Examples about conditional probability and independence

(1-1) Experiment: Throw a “2-one” dice, i.e. no 6 but two 1’s.
Event A: Dice 1 is 1
Event B: Dice 1 is odd
To calculate: \( P(A|B) \) (meaning the probability of Event A occurring under the condition that Event B occurs)

\[
p(A|B) = \frac{p(x=1| x \text{ is odd})}{p(x=\text{odd})} = \frac{p(x=1 \text{ and } x=\text{odd})}{p(x=\text{odd})} = \frac{p(x=1)}{p(x=\text{odd})} = \frac{1/3}{2/3} = \frac{1}{2}
\]
Note that \( P(A|B) \neq P(A) \), so Event A and Event B are dependent (of course they are, by definition).

(1-2) Experiment: Throw a “2-one” dice twice, i.e. no 6 but two 1s.
Event A: Dice 1 is 1
Event B: Dice 2 is odd
To calculate: \( P(A|B) \)

We have \( p(x=1 \text{ and } y=\text{odd}) = p((1,1),(1,3),(1,5)) = p((1,1)) + p((1,3)) + p((1,5)) = 4/36 + 2/36 + 2/36 = 2/9 \) (the original side for “6” also results in “1” for this special dice)

To calculate: \( P(A|B) = P(A|B) = P(x=1| y \text{ is odd}) = P(x=1 \text{ and } y=\text{odd})/P(y=\text{odd}) = (2/9)/(2/3) = 1/3 \)

Note that \( P(A) = p(x=1) = 1/3 \), so we’ve verified that Event A and Event B are independent.

(2) Experiment: Rolling two independent dice. Still the “2-one” dice, i.e. no 6 but two 1’s.
Event A: Dice 1 < 3
Event B: Dice 2 > 3
To calculate: \( P(A|B) \)

It’s trivial that A and B are independent (two different throws have 0 impact on each other).

\[
p(A|B) = p(x1<3|x2>3) = p(x1<3 \text{ and } x2>3) / p(x2>3)
\]

so that for a “2-one” dice:

\[
x1=\{1,2\}, \ x2=\{4,5\}
\]

\[
p(x2>3) = p(x=4 \text{ or } x=5) = p(x=4) + p(x=5) = 1/6 + 1/6 = 1/3
\]

\[
p(x1<3 \text{ and } x2>3) = p((1,4), (1,5), (2,4), (2,5)) = p((1,4)) + p((1,5)) + p((2,4)) + p((2,5)),
\]

where \( p((1,4)) = p(x1=1 \text{ and } x2=4) = 2/36 \), (the original {6,4} also results in {1,4} for this special dice)

similarly \( p((1,5))=2/36 \);

\( p((2,4))=p(x1=2 \text{ and } x2=4)=1/36 \)

similarly \( p((2,5))=1/36 \);

\[
=> p(A \text{ and } B) = 1/18 + 1/18 + 1/36 + 1/36 = 1/6
\]

\[
=> p(A|B) = (1/6) / (1/3) = 1/2
\]

\[
p(A) = p(x=1) + p(x=2) = 1/2 = p(A|B), \text{ so we've verified that A and B are independent.}
\]