An Introduction to the Kalman Filter

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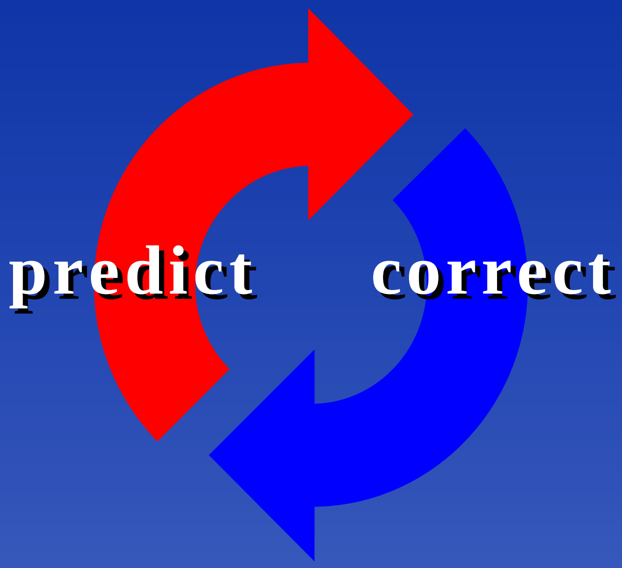
http://www.cs.unc.edu/~welch/kalmanLinks.html

Where We're Going

- Introduction & Intuition
- The Discrete Kalman Filter
- A Simple Example
- Variations of the Filter
- Relevant Applications & References

The Kalman filter

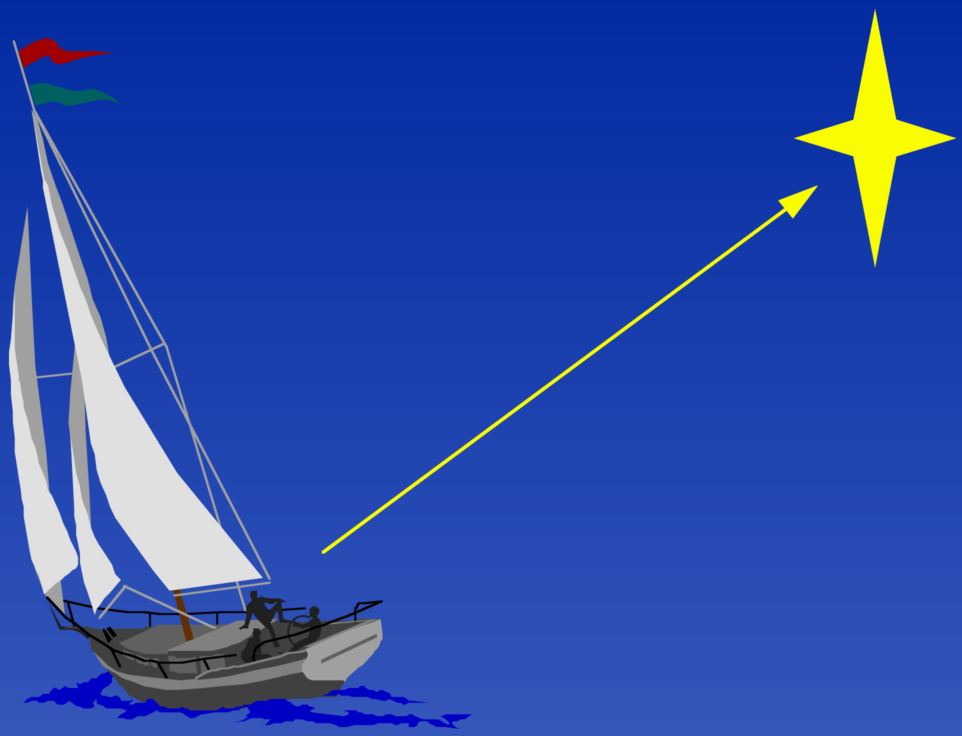
- Seminal paper by R.E. Kalman, 1960
- Set of mathematical equations
- Optimal estimator
  - minimum mean square error
- Versatile
  - Estimation
  - Filtering
  - Prediction
  - Fusion



Why a Kalman Filter?

- Efficient "least-squares" implementation
- Past, present and future estimation
- Estimation of missing states
- Measure of estimation quality (variance)
- Robust
  - forgiving in *many* ways
  - stable given common conditions

Some Intuition



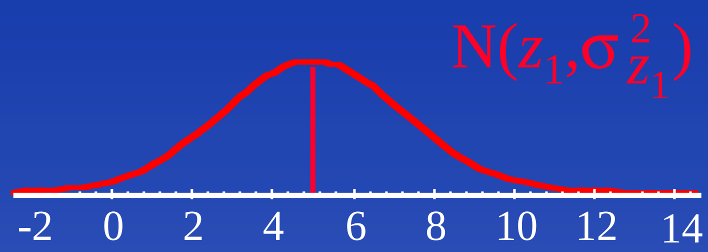
First Estimate

Conditional Density Function

$$z_1, \sigma^2_{z_1}$$

$$\hat{x}_1 = z_1$$

$$\hat{\sigma}^2_1 = \sigma^2_{z_1}$$



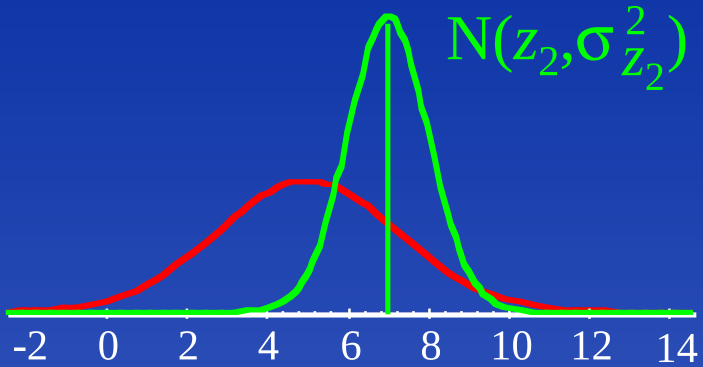
Second Estimate

Conditional Density Function

$$z_2, \sigma_{z_2}^2$$

$$\hat{x}_2 = \ldots ?$$

$$\hat{\sigma}_2^2 = \ldots ?$$



Combine Estimates

$$\hat{x}_2 = \left[\sigma_{z_2}^2 / \left(\sigma_{z_1}^2 + \sigma_{z_2}^2\right)\right] z_1 + \left[\sigma_{z_1}^2 / \left(\sigma_{z_1}^2 + \sigma_{z_2}^2\right)\right] z_2$$

$$= \hat{x}_1 + K_2 \left[z_2 - \hat{x}_1\right]$$

where

$$K_2 = \sigma_{z_1}^2 / \left(\sigma_{z_1}^2 + \sigma_{z_2}^2\right)$$

Combine Variances

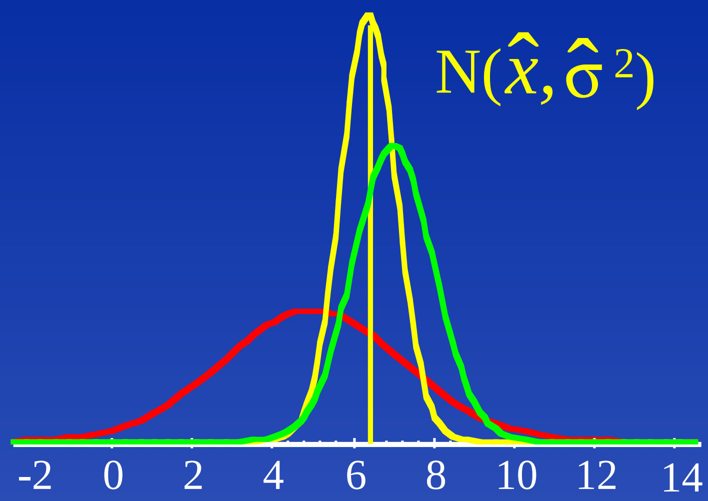
$$1/\sigma_2^2 = \left(1/\sigma_{z_1}^2\right) + \left(1/\sigma_{z_2}^2\right)$$

Combined Estimate Density

$$\hat{x} = \hat{x}_2$$

$$\hat{\sigma}^2 = \sigma_2^2$$

Conditional Density Function



Add Dynamics

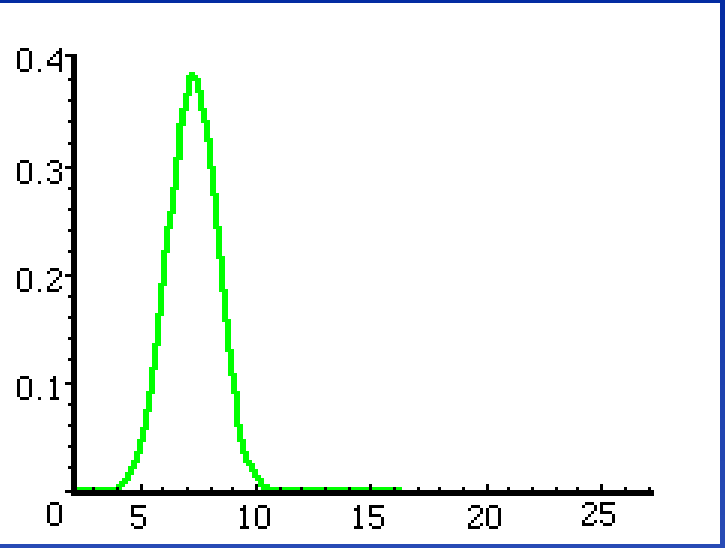
$$dx/dt = v + w$$

where

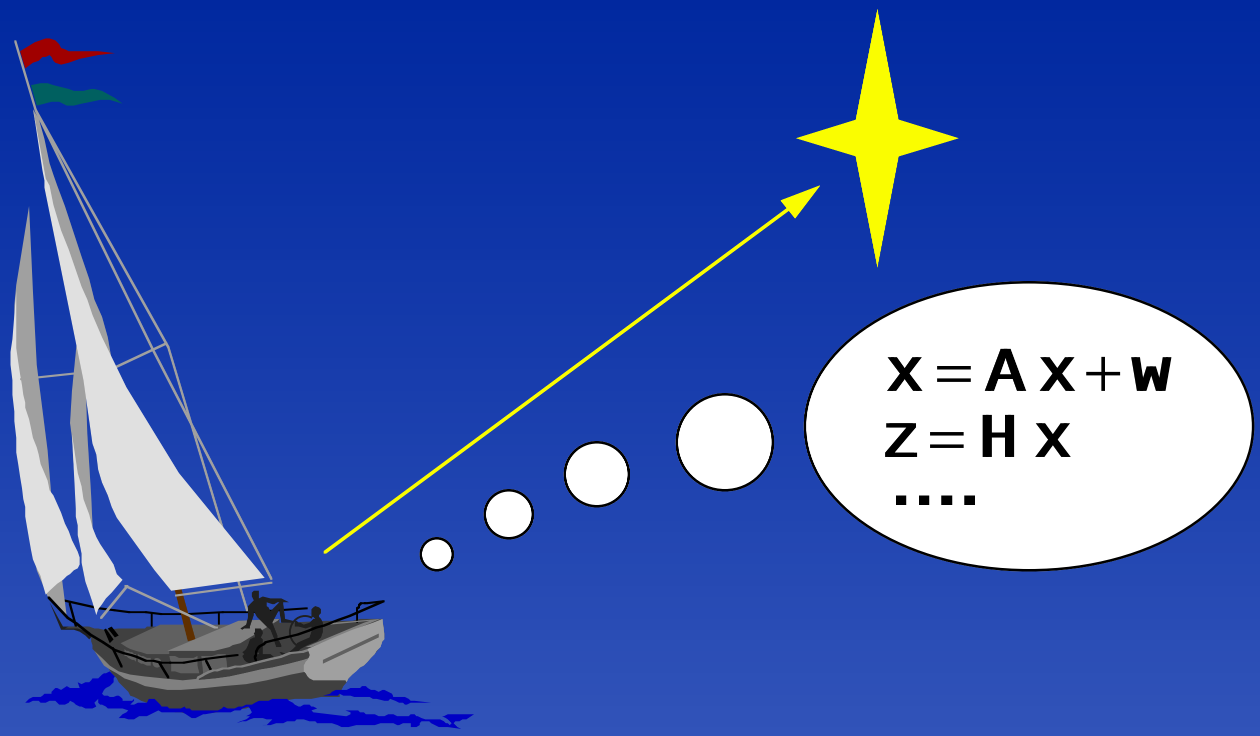
v is the nominal velocity

w is a noise term (uncertainty)

Propagation of Density

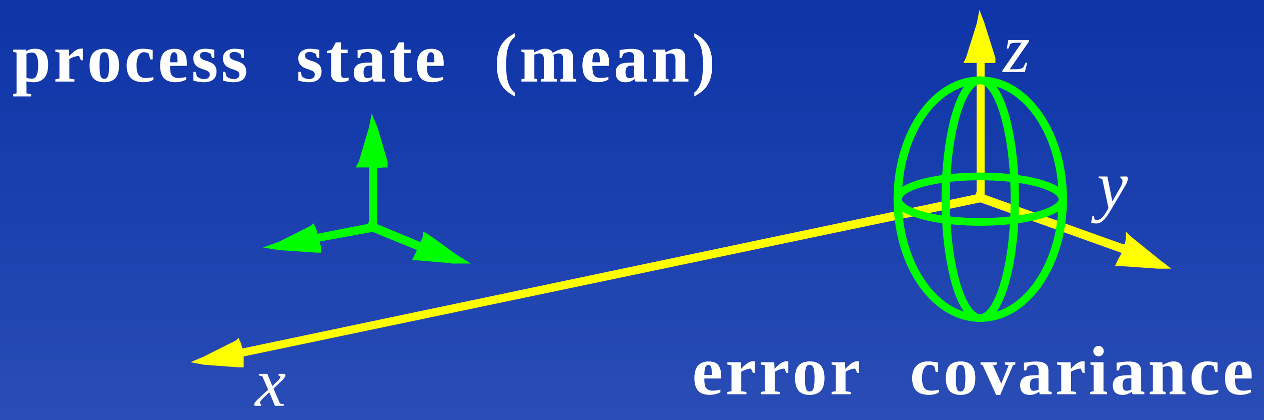


Some Details



Discrete Kalman Filter

Maintains first two statistical moments

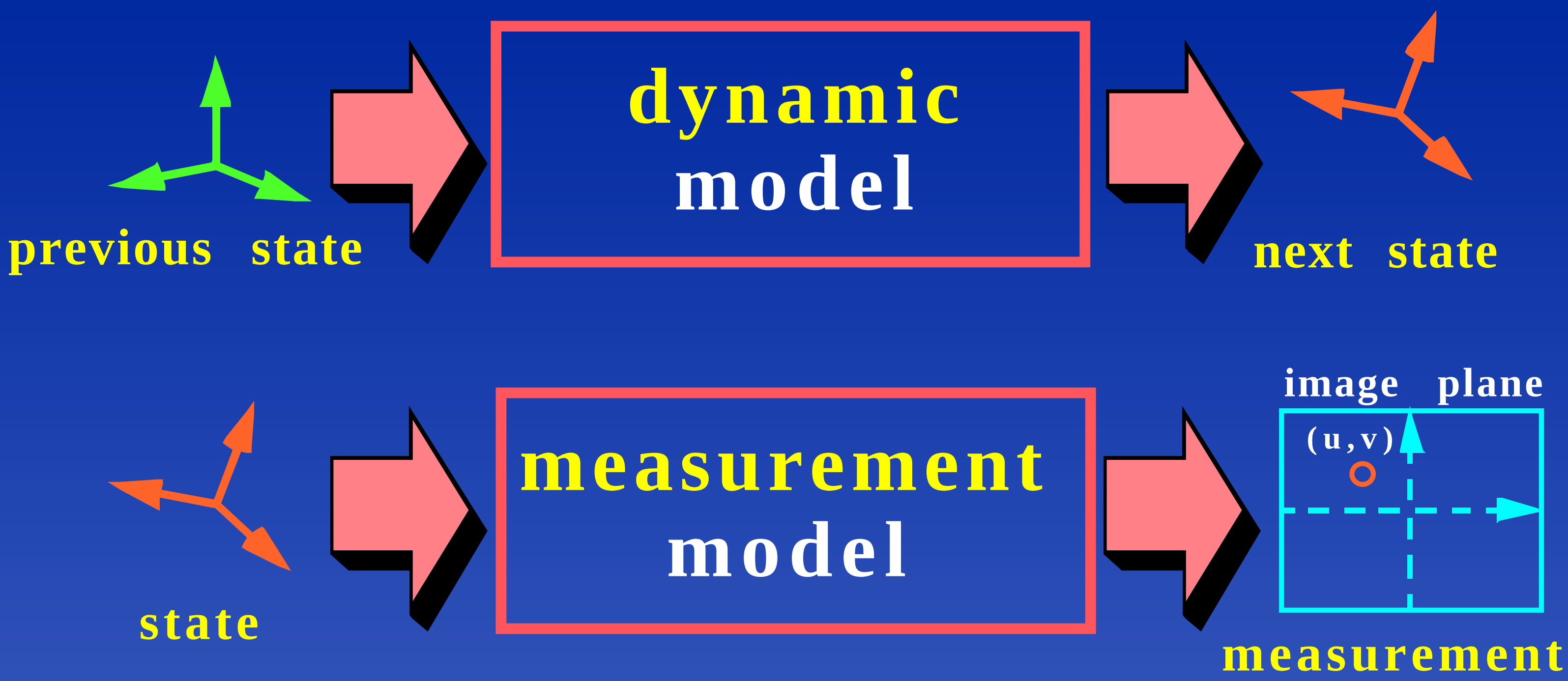


Discrete Kalman Filter

The Ingredients

- A discrete *process model*
  - –change in state over time
  - –linear difference equation
- A discrete *measurement model*
  - –relationship between state and measurement
  - –linear function
- Model Parameters
  - –Process noise characteristics
  - –Measurement noise characteristics

Necessary Models



The Process Model

Process Dynamics

$$x_{k+1} = Ax_k + w_k$$

Measurement

$$z_k = Hx_k + v_k$$

Process Dynamics

$$x_{k+1} = Ax_k + w_k$$

state vector

$x_k \in R^n$ contains the states of the process

Process Dynamics

$$x_{k+1} = Ax_k + w_k$$

state transition matrix

nxn matrix A relates state at time step k to time step $k+1$

Process Dynamics

$$x_{k+1} = Ax_k + w_k$$

process noise

$w_k \in R^n$ models the uncertainty of the process

$$w_k \sim N(0, Q)$$

Measurement

$$z_k = Hx_k + v_k$$

measurement vector

$z_k \in R^m$ is the process measurement

Measurement

$$z_k = H x_k + v_k$$

state vector

Measurement

$$z_k = Hx_k + v_k$$

measurement matrix

*m*x*n* matrix *H* relates state to measurement

Measurement

$$z_k = Hx_k + v_k$$

measurement noise

$z_k \in R^m$ models the noise in the measurement

$$v_k \sim N(0, R)$$

State Estimates

a priori state estimate

$$\hat{x}_k^-$$

a posteriori state estimate

$$\hat{x}_k$$

Estimate Covariances

a priori estimate error covariance

$$P_k^- = E[(x_k - \hat{x}_k^-)(x_k - \hat{x}_k^-)^T]$$

a posteriori estimate error covariance

$$P_k = E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$$

Filter Operation

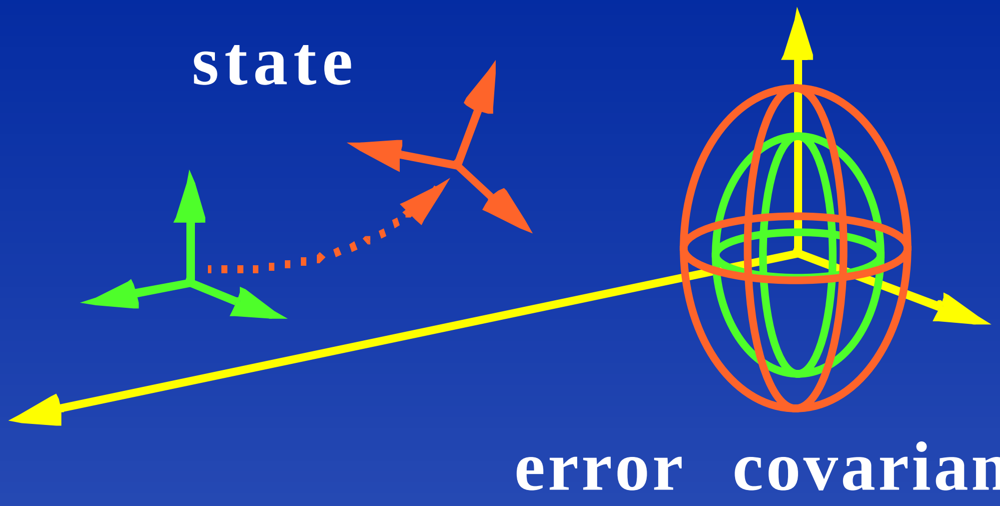
Time update (*a priori* estimates)

- Project state and covariance forward to next time step, i.e. *predict*

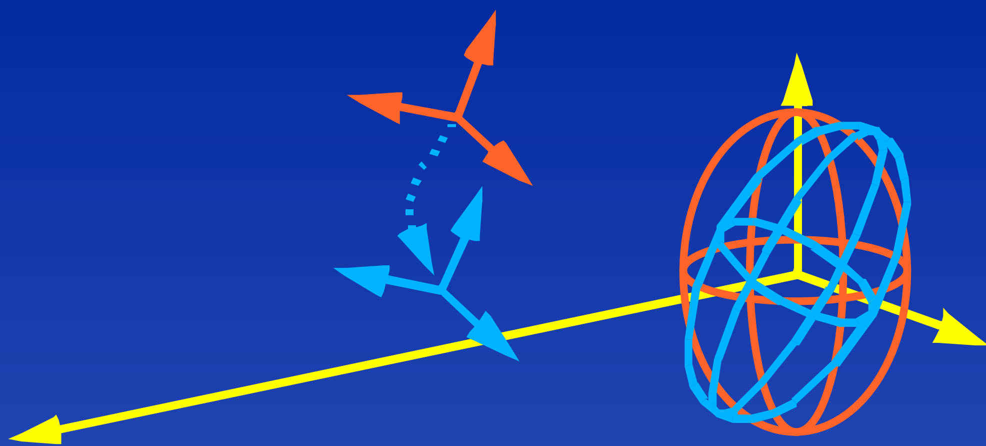
Measurement update (*a posteriori* estimates)

- Update with a (noisy) measurement of the process, i.e. *correct*

Time Update (Predict)



Measurement Update (Correct)



Time Update (Predict)

a priori state and error covariance

$$\hat{x}_{k+1}^{-} = Ax_k$$

$$P_{k+1}^{-} = AP_kA + Q$$

Measurement Update (Correct)

Kalman gain

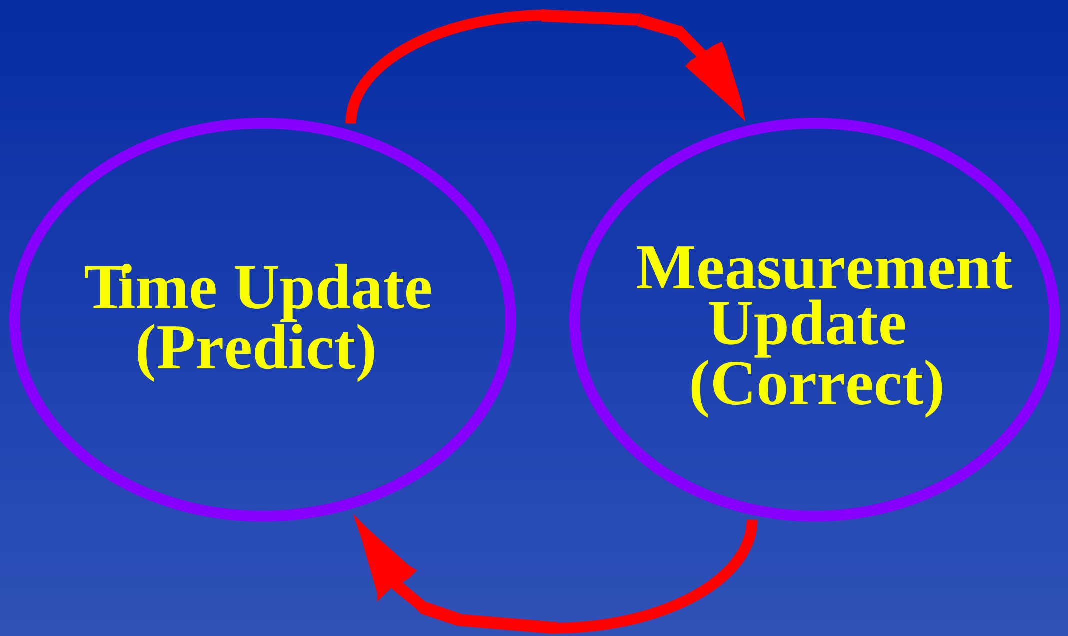
$$K_k = P_k^- H^T (H P_k^- H^T + R)^{-1}$$

a posteriori state and error covariance

$$\hat{x}_k = \hat{x}_k^- + K_k (z_k - H \hat{x}_k^-)$$

$$P_k = (I - K_k H) P_k^-$$

Filter Operation



A Simple Example

Estimating a Constant

Estimating a Constant

A, H, P_k, R, z_k, and K_k are all scalars. In particular,

$$A = 1$$

$$H = 1$$

Process Model

Process Dynamics

$$x_{k+1} = x_k$$

Measurement

$$z_k = x_k + v_k$$

Time Update

a priori state and error covariance

$$\hat{x}^{-}_{k+1} = x_k$$

$$P^{-}_{k+1} = P_k$$

Measurement Update

Kalman gain

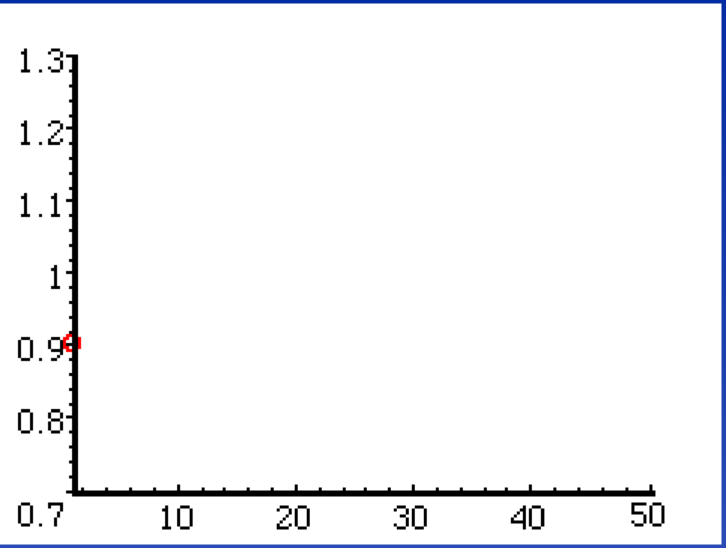
$$K_k = P_k^- / (P_k^- + R)$$

a posteriori state and error covariance

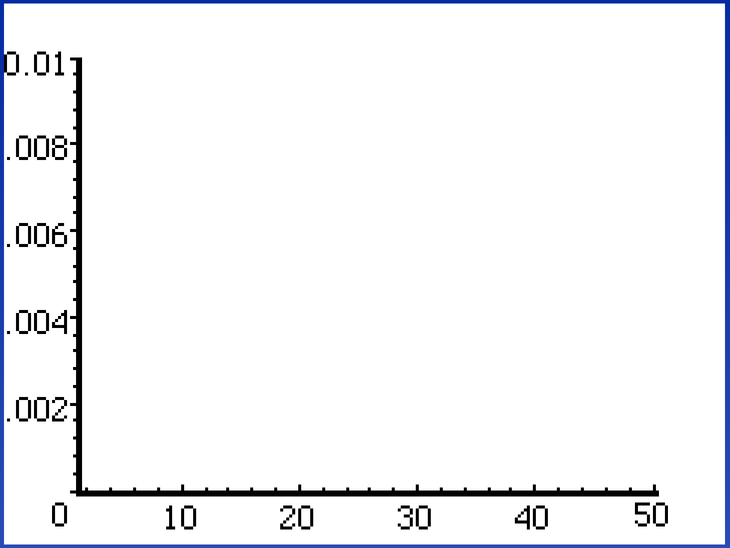
$$\hat{x}_k = \hat{x}_k^- + K_k (z_k - \hat{x}_k^-)$$

$$P_k = (1 - K_k) P_k^-$$

Simulations



z_k and $\hat{x}_k$



P_k

Variations of the Filter

- Discrete-Discrete (a.k.a. “discrete”) ✓
- Continuous-Discrete
- Extended Kalman Filter

Continuous-Discrete

- The Process (Model)
  - – Continuous model of system dynamics
  - – Discrete measurement equation
- Why, When, How
  - – Flexibility in prediction
  - – Irregularly spaced (discrete) measurements
  - – At measurement, integrate state forward (e.g. Runge-Kutta integrator)

Extended Kalman Filter

- Nonlinear Model(s)
  - Process dynamics: A becomes $a(x,w)$
  - Measurement: H becomes $h(x,z)$
- Filter Reformulation
  - Use functions instead of matrices
  - Use Jacobians to project forward, and to relate measurement to state

Relevant Applications

- Rebo

  Rebo, Robert K. 1988. “A Helmet-Mounted Virtual Environment Display System,” M.S. Thesis, Air Force Institute of Technology.

- Azuma

  Azuma, Ronald. 1995. “Predictive Tracking for Augmented Reality,” Ph.D. dissertation, The University of North Carolina at Chapel Hill, TR95-007.

Relevant Applications

- Friedmann et al.

  Friedmann, Martin, Thad Starner, and Alex Pentland. 1992. “Device Synchronization Using an Optimal Filter,” Proceedings of 1992 Symposium on Interactive 3D Graphics (Cambridge MA) 57–62

- Liang et al.

  Liang, Jiandong, Chris Shaw, and Mark Green. “On Temporal-Spatial Realism in the Virtual Reality Environment,” Proceedings of the 4th annual ACM Symposium on User Interface Software & Technology, 19-25

Relevant Applications

- Van Pabst & Krekel

  Van Pabst, Joost Van Lawick, and Paul F. C. Krekel. “Multi Sensor Data Fusion of Points, Line Segments and Surface Segments in 3D Space,” TNO Physics and Electronics Laboratory, The Hague, The Netherlands. [cited 19 November 1995].

SCAAT

- Tracking Latency & Rate
- Simultaneity Assumption
- SCAAT
  - Observability
  - Family of *unobservable* systems
  - Calibration
  - VE Tracking, GPS, etc.

Kalman Filter Papers...

- Kalman

  Kalman, R. E. 1960. “A New Approach to Linear Filtering and Prediction Problems,” Transaction of the ASME—Journal of Basic Engineering, pp. 35-45 (March 1960).

- Sorenson

  Sorenson, H. W. 1970. “Least-Squares estimation: from Gauss to Kalman,” IEEE Spectrum, vol. 7, pp. 63-68, July 1970.

Some Books

Brown

Introduction to Random Signals and Applied Kalman Filtering (2nd)

Gelb

Applied Optimal Estimation

Jacobs

Introduction to Control Theory

Lewis

Optimal Estimation with an Introduction to Stochastic Control Theory

Maybeck

Stochastic Models, Estimation, and Control, Volume 1

Further Information

- Welch & Bishop

  Welch, Greg and Gary Bishop. 1995. “An Introduction to the Kalman Filter,” The University of North Carolina at Chapel Hill, TR95-041

http://www.cs.unc.edu/~welch/kalmanLinks.html