $O(nr)$ from all types except type 0 and type 1. Lemmas 2.5 and 2.6 show that the contributions of type 0 and type 1 are $O(m(n/r)^{1/2} + nr)$.

**Lemma 2.4** *At most $O(nr)$ segments bound faces in the trapezoid/face pairs of types $X$, $I$, $C$, 2, 3, and 4.*

**Proof:** Let us look at the cases.

type $X$: If a trapezoid $t$ does not intersect the interior of the face $f$, then $t$ makes no contribution to $f$. Therefore, the total type $X$ contribution is zero.

type $I$, $C$, 4, or 3: Because faces are disjoint and convex, any given trapezoid $t$ can be involved in pairs of these types with only a constant number of faces. (One face of type $I$, four of type $C$, one of type 4 and/or two of type 3.) Furthermore, the contribution to a given face $f$ in $t$ lies on a convex curve involving only segments of the $O(n/r)$ lines that intersect $t$. Since there are $r^2$ trapezoids, the total contribution from these types is $O(r^2) \cdot O(n/r) = O(nr)$.

type 2: For any trapezoid $t$, we can partition the faces that intersect two edges of $t$ into six classes, depending on which pair of edges they intersect. By the argument of Lemma 2.3, no two faces touching the same two edges of the trapezoid can touch the same side of a line $\ell$. Thus, line $\ell$ can contribute to at most two faces of each class and the number of segments contributed by type 2 pairs involving $t$ is $6 \cdot O(n/r)$. The total type 2 contribution is $O(nr)$.

Combining these results proves the lemma. ∎

**Lemma 2.5** *The total type 0 contribution is $O(m(n/r)^{1/2} + nr)$.*

**Proof:** Consider a trapezoid $t$ that is in $m_i$ type 0 pairs, meaning that it completely contains $m_i$ faces. We form an erased arrangement that involves only the $O(n/r)$ lines that cut $t$ by extending the segments inside $t$ that intersect the boundary of $t$ to infinity. (This may cause more intersections, but won't change the arrangement inside $t$.)

The contribution of type 0 faces in $t$ is $cm_i(n/r)^{1/2}+cn/r$, for some constant $c$, by Lemma 2.3. Summing the contributions over all trapezoids gives

$$\sum_{i \le r^2} cm_i \left(\frac{n}{r}\right)^{1/2} + c\frac{n}{r} = cnr + c\left(\frac{n}{r}\right)^{1/2} \sum_{i \le r^2} m_i = O(m(n/r)^{1/2} + nr)$$

and establishes the lemma. ∎

**Lemma 2.6** *The total type 1 contribution is $O(m(n/r)^{1/2} + nr)$.*

**Proof:** We prove first that the number of type 1 pairs is $O(m)$, then that their total contribution is $O(m(n/r)^{1/2} + nr)$.

For each face $f$, form a graph $G_f$ on trapezoids: join the trapezoids $t$ and $t'$ by an edge of $G$ iff $t$ and $t'$ are neighboring trapezoids in the trapezoidation and the interior of $f$ intersects the segment common to $t$ and $t'$. Since the face $f$ is connected, $G_f$ is connected.

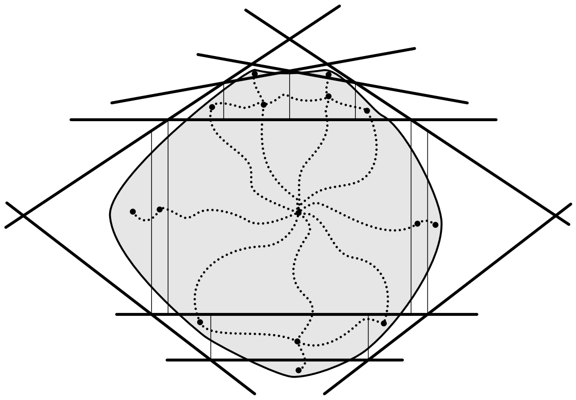

Figure 5: Graph $G_f$ of the shaded face $f$

By a slight abuse of terminology, we will say that a trapezoid $t$ is a type $k$ node in $G_f$ if it is in a type $k$ pair $(t, f)$. If $t$ is a type 2 node in $G_f$, then $t$ has neighbors on only two sides. It may have up to $\Omega(r)$ neighbors on either side, but if a neighbor is of type 1 or 2 in $G_f$, then it is the only neighbor on its side of the trapezoid $t$. Thus, we can group the type 2 nodes of $G_f$ into maximal-length chains. A chain is incident to at most one type 1 node on each end.

The graph $G_f$ can now have one of two forms. It can consist of a chain of zero or more type 2 nodes with type 1 nodes on each end. If all faces had such graphs there would be $2m$ type 1 pairs. Alternatively, $G_f$ can contain a type 3, 4, $C$, or $I$ node. Each node of these types has at most four neighbors of type 1 or 2, so the total number of type 1 nodes in all such graphs is bounded by four times the number of trapezoid/face pairs of types 3, 4, $C$, and $I$. Since each trapezoid can be in only a constant number of pairs of types 3, 4, $C$, and $I$ (see Lemma 2.4), there are $O(r^2)$ such nodes. Combining these bounds shows that the total number of type 1 pairs is $O(m + r^2) = O(m)$.

Now we use the Canham bound, as in the proof of Lemma 2.5, to bound the type 1 contribution along each edge of a trapezoid. Again, consider a trapezoid $t$ that is involved in $m_i$ type 1 pairs, meaning that $m_i$ faces cross one edge of $t$. We form an erased arrangement without decreasing the complexity of these $m_i$ faces by extending the segments inside $t$ that intersect the boundary of $t$ to infinity. Then Lemma 2.3 states that the contribution from these faces is $O(m_i(n/r)^{1/2} + n/r)$ and summing the contributions over all trapezoids gives $O(m(n/r)^{1/2} + nr)$ as before. ∎

We have shown that the total complexity of $m$ faces is $O(m(n/r)^{1/2} + nr)$. By choosing $r = (m^{2/3}n^{-1/3})$, we balance the two terms of the sum and establish the following upper bound theorem. Notice that $r^2 = m \cdot (m/n^2)^{1/3} = O(m)$ as promised, since $m = O(n^2)$.

**Theorem 2.7** *In any erased arrangement of $n$ lines, the total complexity of any set of $m$ faces is $O(m^{2/3}n^{2/3} + n)$.*

## 3 Polyhedral decomposition

Chazelle's notch-cutting procedure, which was described in Section 1.1, gives a decomposition of a polyhedron $P$ whose complexity depends on $n$, the number of $P$'s faces, and $r$, the number of $P$'s reflex edges or notches. We prove lower and upper bounds on the decomposition complexity in two subsections.

### 3.1 The lower bound on decomposition complexity

In this section, we build a polyhedron with $\Theta(n)$ faces and $\Theta(r)$ reflex edges and give a sequence of cuts such that the complexity of the resulting convex decomposition is $\Omega(nr+r^{7/3})$. Our polyhedron can even be made homeomorphic to a ball.

Imagine the $x$ axis extending to the right, the $y$ axis extending away and the $z$ axis extending upward. On the $xy$ plane, sketch a template erased arrangement of $r$ lines for which the line $y=-1$ intersects $r$ faces and has horizon complexity $O(r^{4/3})$, as described in Section 2.1.

Form a rectangular box $[x_{\mathrm{lo}}, x_{\mathrm{hi}}] \times [0, y_{\max}] \times [0, 3]$, where $x_{\mathrm{lo}}$, $x_{\mathrm{hi}}$, and $y_{\max}$ are chosen so that the box covers the horizon complexity of the template on the $xy$ plane.

We create several gadgets, which figure 6 attempts to illustrate. Round off the left near edge of the box by $n$ faces parallel to the $z$ axis that form no reflex edge nor intersect the erased arrangement in the projection on the $xy$ plane.

In the upper third of the box, $2 < z < 3$, we place $r$ *needles* above the lines of the template erased arrangement. The needle for the line $\ell$ is formed by subtracting a long, thin, rectangular box from the outer box; the needle has a vertical side (parallel to the $z$ axis) whose plane intersects the $xy$ plane along one of the lines of the template. To keep the box homeomorphic to a ball, each needle starts from one side of the outer box and comes near but does not touch another. The $z$ coordinate ranges of the needles are different, so no two needles intersect. Each needle contributes 8 reflex edges.

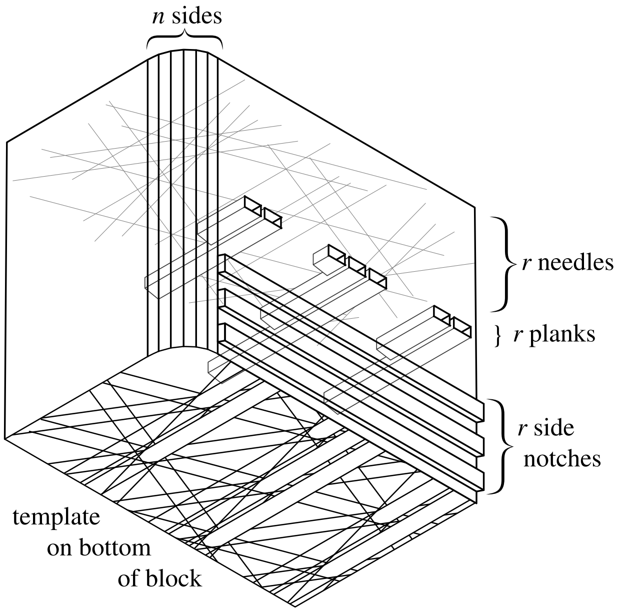

Figure 6: The lower bound construction

In the middle third, $1 < z < 2$, we place $r$ *planks* above the $r$ channels in the template. For the channel bounded by tangents to the polygon $p_i$ in the template, plank $i$ is formed by subtracting from the outer box all points with $z$ coordinates in $[5/4, 7/4]$ whose $xy$ projection lies in the trapezoid bounded by the channel walls, the $x$ axis, and the segment joining the points of tangency of $p_i$. Each plank also contributes 8 reflex edges.

Finally, in the lower third of the box, place $r$ *side notches* parallel to the $x$ axis and having rectangular cross-sections. These notches will be removed by horizontal cuts (parallel to the $xy$ plane).

We now make three families of cuts. First we cut with the sides of the planks that form channels.

The side notches below the planks and the needles above are all cut, but they do not block any cuts.

Next, on each needle, we cut along the vertical face that is above a line of the template erased arrangement. By the placement of the planks, such a cut extends to the lower third of the box only where segments are drawn on the template in the $xy$ plane. Notice that other needles and their cutting planes do not interfere with the propagation of the cut.

When we finish cutting vertically along each needle, the lower third of the box has been decomposed into vertical prisms—one for each face of the erased arrangement of the template. A cut from a single side notch has $\Omega(n)$ complexity from the near left corner and $\Omega(r^{4/3})$ from the horizon complexity. Summing over all side notches gives the $\Omega(nr + r^{7/3})$ lower bound. The remaining cuts can then be made arbitrarily—they can only increase the complexity.

Thus, this example establishes

**Theorem 3.1** *There is a polyhedron with $n$ faces and $r$ reflex edges, and a sequence of cuts resolving the reflex edges, such that Chazelle's algorithm gives a decomposition into convex pieces with complexity $\Omega(nr + r^{7/3})$.*

### 3.2 The upper bound on decomposition complexity

In this section we bound the complexity of Chazelle's decomposition algorithm by analyzing the cuts along a notch plane. Suppose that $P$ is a polyhedron with notches (reflex edges) $e_1, e_2, \ldots, e_r$, which we resolve in this order as described in Section 1.1. Let us take a closer look at how the $k$th notch $e_k$ and its subnotches are resolved.

The set of notch planes, $\pi_1, \pi_2, \ldots, \pi_r$, induces an arrangement of lines on the notch plane $\pi_k$. When a notch $e_j$, with $j < k$, is resolved by cutting, some segment of $\pi_j \cap \pi_k$ may appear in $\pi_k$. These segments end on $P \cap \pi_k$ or on $\pi_i \cap \pi_k$ for some $i < j$, just as they would in an incrementally-constructed erased arrangement. This subdivision is not an erased arrangement, however, because the vertices of $P \cap \pi_k$ do not accord with Observation 1.4.

Given a plane polygon $Q = P \cap \pi_k$, we define an *erased subdivision of $Q$ based on $l$ lines* to be a subdivision in which segments that intersect the interior of $Q$ lie on the $l$ lines and end on the boundary of $Q$ or on the interior of another segment. It is no longer the case that all faces of an erased subdivision of $Q$ are convex, but the only reflex angles are those of $Q$. We can now state and prove the main result of this section, Theorem 3.2.

**Theorem 3.2** *Let $Q$ be a connected $n$-gon with $r$ reflex vertices. The complexity of the horizon of a segment $e$ in an erased subdivision of $Q$ based on $r$ lines is $O(n + r^{4/3})$.*

In the proof of Theorem 3.2, we use a simple combination lemma:

**Lemma 3.3** *A single connected component of the intersection of a connected $j$-gon and a connected $k$-gon has $O(j + k)$ sides.*

**Proof:** Let $T$ and $U$ be connected, but not necessarily simply-connected, $j$-gons and $k$-gons, respectively. Let $I$ be a connected component of $T \cap U$. The vertices of $I$ come from $T$, from $U$, and from the intersection of edges of $T$ with edges of $U$. We call an edge of $I$ *bad* if its endpoints are both vertices of the last type. Clearly, the number of non-bad edges is at most $2j + 2k$.

Let $\mathcal{A}$ be the arrangement of edges of $T$ together with the bad edges of $I$ that come from $U$. We count the bad edges from $U$ by analyzing the face $F$ of $\mathcal{A}$ that contains $I$. In Figure 7, the bad edges from $U$ are dark and the face $F$ containing $I$ is lightly shaded.

Face $F$ is a connected polygon that may contain holes. Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be the boundary cycles of $F$: i.e., $\sigma_i$ is the list of edges of $T$ and bad edges of $U$ that appear around the $i$th connected component of the boundary of $F$. Because $F$ is connected, each edge on its boundary appears in a unique boundary cycle (although it may appear more than once).

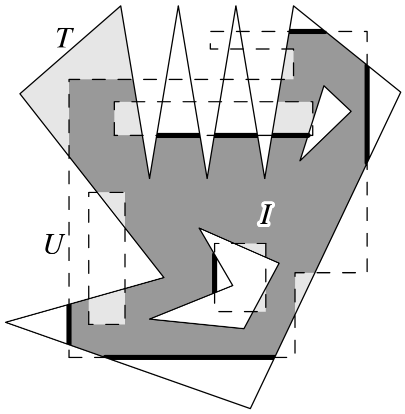

Figure 7: Counting bad edges from $U$

The bad edges from $U$ on cycle $\sigma_i$ cannot appear consecutively, thus we can bound them by the number of edges from $T$. The list of edges of $T$, however, can have no alternating $\ldots a \ldots b \ldots a \ldots b \ldots$ subsequence—an alternation would indicate that two edges of $T$ intersected. Such a sequence can have total length at most $2j$ by Davenport-Schinzel sequence bounds [17].

A symmetric argument counts bad edges on $T$ and establishes the $O(j+k)$ total bound. ∎

We prove Theorem 3.2 in three steps. First, we find an erased subdivision of the $n$ vertex polygon $Q$ based on a set of $O(r)$ lines such that the horizon complexity of the segment $e$ is contained in $s = O(r)$ convex faces, $F_1, F_2, \ldots, F_s$. Second, we find an erased arrangement of $O(r)$ lines whose intersection with the polygon $Q$ contains the faces $F_1, F_2, \ldots, F_s$. Third, we form a horizon polygon $R$ in this erased arrangement that has $O(r^{4/3})$ complexity by Theorem 2.7. Applying the combination lemma to $Q \cap R$ gives the desired $O(n + r^{4/3})$ bound.

To begin, let $S$ be the set of segments of the erased subdivision of $Q$ that are not segments of the boundary $\partial(Q)$ and let $L$ be the set of $r$ lines that contain $S$.

**Step 1:** Inside $Q$, the reflex angles in the subdivision induced by $Q$ and $S$ are those of $Q$. To remove them, we process reflex vertices in some order; we extend one of the edges incident to the current vertex until it hits a segment of $S$ or of the boundary $\partial(Q)$. Since this cutting introduces no new reflex angles and forms only $O(r)$ new faces, the horizon complexity is now contained in $s = O(r)$ convex faces $F_1, F_2, \ldots, F_s$.

Before we proceed to step 2, forming an erased arrangement of $O(r)$ lines, let us explain why we need it. We currently have $s$ convex faces and we could make them lie in an erased arrangement by extending edges of $Q$ through all convex corners into the exterior of $Q$. This, however, would be an erased arrangement of $n + r$ lines, leading to an $O(n^{2/3}r^{2/3} + n)$ bound on the complexity of the horizon. By placing the faces $F_1, F_2, \ldots, F_m$ inside an erased arrangement of $O(r)$ lines and applying the combination lemma, we obtain a tight bound of $O(n + r^{4/3})$.

It is tempting try to cut off the contributions of $\partial(Q)$ to $F_i$ by segments in $F_i$. Unfortunately, this cutting may require $\Omega(r^{4/3})$ segments over all the $F_i$'s even though $Q$ has only $r$ reflex angles. The only way to prove an upper bound is to form an erased arrangement of at most $O(r)$ segments.

**Step 2:** We apply Wenger's construction [24], which encloses $m$ convex polygons inside convex polygons with a total of $12m$ sides, to the faces $F_1, F_2, \ldots, F_m$. First, enclose the faces in a triangle and triangulate the complement of their interiors inside the triangle. Next, discard triangulation edges, which join pairs of faces, until every region is bounded by three triangulation edges. The *region of a face* $F_i$ is the union of all regions that border $F_i$.

Triangulation edges are undirected, so the ordering of faces and subscripts in the next sentences does not matter. For each triangulation edge $(F_i, F_j)$, choose a line $\ell_{ij}$ that separates the interiors of faces $F_i$ and $F_j$. Then, for all $i$, form the polygon $R_i$ by intersecting the region of $F_i$ with the halfspace defined by $\ell_{ij}$ that contains $F_i$, for all edges $(F_i, F_j)$ incident to $F_i$. As Wenger proves, the polygons $R_1, R_2, \ldots, R_m$ have disjoint interiors and have $12s$ edges in total [24].

To convert this into an erased arrangement of $O(r)$ lines, place the non-$Q$ edges of $F_i$ inside $R_i$. Extend those that ended on $Q$ to end on $R_i$ and extend all edges of $R_i$ until they end on the boundary of some other polygon $R_k$.

**Lemma 3.4** *The construction above gives an erased arrangement based on $O(r)$ lines whose intersection with $Q$ has the faces $F_1, F_2, \ldots, F_m$.*

**Proof:** If all lines used are in general position, then each of the segments stops on some other line segment. If they are not in general position, then they can be perturbed without changing the face complexity and points where three of more lines meet can be replaced by constructions like the "vortex" in Figure 1. Thus, the construction gives an erased arrangement.

Since the segments of $R_1, R_2, \ldots, R_m$ and the non-$Q$ segments of $F_1, F_2, \ldots, F_m$ are the only ones used, the number of lines in this erased arrangement is $O(r)$.

To argue that the intersection with $Q$ contains the faces $F_1, F_2, \ldots, F_m$, we first show that no segment was extended into the interior of $F_i$. Because $F_i$ is convex, none of its own extended segments intersected its interior; they all stop on $R_i$. But this also implies that the extended segments of $F_j$ stop on $R_j$, which is disjoint from the interior of $R_i$. Finally, the segments of $R_1, R_2, \ldots, R_m$ were all extended in the exterior of $\bigcup_{k \le m} R_k$, so they also did not affect the interior of $F_i$.

All non-$Q$ boundary segments of $F_i$ were included in the erased arrangement, however, and the rest are contributed by $Q$ in the intersection. ■

**Step 3:** Applying Theorem 2.7 to the faces containing $F_1, F_2, \ldots, F_m$ in the erased arrangement constructed in step 2 gives us an $O(r^{4/3})$ bound on the complexity of those faces.

In the construction, moreover, the only segments added inside the original horizon were the cuts through the reflex vertices of $Q$ that formed $F_1, F_2, \ldots, F_m$. Removing these cuts from the erased arrangement gives a subdivision (not an erased arrangement) in which the horizon of segment $e$ has $O(r^{4/3})$ complexity spread across the faces incident to $e$. If we now clip the at most $r$ segments that intersect $e$ just before they touch $e$ and double them, we obtain a polygon that encloses the horizon and has the same complexity bound. Finally, applying the combination lemma gives the desired $O(n + r^{4/3})$ bound and establishes Theorem 3.2.

Because of the discussion at the beginning of this section, this theorem has an immediate corollary.

**Corollary 3.5** *The complexity of Chazelle's decomposition is $O(nr + r^{7/3})$.*

**Proof:** Every vertex in the decomposition is a vertex of the polyhedron $P$ or is introduced by some cut.

The $k$th cut, in the notch plane $\pi_k$, is the horizon of segment $e_k$ in an erased subdivision of $P \cap \pi_k$ based on the set of $k-1$ lines $\pi_j \cap \pi_k$ with $j < k$. Since $P$ has $r$ reflex edges, $P \cap \pi_k$ has at most $r$ reflex vertices and Theorem 3.2 gives a bound of $O(n + r^{4/3})$ for the complexity of the cut. Summing over all cuts gives $O(nr + r^{7/3})$ total complexity. ∎

### 3.3 Computing the decompostion

In this section, we sketch an algorithm to compute the notch cutting decomposition in $O(nr \log r + r^3)$ time and space, which improves on the algorithms of Chazelle [5] and Dey [9]. We assume that the surface of the original polyhedron $P$ is stored in a winged edge [1], quadedge [14], or equivalent data structure. That is, that vertices and faces can access their incident edges (and vice versa) in sorted order. We produce quadedge structures for each convex piece in the final decomposition.

**Theorem 3.6** *One can compute the notch-cutting decomposition of a polyhedron $P$ with $n$ edges, $r$ of them reflex, in $O(nr + r^3)$ time and space.*

**Proof:** First, for each notch plane $\pi_i$, compute the connected component $Q_i \subseteq P \cap \pi_i$ that contains notch $e_i$ as follows. Determine the intersection of the boundary of $P$ with $\pi_i$ by traversing the edges and faces of $P$. This gives a set of closed polylines that can be broken into $O(r)$ convex pieces that are monotone with respect to a given direction. (There are at most $r$ connected pieces and $r$ reflex vertices, and any polyline that winds more than 360° will need at least one reflex per 360° to unwind.) One can sweep these chains to determine their nesting structure in $O(n \log r)$ time per notch plane. $Q_i$ is then the one polygon (possibly with holes) whose boundary includes the notch edge $e_i$.

Next, compute the arrangement of all notch planes in space by incremental construction [4, 11] using $O(r^3)$ time. We can assume that within a notch plane $\pi_i$, the arrangement is represented by a quadedge structure.

We can stitch the $O(r)$ convex chains bounding $Q_i$ into the arrangement in $\pi_i$ by walking through the *horizon* or *zone* of each piece on the outer side, as illustrated in figure 8. Because any line intersects such a convex piece at most two times, we can form a pseudoline arrangement by breaking lines inside the convex piece. We can therefore use a horizon theorem [2, 11] to show that each walk takes time proportional to $r$ plus the number of vertices in the convex piece, leading to $O(nr + r^3)$ total stitching time over all notch planes. Jordan sorting [18] along each edge of the arrangement gives the intersections in edge order, instead of order along $Q$.

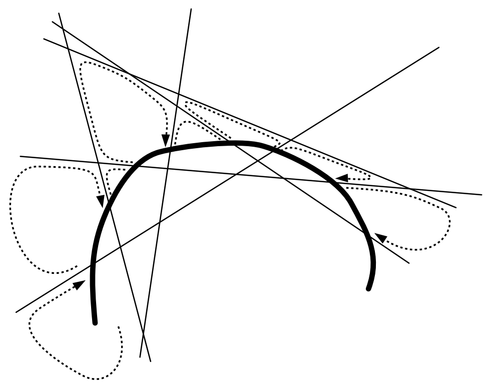

Figure 8: Stitching in a convex chain

With this preparation, notch cutting is easy. One can just mark cells and edges of the arrangement that are connected to the notch. The total time spent marking is bounded by the size of the arrangement: $O(nr + r^3)$. ∎

## 4 Conclusions

We have given tight combinatorial bounds for the complexity of $m$ convex polygons defined by $n$ lines and for the complexity of Chazelle's convex decompositions of polyhedra [5]. We have also given an algorithm that runs in $O(nr \log r + r^3)$ time and space.

Our algorithm has the disadvantage of always taking at least $\Omega(nr + r^3)$ time and space, even though the decomposition produced may be linear in $n$ and $r$. Ideally, one would like an *output-sensitive* algorithm—one whose running time was proportional to the size of the decomposition. Reducing the time to $O(nr + r^{7/3})$ also remains open. There also remain several open questions on how best to choose notch planes in practical examples to obtain few or well-shaped convex pieces [19].

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