Course 8

An Introduction to the Kalman Filter

SIGGRAPH 2001
EXPLORE INTERACTION AND DIGITAL IMAGES

Speakers

Greg Welch

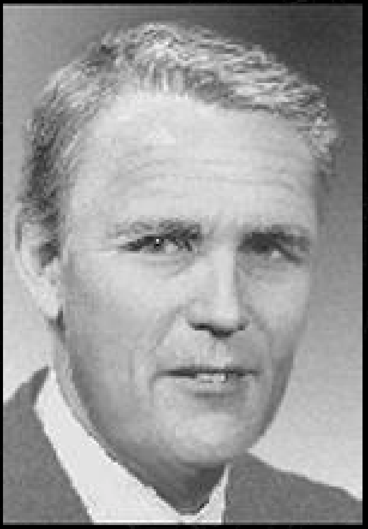
Gary Bishop

Kalman Filters in 2 hours?

- Hah!
- No magic.
- Pretty simple to apply.
- Tolerant of abuse.
- Notes are a standalone reference.
- These slides are online at
  http://www.cs.unc.edu/~tracker/ref/s2001/kalman/

Rudolf Emil Kalman

- Born 1930 in Hungary
- BS and MS from MIT
- PhD 1957 from Columbia
- Filter developed in 1960-61
- Now retired



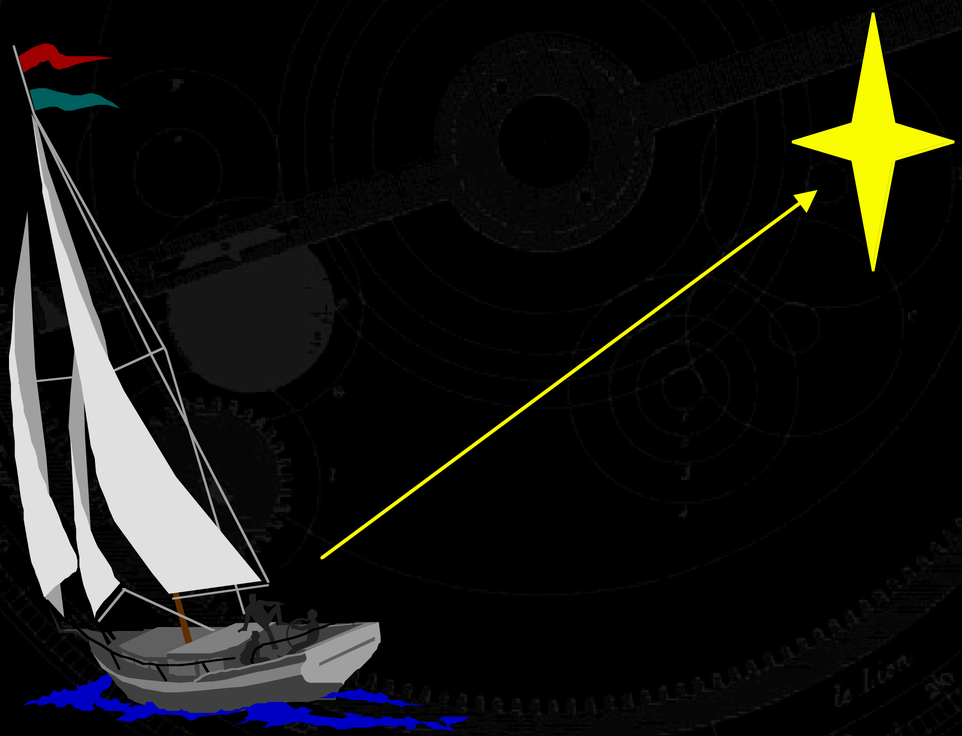
What is a Kalman Filter?

- Just some applied math.
- A linear system: f(a+b) = f(a) + f(b).
- Noisy data in → hopefully less noisy out.
- But delay is the price for filtering...
- Pure KF does not even adapt to the data.

What is it used for?

- Tracking missiles
- Tracking heads/hands/drumsticks
- Extracting lip motion from video
- Fitting Bezier patches to point data
- Lots of computer vision applications
- Economics
- Navigation

A *really* simple example



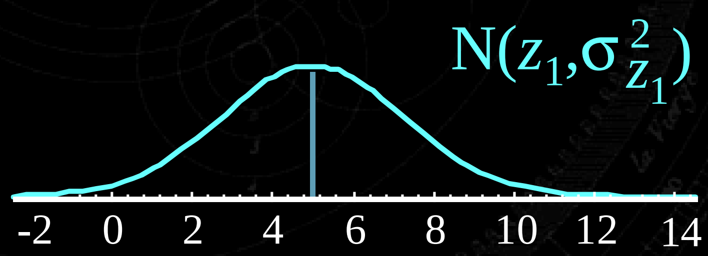
Gary makes a measurement

$$z_1, \sigma_{z_1}^2$$

$$\hat{x}_1 = z_1$$

$$\hat{\sigma}_1^2 = \sigma_{z_1}^2$$

Conditional Density Function



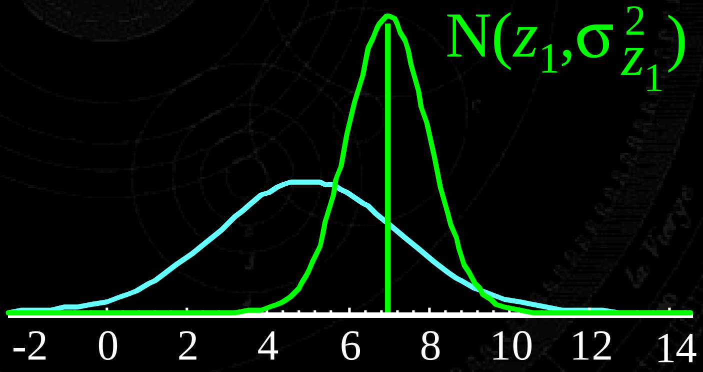
Greg makes a measurement

$$z_2, \sigma_{z_2}^2$$

$$\hat{x}_2 = \ldots ?$$

$$\hat{\sigma}_2^2 = \ldots ?$$

Conditional Density Function



Combine estimates

$$\hat{x}_2 = \hat{x}_1 + K_2\left(z_2 - \hat{x}_1\right)$$

$$K_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_{z_2}^2}$$

Combine variances

$$\frac{1}{\sigma_2^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_{z_2}^2}$$

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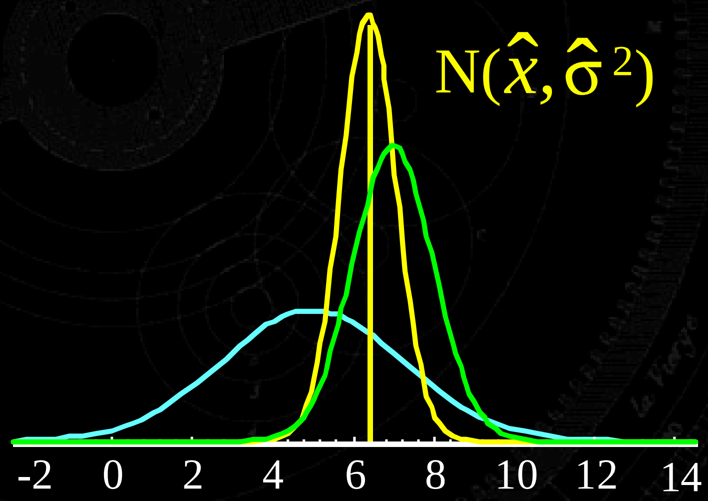
Combined Estimates

Conditional Density Function



$$\hat{x} = \hat{x}_2$$

$$\hat{\sigma}^2 = \sigma_2^2$$

Online weighted average!

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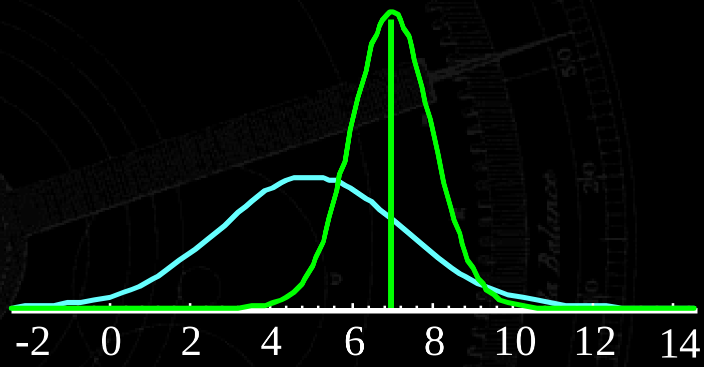
But suppose we're moving



- Not *all* the difference is error
- Some may be motion
- KF can include a motion model
- Estimate velocity and position

Process Model

- Describes how the *state* changes over time
- The *state* for the first example was scalar
- The *process model* was "nothing changes"

A better model might be

- State is a 2-vector [position, velocity]
- $\text{position}_{n+1} = \text{position}_n + \text{velocity}_n * \text{time}$
- $\text{velocity}_{n+1} = \text{velocity}_n$

Measurement Model

"What you see from where you are"

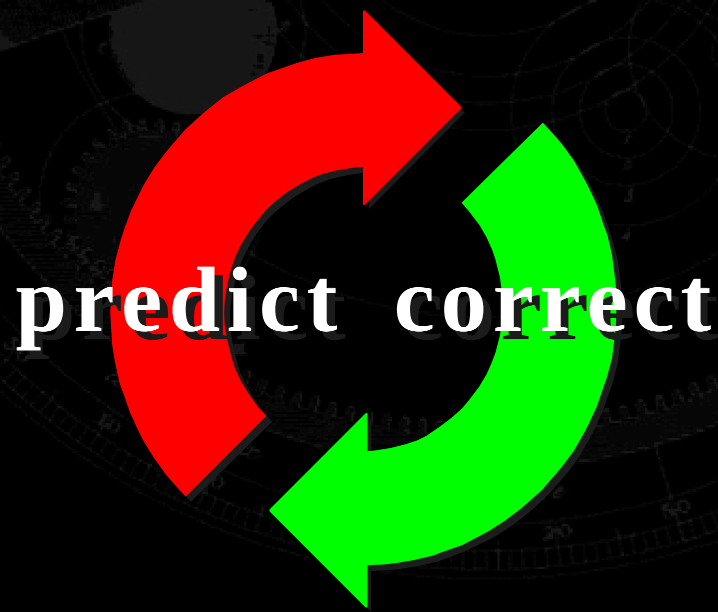
not

"Where you are from what you see"

Predict → Correct

KF operates by

- Predicting the new state and its uncertainty
- Correcting with the new measurement



Example: 2D Position-Only

(Greg Welch)

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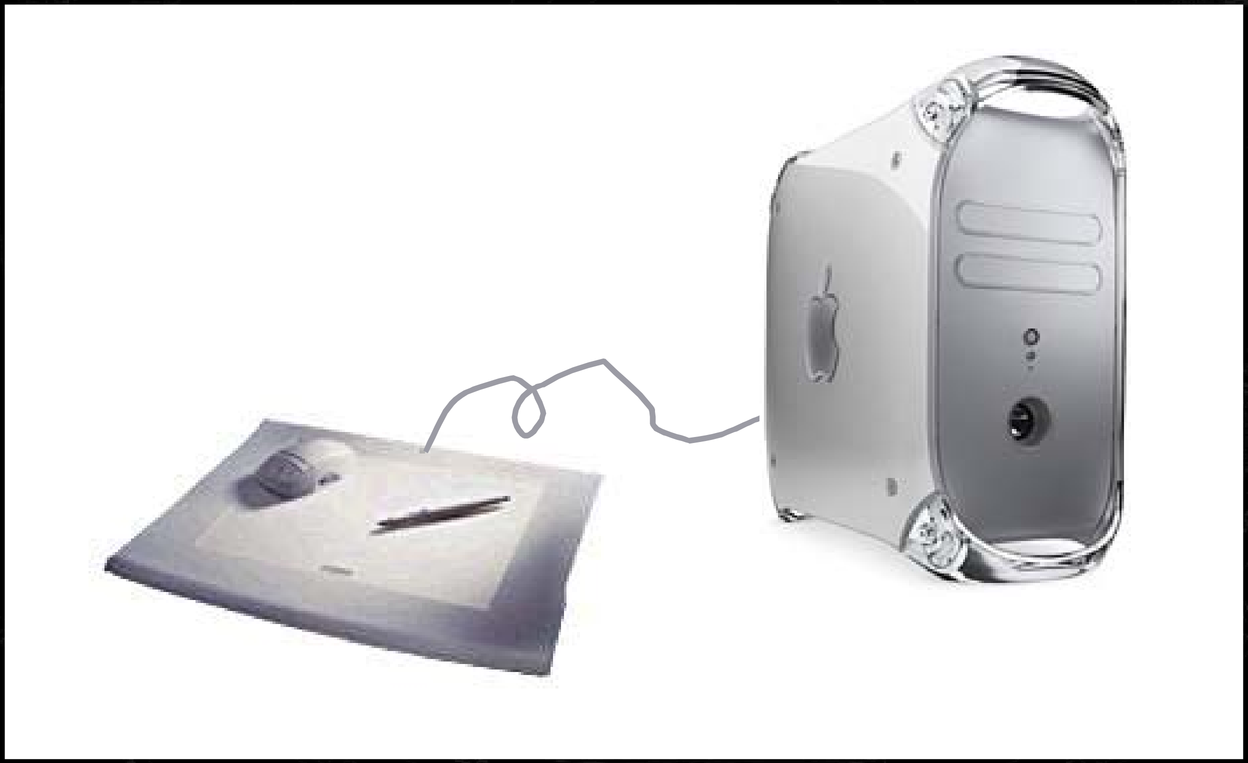
Apparatus: 2D Tablet



Process Model

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} + \begin{bmatrix} \sim x_{k-1} \\ \sim y_{k-1} \end{bmatrix}$$

$\bar{x}_k$	A	$\bar{x}_{k-1}$	$\bar{w}_{k-1}$
state	*state transition*	*state*	*noise*

$$\bar{x}_k = A\bar{x}_{k-1} + \bar{w}_{k-1}$$

Measurement Model

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} H_x & 0 \\ 0 & H_y \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} \sim u_k \\ \sim v_k \end{bmatrix}$$

$\bar{z}_k$ *measurement* $\quad$ H *measurement matrix* $\quad$ $\bar{x}_k$ *state* $\quad$ $\bar{v}_k$ *noise*

$$\bar{z}_k = H\bar{x}_k + \bar{v}_k$$

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Preparation

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

State Transition

$$Q = E\{\overline{w} * \overline{w}^T\} = \begin{bmatrix} Q_{xx} & 0 \\ 0 & Q_{yy} \end{bmatrix}$$

Process Noise Covariance

$$R = E\{\overline{v} * \overline{v}^T\} = \begin{bmatrix} R_{xx} & 0 \\ 0 & R_{yy} \end{bmatrix}$$

Measurement Noise Covariance

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Initialization

$$\bar{x}_0 = H\bar{z}_0$$

$$P_0 = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}$$

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PREDICT

$$\bar{x}_k^- = A\bar{x}_{k-1}$$

$$P_k^- = AP_{k-1}A^T + Q$$

transition

uncertainty

CORRECT

$$\bar{x}_k = \bar{x}_k^- + K\left(\bar{z}_k - H\bar{x}_k^-\right)$$

$$P_k = \left(I - KH\right)P_k^-$$

actual

predicted

$$K = P_k^- H^T \left(H P_k^- H^T + R\right)^{-1}$$

“denominator”
(measurement space)

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Summary

$$\bar{x}_k^- = A\bar{x}_{k-1}$$

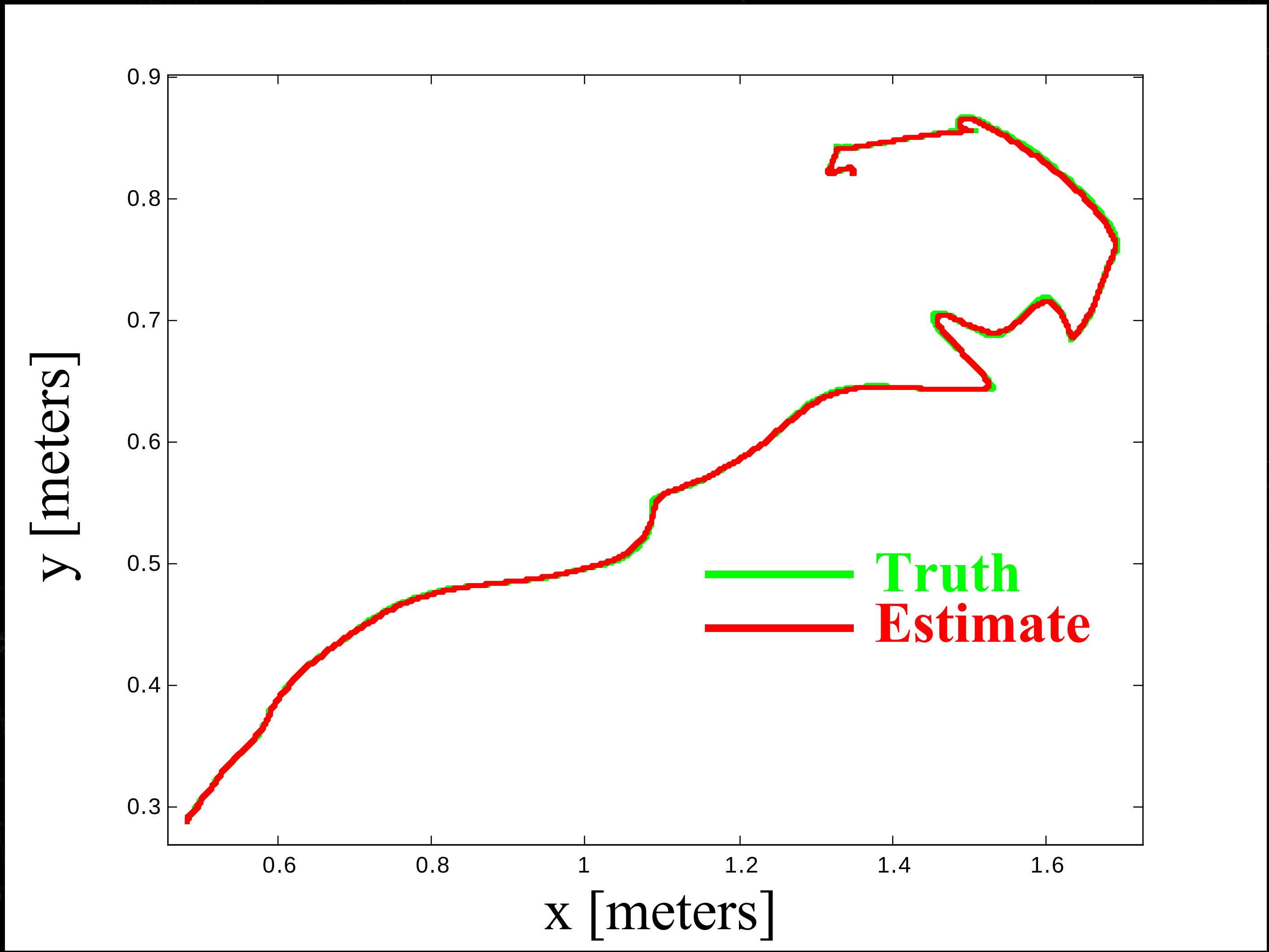
$$P_k^- = AP_{k-1}A^T + Q$$

$$K = P_k^- H^T \left(HP_k^- H^T + R\right)^{-1}$$

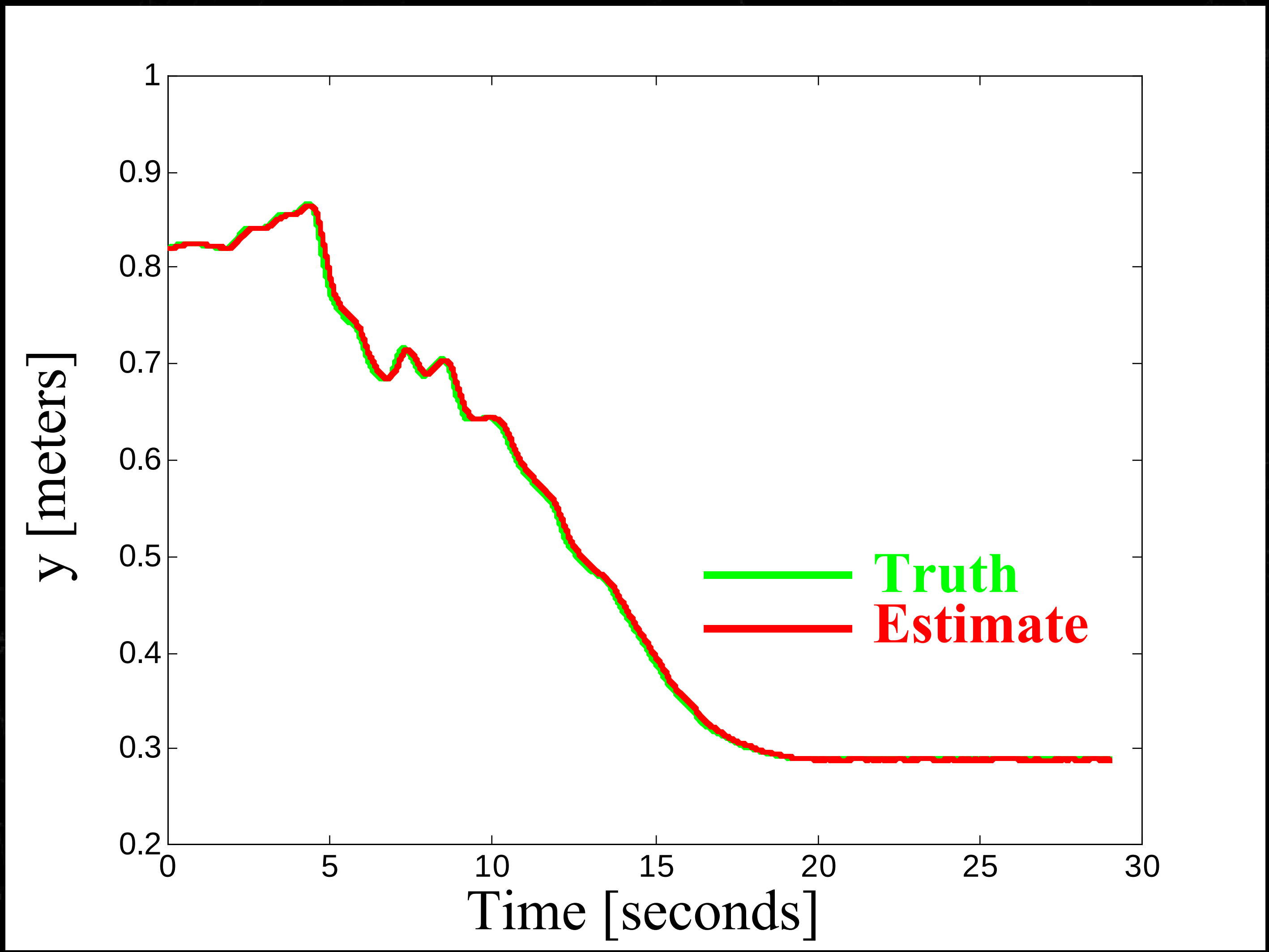
$$\bar{x}_k = \bar{x}_k^- + K\left(\bar{z}_k - H\bar{x}_k^-\right)$$

$$P_k = \left(I - KH\right)P_k^-$$

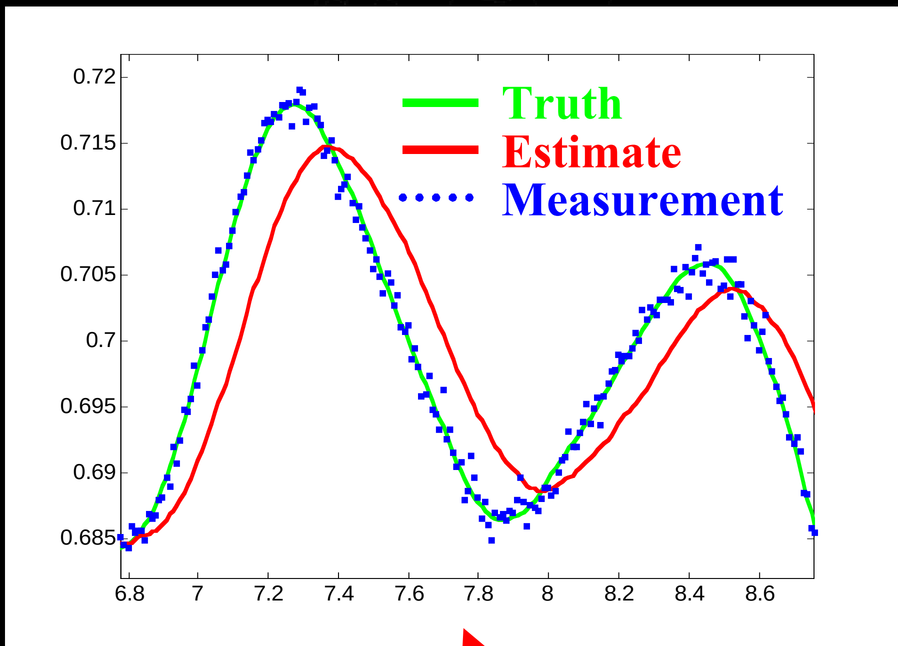
Results: XY Track

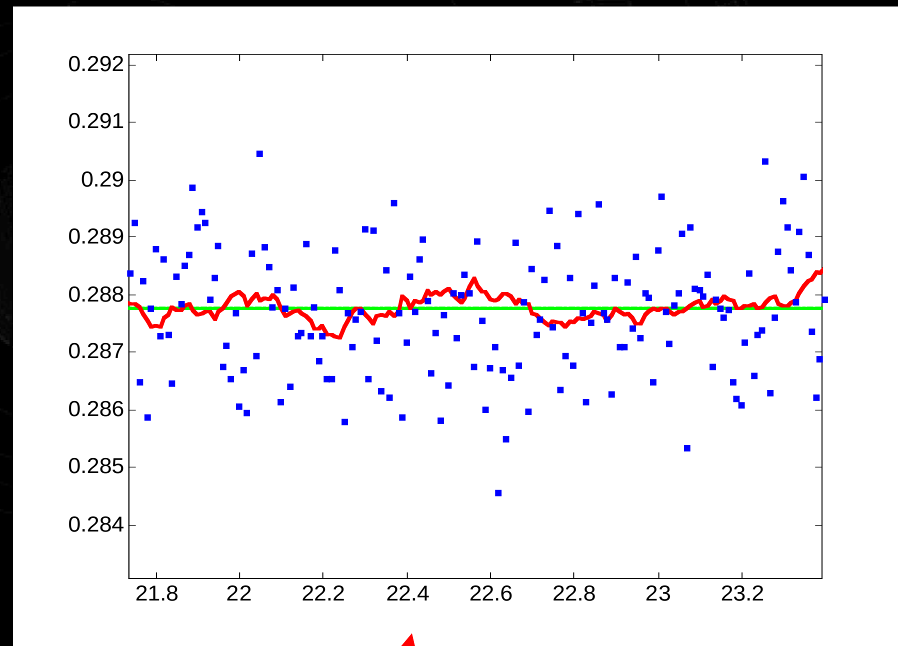


Y Track: Moving then Still

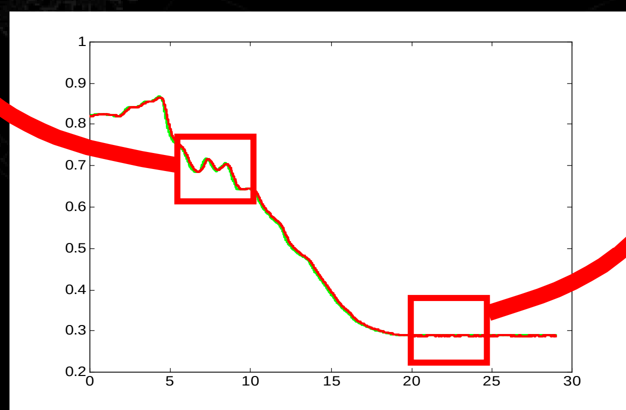


Motion-Dependent Performance





significant *latency* when moving…



…relatively *smooth* when not

Example: 2D Position-Velocity

(PV Model)

Process Model (PV)

state transition state

$$
\begin{bmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} x \\ y \\ dx/dt \\ dy/dt \end{bmatrix}
$$

Measurement Model (Same)

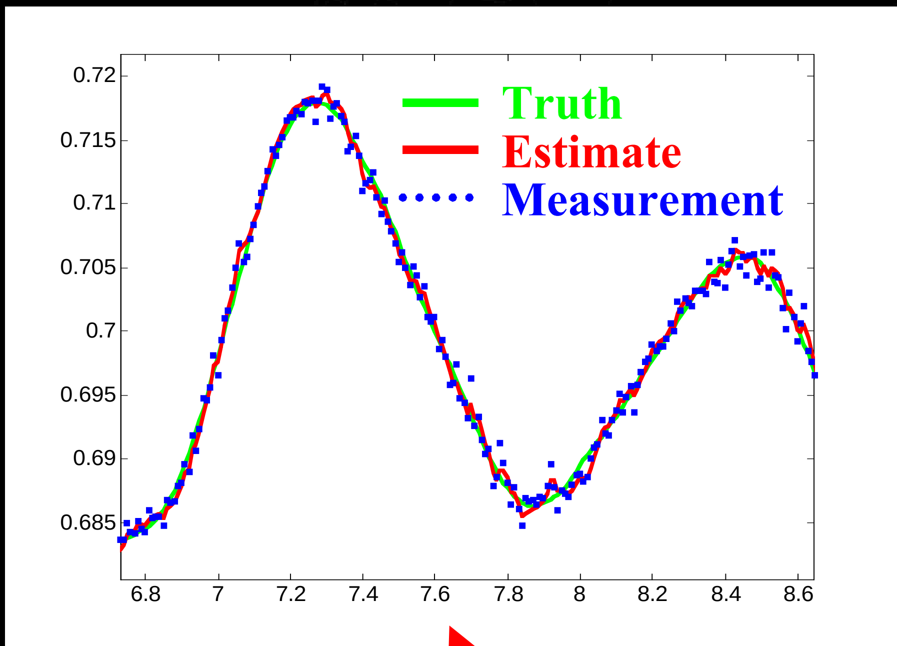
measurement matrix

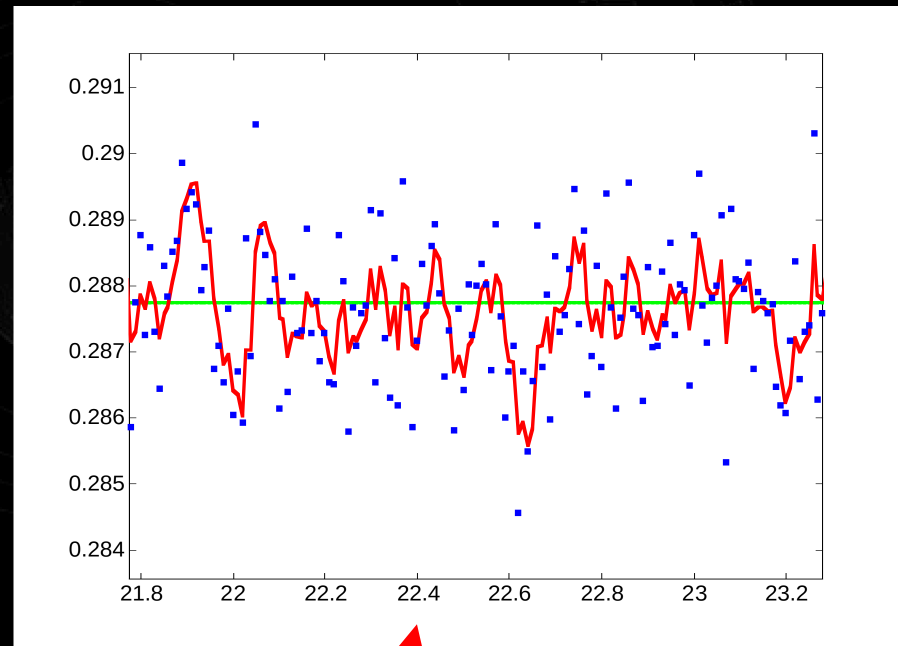
$$\begin{bmatrix} H_x & 0 & 0 & 0 \\ 0 & H_y & 0 & 0 \end{bmatrix}$$

state

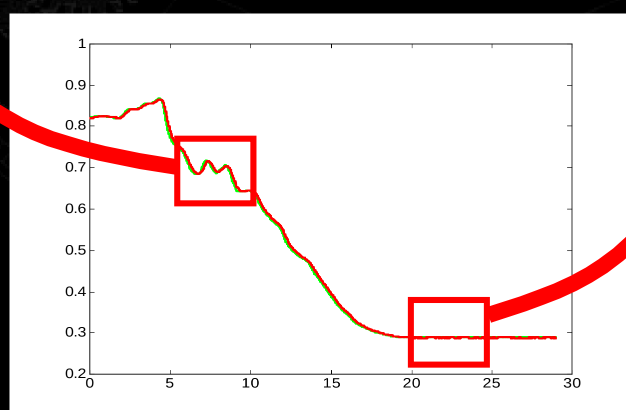
$$\begin{bmatrix} x \\ y \\ dx/dt \\ dy/dt \end{bmatrix}$$

Different Performance





improved latency when moving…



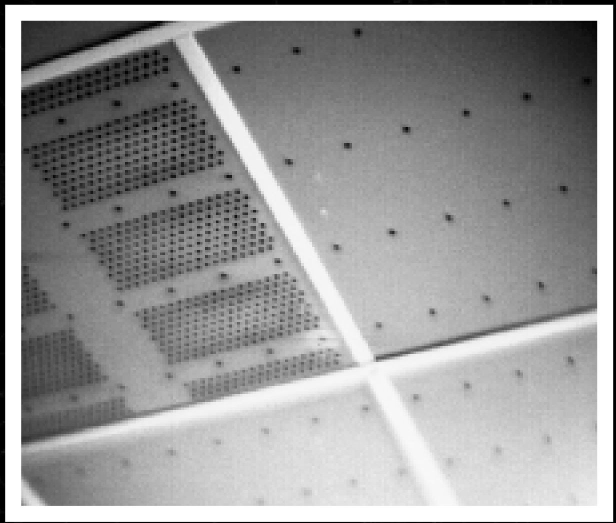
…relatively *noisy* when not

Example: 6D HiBall Tracker

(x, y, z, roll, pitch, yaw)

Apparatus

HiBall with six optical sensors



Ceiling panel with LEDs

State Vector (PV)

$$\bar{x} = \left[\bar{\tau} \quad \bar{\rho} \quad {}^{d\bar{\tau}}/_{dt} \quad {}^{d\bar{\rho}}/_{dt} \quad \bar{\lambda} \right]^T$$

$\bar{\tau} =$ translation (3D)

$\bar{\rho} =$ rotation (3D)

${}^{d\bar{\tau}}/_{dt} =$ linear velocity (3D)

${}^{d\bar{\rho}}/_{dt} =$ angular velocity (3D)

$\bar{\lambda} =$ LED position (3D)

Non-Linear Measurement Model

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = V \cdot \text{rotate}(\bar{\rho}) \cdot (\bar{\lambda} - \bar{\tau})$$

view matrix

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c_x / c_z \\ c_y / c_z \end{bmatrix}$$

SCAAT vs. MCAAT

- *Single or Multiple Constraint(s) at a Time*
- **Dimension of the measurement**
  - Nothing about KF mathematics restricts it
  - Can process in "batch" or sequential mode
- **SCAAT**
  - Estimate 15 parameters with 2D measurements
  - Temporal improvements
  - Autocalibration of LED positions

HiBall Initialization

- Initialize pose using a brute-force (relatively slow) MCAAT approach
- Initial velocities = 0
- Initial process covariance P_0 = ~cm/degrees
- Transition to SCAAT Kalman filter

Nonlinear Systems

(Gary Bishop)

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Kalman Filter assumes linearity

- Only matrix operations allowed
- Measurement is a linear function of state
- Next state is linear function of previous state
- Can’t estimate gain
- Can’t handle rotations (angles in state)
- Can’t handle projection

Extended Kalman Filter

Nonlinear Process (Model)

- Process dynamics: A becomes $a(x)$
- Measurement: H becomes $h(x)$

Filter Reformulation

- Use functions instead of matrices
- Use Jacobians to project forward, and to relate measurement to state

Jacobian?

- Partial derivative of measurement with respect to state
- If measurement is a vector of length M
- And state has length N
- Jacobian of measurement function will be MxN matrix of numbers (not equations)
- Often evaluating h(x) and Jacobian(h(x)) at the same time cost only a little extra

Tips

- Don't compute giant symbolic Jacobian with a symbolic algebra package
- Do use an automatic method during development
- Check out tools from optimization packages
- Differentiating your function line-by-line is usually pretty easy

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New Approaches

Several extensions are available that work better than the EKF in some circumstances

System Identification

Model Form and Parameters
(Greg Welch)

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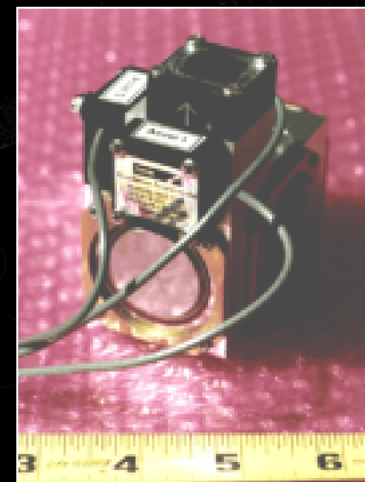
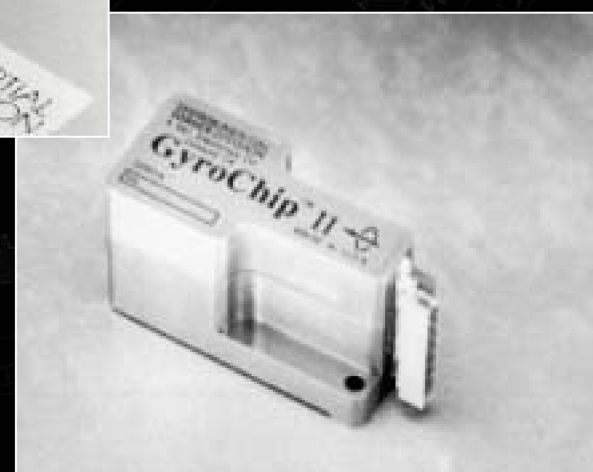
Measurement Noise (R)





Sampled Process Noise (Q)

For continuous model

$$\frac{d\overline{x}}{dt} = F\overline{x} + Q_c$$

The sampled (discrete) Q is

$$Q_d = \int_0^{dt} e^{F\tau} Q_c e^{F^T \tau} d\tau$$

Example: 2D PV Model

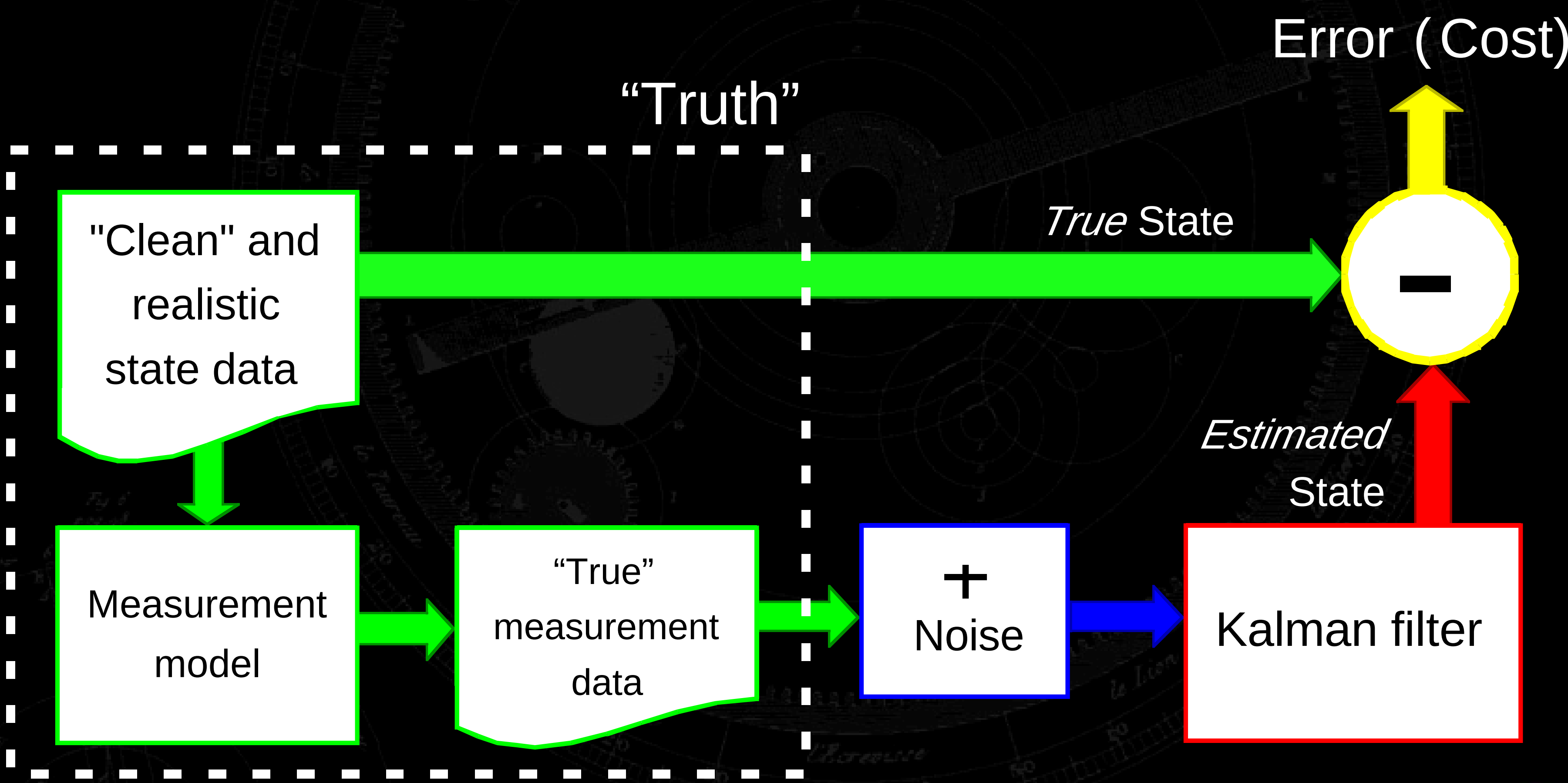
For continuous model

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}$$

The sampled (discrete) Q is

$$Q_d = \begin{bmatrix} dt^3 q/3 & dt^2 q/2 \\ dt^2 q/2 & dt\, q \end{bmatrix}$$

Parameter Optimization



Multiple-Model Configurations

Off or On-Line Model Selection

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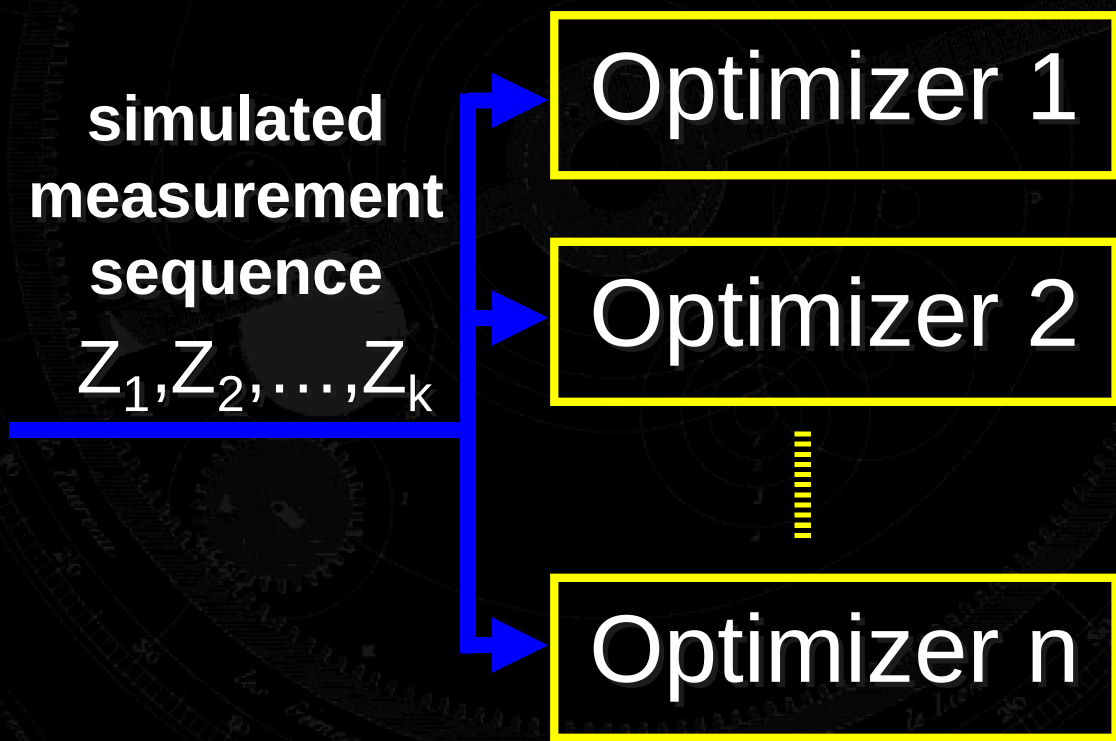
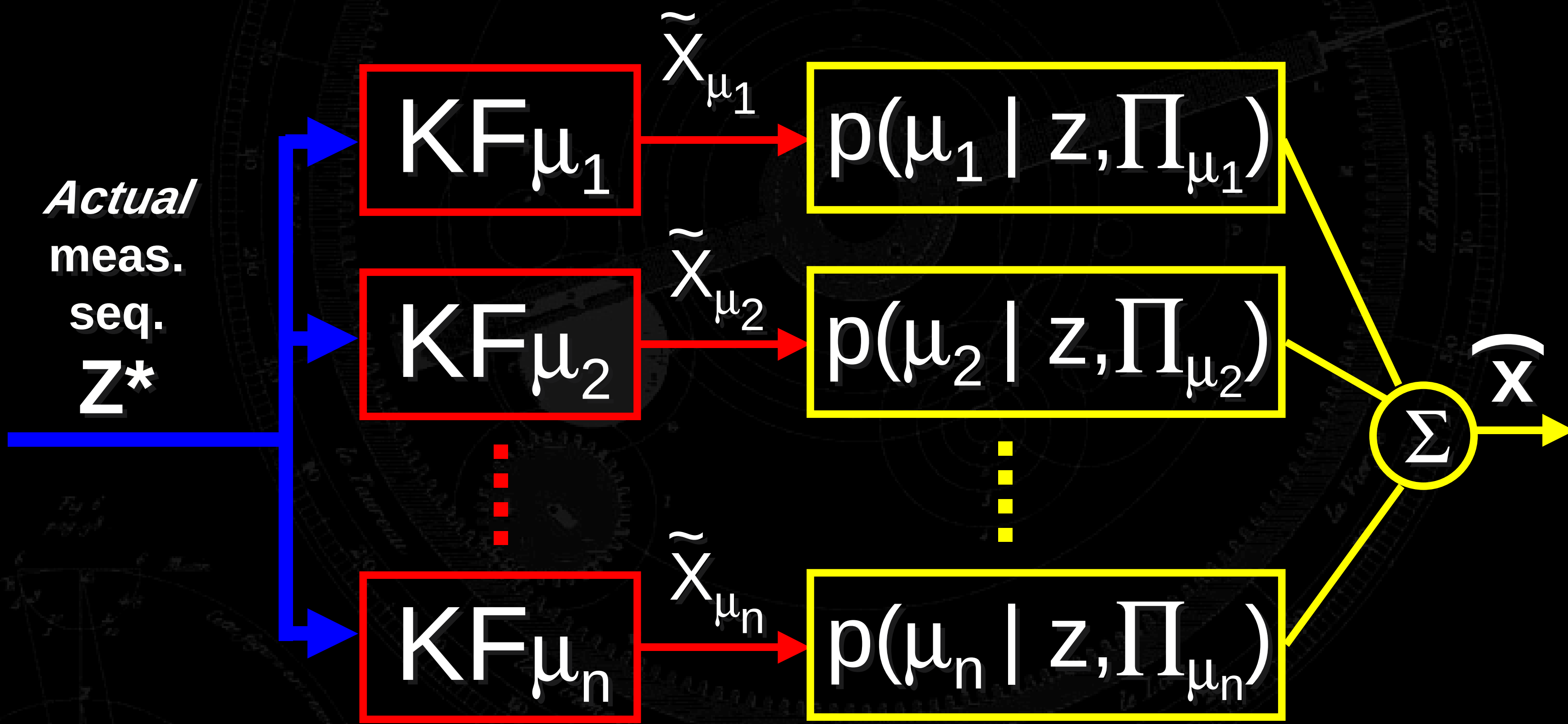
Off-Line Model Selection



On-Line Multiple-Model Estimation



$$\Pi_{\mu_n} = \{ \bar{x}_k, P_k, H, R \}$$

Probability of Model μ

For model μ with $\Pi_\mu = \{x, P, H, R\}$

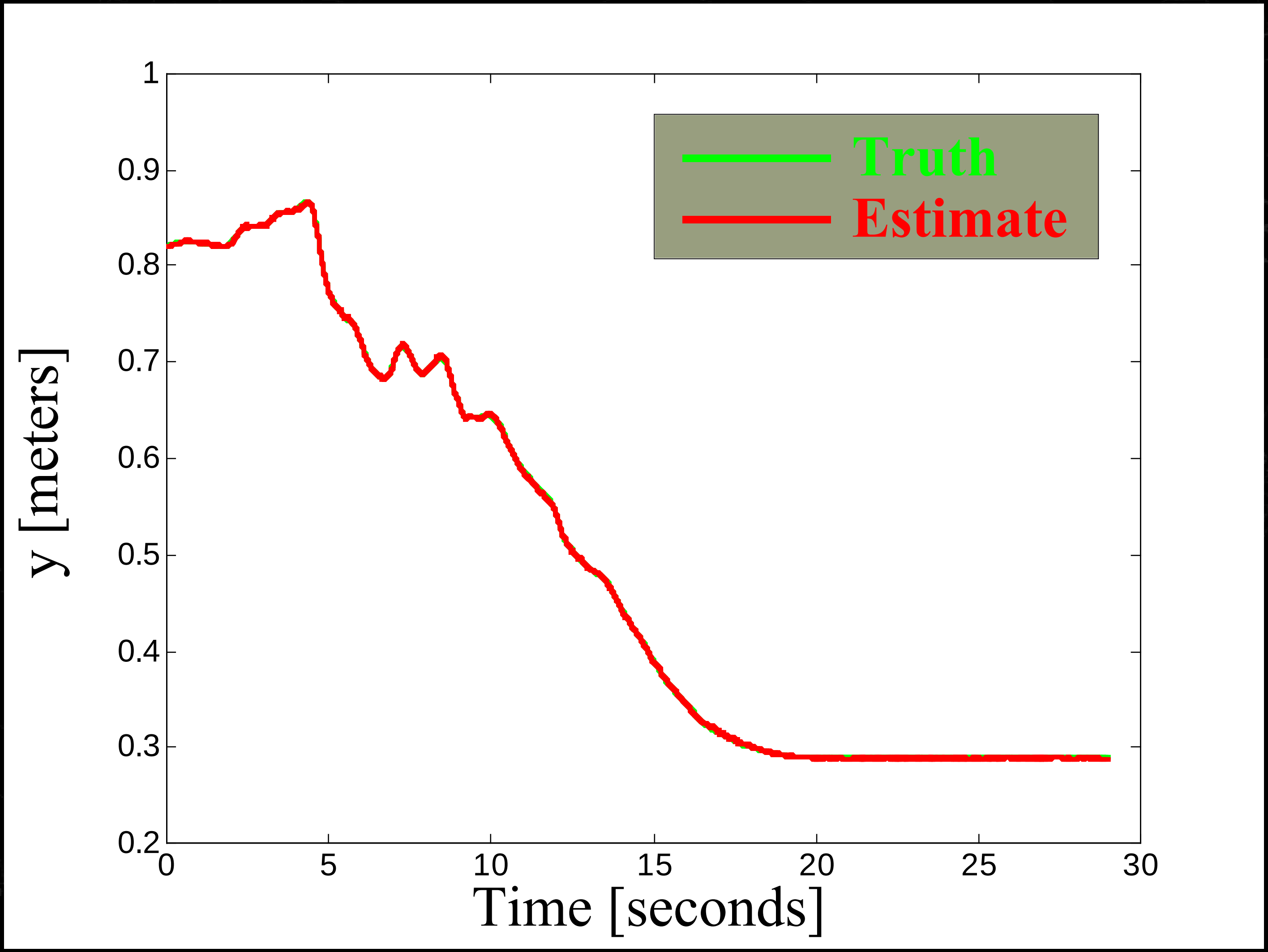
$$p(\mu | z, \Pi_\mu) = \frac{1}{(2\pi |C|)^{\frac{n}{2}}} e^{-\frac{1}{2}(z-Hx)^T C^{-1} (z-Hx)}$$

where $C = HPH^T + R$

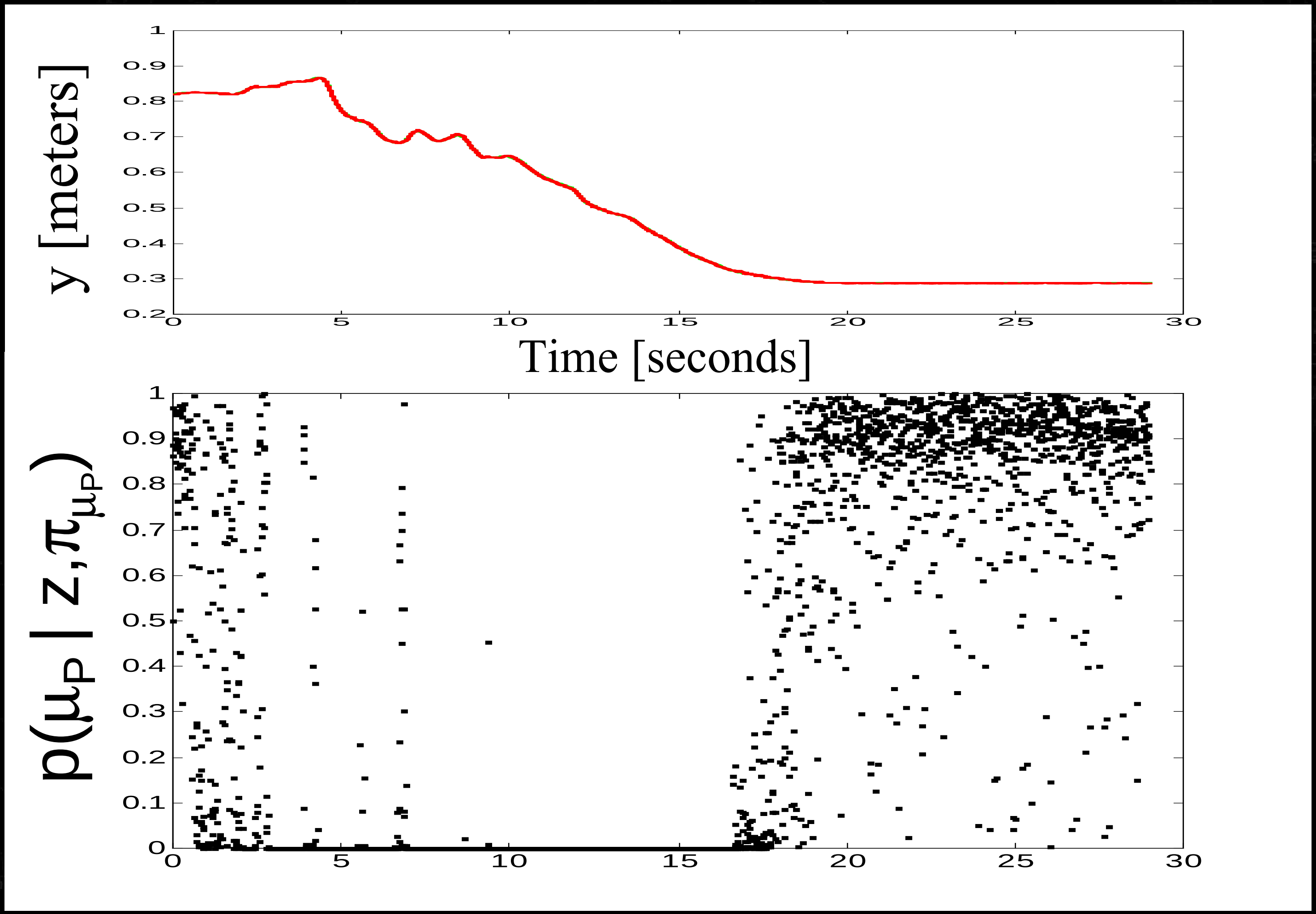
Final Combined Estimate

$$\hat{x} = \sum_{\mu} x_{\mu} \frac{p(\mu | z, \Pi_{\mu})}{\sum_{\nu} p(\nu | z, \Pi_{\nu})}$$

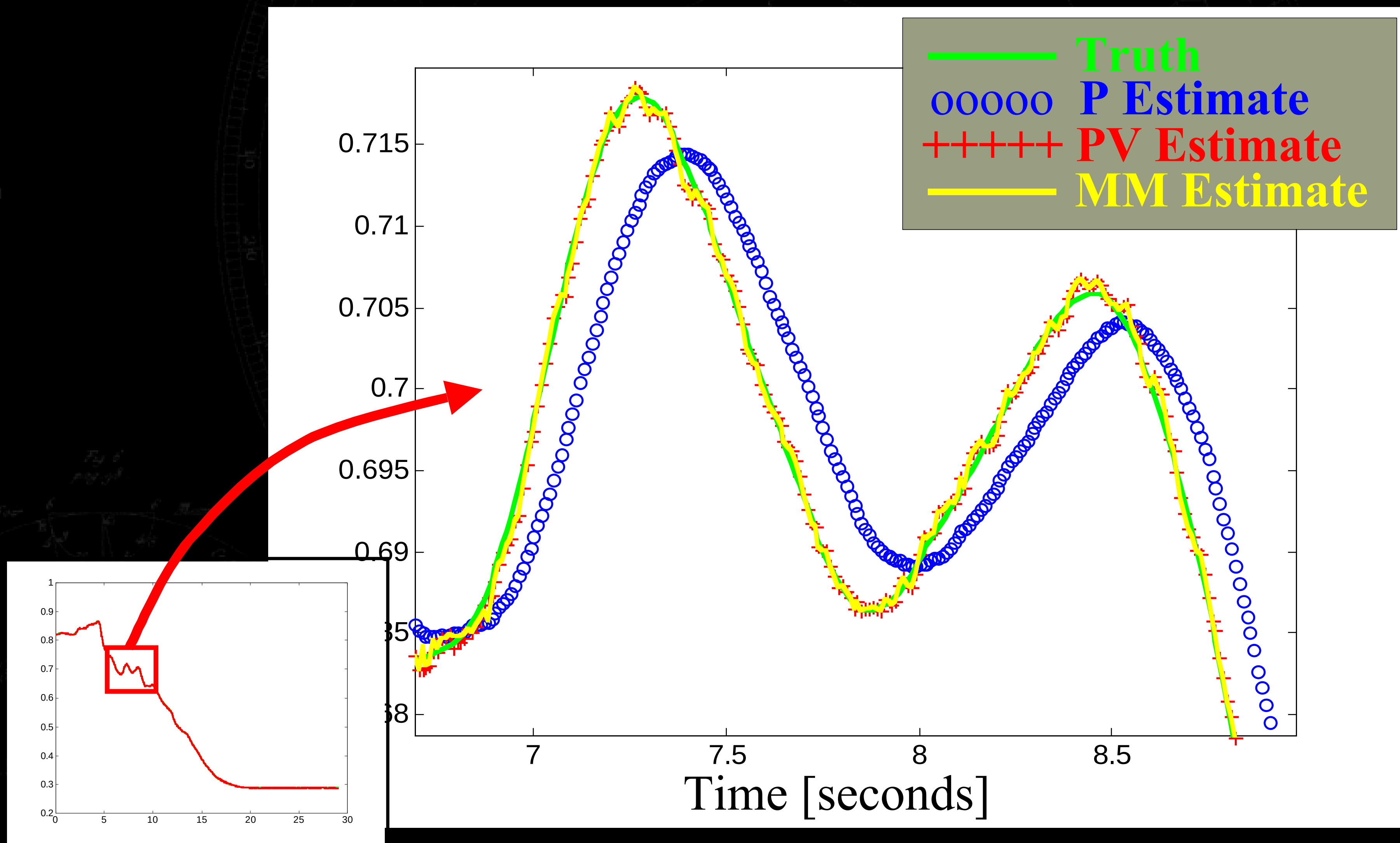
Example: P/PV Multiple-Model



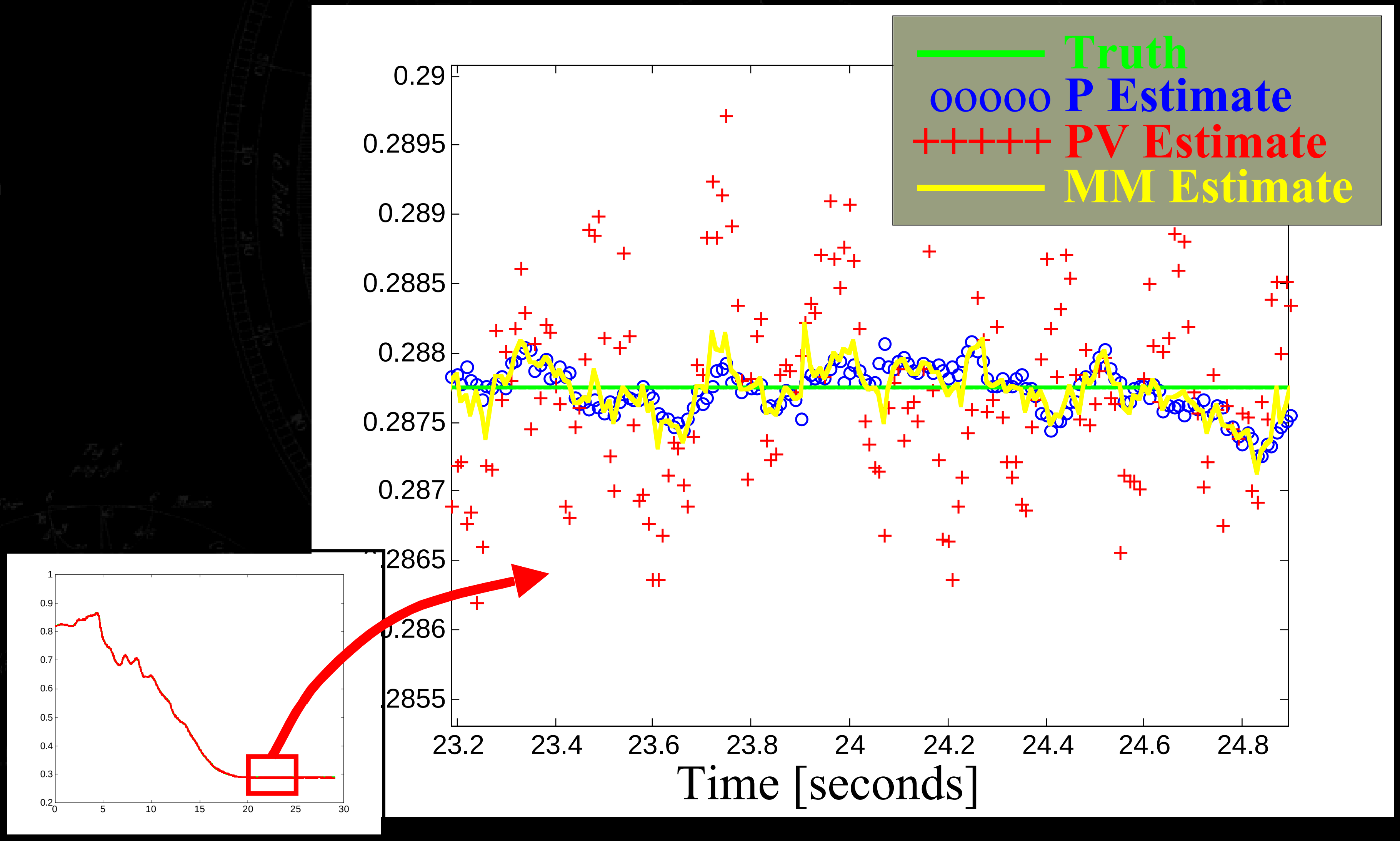
MME Weighting



Low-Latency During Motion



Smooth When Still



Conclusions

Suggestions and Resources
(Greg Welch)

Many Applications (Examples)

- Engineering
  - Robotics, spacecraft, aircraft, automobiles
- Computer
  - Tracking, real-time graphics, computer vision
- Economics
  - Forecasting economic indicators
- Other
  - Telephone and electricity loads

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Kalman Filter Web Site

http://www.cs.unc.edu/~welch/kalman/

- **Electronic and printed references**
  - Book lists and recommendations
  - Research papers
  - Links to other sites
  - Some software
- **News**

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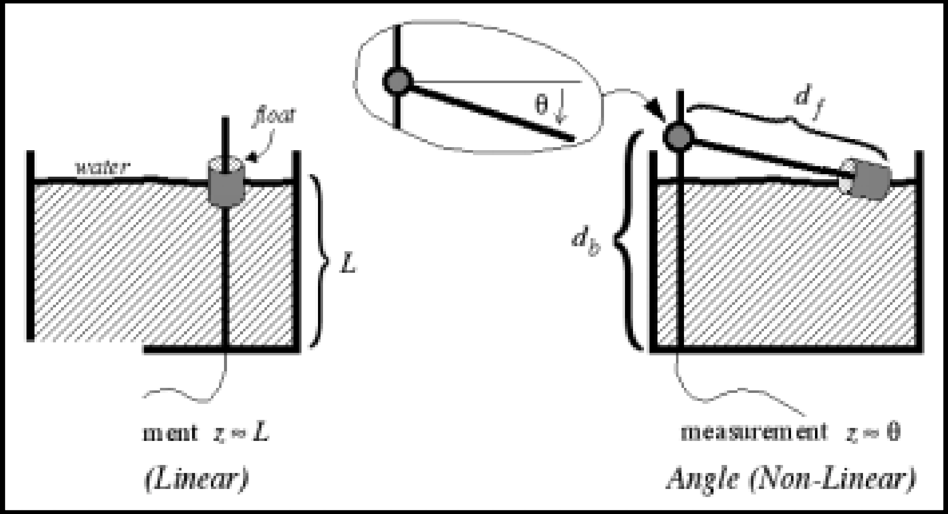
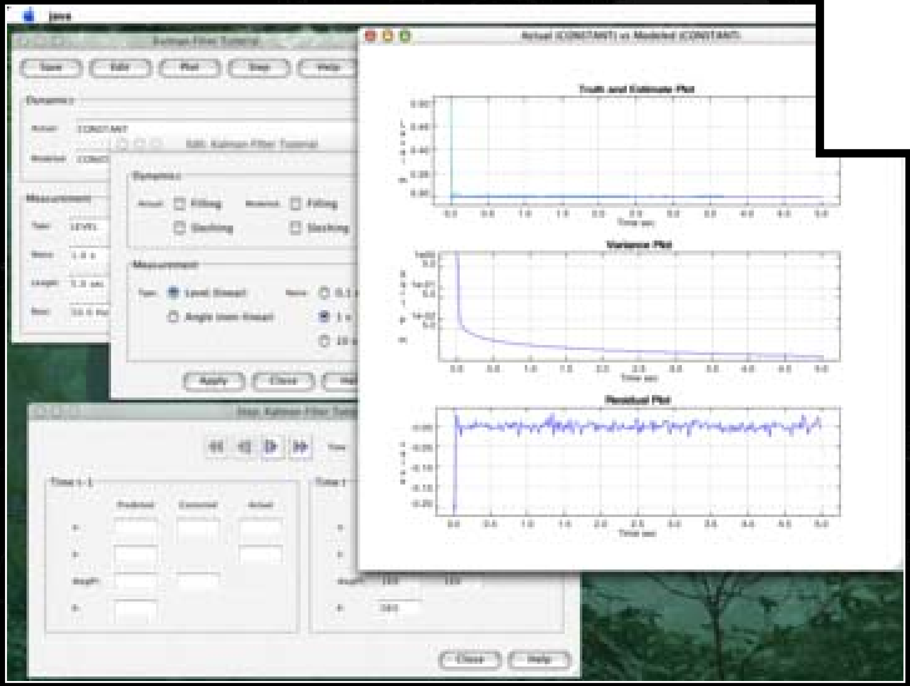
Java-Based KF Learning Tool

- On-line 1D simulation
- Linear and non-linear
- Variable dynamics





http://www.cs.unc.edu/~welch/kalman/

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KF Course Web Page

http://www.cs.unc.edu/~tracker/ref/s2001/kalman/index.html

(http://www.cs.unc.edu/~tracker/)

- Electronic version of course pack (updated)
- Java-Based KF Learning Tool
- KF web page

- *See also notes for Course 11 (Tracking)*

Closing Remarks

- **Try it!**
  - Not too hard to understand or program
- **Start simple**
  - Experiment in 1D
  - Make your own filter in Matlab, etc.
- **Note: the Kalman filter "wants to work"**
  - Debugging can be difficult
  - Errors can go un-noticed

End