COMP 550, Spring 2015
Quiz 1 (open book)
Jan 26, 2015

1) (80') Name:______ANSWER_________ PID:______________________

2) (10') Show the solution of $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$ is $O(n \lg n)$ using the substitution method.

   By induction, we have $T(k) \leqslant c \cdot n\lg k$ for all $k<n$. Thus,

   $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$

   $\leqslant 2\ c \cdot (\lfloor n/2 \rfloor + 17)\lg(\lfloor n/2 \rfloor + 17) + n$

   $\leqslant 2\ c \cdot (n/2 + 17)\lg(n/2 + 17) + n$

   $= c \cdot (n+34)\ \lg((n+34)/2) + n$

   $= c \cdot (n+34)\ (\lg(n+34) - \lg 2) + n$

   $= c \cdot n \lg(n + 34) + 34c \cdot \lg(n+34) - \lg 2 \cdot c \cdot (n+34) + n$

   $= (c \cdot n\lg n - c \cdot n\lg n) + c \cdot n \lg(n+34) + 34 \cdot c \lg(n+34) - \lg 2 \cdot c \cdot n - \lg 2 \cdot 34 \cdot c + n$

   $= c \cdot n\lg n + cn(\lg(n + 34) - \lg n) + 34\ c\lg(n + 34) - (\lg 2 \cdot c - 1)n + \lg 2 \cdot 34 \cdot c$

   $= c \cdot n\lg n + \underline{cn\lg((n + 34)/n) + 34\ c\lg(n + 34) - (\lg 2 \cdot c - 1)n + \lg 2 \cdot 34 \cdot c}$

   We have a complicated residual. However, $\underline{\lg((n + 34)/n)} \to 0$ when n is large, and the dominating term is $-(\lg 2 \cdot c + 1)n$ which is negative for some proper c – that's a good sign. On the other hand, we have a $\underline{34\ c\lg(n + 34)}$ term, that is not dominating, but may be bothersome when n is not large enough.

   So we set $n_0 = 990$, so we have

   $34\ c\lg(n + 34) = 34\ c \cdot \lg 1024 = 340\ c$.

   $cn\lg((n + 34)/n) \leqslant cn/5$ (since $\lg(1024/990) < 0.2$)

   Thus the residual

   $\underline{cn\lg((n + 34)/n) + 34\ c\lg(n + 34) - (\lg 2 \cdot c - 1)n + \lg 2 \cdot 34 \cdot c}$

   $\leqslant 340\ c + cn/5 - \lg 2 \cdot c \cdot n + n + \lg 2 \cdot 34 \cdot c \leqslant 0$, when $c>5$ and $n>n_0=990$

   So we've shown for some $c>5$ and $n>n_0=990$, $T(n) \leqslant cn\lg n$ holds.

3) (10') Solve $T(n) = 2T(n^{1/2}) + 1$ by making a change of variables. Don't worry whether values are integral.

Let $m = \lg n$, so $n = 2^m$.

$T(n) = 2\,T(n^{0.5})+1 \Rightarrow T(2^m) = 2\,T(2^{0.5m})+1$

Let $S(m) = T(2^m)$, then $S(m) = 2\,S(0.5m) + 1$

(1) If we use the Master's Thm, a=2, b=2, this belongs to case 1,

thus $S(m) = O(m)$

(2) If using the recursion tree method, we will end up with a full binary tree, with height of $\lg_2 m$, and each element being 1. The sum for each level is $1,2,2^2,2^3,\ldots,2^{\lg 2m}=m$. Thus total cost is $2m=O(m)$

(3) If using the substitution method, we may guess $S(m) = O(m)$.

So Guess $S(m) = cm - b$

(Similar to the example we've seen on the white board, guessing $S(m) = cm$ will not work easily – lower order terms matter)

$S(m) = 2(c\,(m/2) - b) + 1$

$= cm - 2b + 1$

$\leqslant cm - b$ if $b > 1$, (any c will work, so let's just say c>1)

So we have $S(m) = O(m)$, which implies $T(2^m)=O(m)$, which is $T(n) = O(\lg m)$

4) (Bonus 10') Any suggestions to the course and/or the instructor (you may use the other side of the page)

I very much appreciated the suggestive comments!

Don't get too worried about not being very familiar with the math facts – in HWs you can take the time and search for the formulas, while in exams they will be provided. But, it is still important to at least work out some messy details (e.g., of HW problems) on your own.