Statistics of Shape: Eigen Shapes
“PCA and PGA”

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Two approaches to study the shape variation of the hippocampus in populations:

- Statistics of deformation fields using “Principal Components Analysis”

- Statistics of medial descriptions using Lei Groups: “Principal Geodesic Analysis”
Statistics of Deformation Fields
Hippocampal Mapping

Atlas

Patients
Hippocampal Mapping

Atlas

Subjects
Shape of 2-D Sub-Manifolds of the Brain: Hippocampus.

The provisory template hippocampal surface $M_0$ is carried onto the family of targets:

$$
M_0 \xleftarrow{h_1} M^1, \quad M_0 \xleftarrow{h_2} M^2, \quad \cdots, \quad M_0 \xleftarrow{h_N} M^N.
$$
Shape of 2-D Sub-Manifolds of the Brain: Hippocampus.

- The mean transformation and the template representing the entire population:

$$\bar{h} = \frac{1}{N} \sum_{i=1}^{N} h_i , \quad M_{\text{temp}} = \bar{h} \circ M_0 .$$

The mean hippocampus of the population of thirty subjects.
Shape of 2-D Sub-Manifolds of the Brain: Hippocampus.

- Mean hippocampus representing the control population:
  \[
  \bar{h}_{control} = \frac{1}{N_{control}} \sum_{i=1}^{N_{control}} h_i^{control}, \quad M_{control} = \bar{h}_{control} \circ M_0.
  \]

- Mean hippocampus representing the Schizophrenic population:
  \[
  \bar{h}_{schiz} = \frac{1}{N_{schiz}} \sum_{i=1}^{N_{schiz}} h_i^{schiz}, \quad M_{schiz} = \bar{h}_{schiz} \circ M_0.
  \]
Gaussian Random Vector Fields on 2-D Sub-Manifolds.

- Hippocampi $\mathcal{M}^i$, $i = 1, \cdots, N$ deformation of the mean $\mathcal{M}_{\text{temp}}$: 
  $$\mathcal{M}^i : \{y | y = x + u_i(x), x \in \mathcal{M}_{\text{temp}}\}$$

  $$u_i(x) = h_i(x) - x, x \in \mathcal{M}_{\text{temp}}.$$ 

  Vector field $u_i(x)$ shown in red.

- Construct Gaussian random vector fields over sub-manifolds.
Gaussian Random Vector Fields on 2-D Sub-Manifolds.

- Let $\mathcal{H}(\mathcal{M})$ be the Hilbert space of square integrable vector fields on $\mathcal{M}$. Inner product on the Hilbert space $\mathcal{H}(\mathcal{M})$:

$$ \langle f, g \rangle = \sum_{i=1}^{3} \int_{\mathcal{M}} f^i(x) g^i(x) d\nu(x) $$

where $d\nu$ is a measure on the oriented manifold $\mathcal{M}$.

**Definition 1** The random field $\{U(x), x \in \mathcal{M}\}$ is a Gaussian random field on a manifold $\mathcal{M}$ with mean $\mu_u \in \mathcal{H}(\mathcal{M})$ and covariance operator $K_u(x, y)$ if $\forall f \in \mathcal{H}(\mathcal{M})$, $\langle f, \cdot \rangle$ is normally distributed with mean $m_f = \langle \mu_u, f \rangle$ and variance $\sigma_f^2 = \langle K_u f, f \rangle$

- Gaussian field is completely specified by it’s mean $\mu_u$ and the covariance operator $K_u(x, y)$.

- Construct Gaussian random fields as a quadratic mean limit using a complete $\mathbb{R}^3$-valued orthonormal basis

$$ \{ \phi_k, k = 1, 2, \cdots \}, \quad \langle \phi_i, \phi_j \rangle = 0, \quad i \neq j $$
**Gaussian Random Vector Fields on 2-D Sub-Manifolds.**

**Theorem 1** Let \( \{U(x), x \in \mathcal{M}\} \) be a Gaussian random vector field with mean \( m_U \in \mathcal{H} \) and covariance \( K_U \) of finite trace. There exists a sequence of finite dimensional Gaussian random vector fields \( \{U_n(x)\} \) such that

\[
U(x)^{q,m} \xrightarrow{n \to \infty} U_n(x)
\]

where

\[
U_n(x) = \sum_{k=1}^{n} Z_k(\omega) \phi_k(x) ,
\]

\( \{Z_k(\omega), k = 1, \cdots \} \) are independent Gaussian random variables with fixed means \( E\{Z_k\} = \mu_k \) and covariances \( E\{|Z_i|^2\} - E\{Z_i\}^2 = \sigma_i^2 = \lambda_i, \Sigma_i \lambda_i < \infty \) and \( (h_k, \lambda_k) \) are the eigen functions and the eigen values of the covariance operator \( K_U \):

\[
\lambda_i \phi_i(x) = \int_{\mathcal{M}} K_U(x, y) \phi_i(y) d\nu(y) ,
\]

where \( d\nu \) is the measure on the manifold \( \mathcal{M} \).

If \( d\nu \), the surface measure on \( \tilde{\mathcal{M}}_{\text{temp}} \) is atomic around the points \( x_k \) then \( \{\phi_i\} \) satisfy the system of linear equations

\[
\lambda_i \phi_i(x_k) = \sum_{j=1}^{M} \hat{K}_U(x_k, y_j) \phi_i(y_j) \nu(y_j) , i = 1, \cdots, N,
\]

where \( \nu(y_j) \) is the surface measure around point \( y_j \).
Eigen Shapes of the Hippocampus.

- Eigen shapes $\mathcal{E}^i, i = 1, \cdots, N$ defined as:
  \[
  \mathcal{E}^i = \{x + (\lambda_i)\phi_i(x) : x \in \tilde{M}_{temp}\}.
  \]

- Eigen shapes completely characterize the variation of the sub-manifold in the population.
Statistical Significance of Shape Difference Between Populations.

- Assume that \{u_j^{schiz}, u_j^{control}\}, \(j = 1, \ldots, 15\) are realizations from a Gaussian process with mean \(\bar{u}_{schiz}\) and \(\bar{u}_{control}\) and common covariance \(K_U\).

  Statistical hypothesis test on shape difference:
  
  \[ H_0 : \bar{u}_{norm} = \bar{u}_{schiz} \]
  \[ H_1 : \bar{u}_{norm} \neq \bar{u}_{schiz} \]

- Expand the deformation fields in the eigen functions \(\phi_i\):
  
  \[ u_N^{schiz}(j)(x) = \sum_{i=1}^{N} Z_i^{schiz}(j) \phi_i(x) \]
  \[ u_N^{control}(j)(x) = \sum_{i=1}^{N} Z_i^{control}(j) \phi_i(x) \]

- \(\{Z_j^{schiz}, Z_j^{control}, j = 1, \ldots, 15\}\) Gaussian random vectors with means \(\bar{Z}_{schiz}\) and \(\bar{Z}_{control}\) and covariance \(\Sigma\).

  Hotelling’s \(T^2\) test:
  
  \[ T_N^2 = \frac{M}{2}(\hat{Z}_{norm} - \hat{Z}_{schiz})^T \Sigma^{-1}(\hat{Z}_{norm} - \hat{Z}_{schiz}). \]

<table>
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<th>N</th>
<th>(T_N^2)</th>
<th>p-value : (P_N(H_0))</th>
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<td>3</td>
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<td>6</td>
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\(N\): number of eigen functions.
Bayesian Classification on Hippocampus Shape Between Population.

- Bayesian log-likelihood ratio test: $H_0$: normal hippocampus, $H_1$: schizophrenic hippocampus.

$$\Lambda_N = -(Z - \hat{Z}_{schiz})^\dagger \hat{\Sigma}^{-1}(Z - \hat{Z}_{schiz})$$

$$+ (Z - \hat{Z}_{norm})^\dagger \hat{\Sigma}^{-1}(Z - \hat{Z}_{norm}) \begin{cases} H_0 \leq 0 \\ H_1 > 0 \end{cases}$$

- Use Jack Knife for estimating probability of classification:
Statistics of Medial descriptions

- Each figure a quad mesh of medial atoms:

\[ \{m_{i,j}^0 : i = 1 \cdots N, j = 1 \cdots M \} \]

\[ m_{i,j}^0 = (x_{i,j}, r, F, \theta) \]

- Medial atom parameters include angles and rotations.
- Medial atoms do not form a Hilbert Space
  – Cannot use “Eigen Shape” for statistical characterization!!
Statistics of Medial descriptions

- Set of all Medial Atoms forms Lie-Group

\[ m = (x_{i,j}, r, F, \theta) \]
\[ m \in \mathbb{R}^3 \times \mathbb{R}^+ \times SO(3) \times SO(2) \]

- \( \mathbb{R}^3 \): Position \( x \)
- \( \mathbb{R}^+ \): Radius \( r \)
- \( SO(3) \): Frame
- \( SO(2) \): Object angle
A Lie group is a group $G$ which is also a differential manifold where the group operations are differential maps.

- Both composition and the inverse are differential maps

$\mu : (x, y) \mapsto xy : G \times G \mapsto G$

$\iota : x \mapsto x^{-1} : G \mapsto G$

$\mathbb{R}^3 : \mu(x, y) = x + y, x^{-1} = -x$

$\mathbb{R}^+ : \text{Multiplicative reals} \quad \mu(x, y) = xy, x^{-1} = \frac{1}{x}$

$SO(3) : 3 \times 3 \text{ Orthogonal Matrix Group}$

$SO(2) : 2 \times 2 \text{ Orthogonal Matrix Group}$
Lie Group Means

- Algebraic mean not defined on Lie Groups
- Use geometric definition:
  - Remanian Distance well defined on a Manifold.
- Given N medial atoms \( \{m_i : i = 1 \ldots N\} \)
  the mean \( \bar{m} \) is defined as the group element that minimizes the average squared distance to the data.

\[
\bar{m} = \arg \min_{m} \frac{1}{N} \sum_{i=1}^{N} \left| d(m, m_i) \right|^2
\]

- No closed form solution need to use Lie-Group optimization techniques.
Geodesic Curves

- Medial manifold is curved and hence no straight lines.
- Distance minimizing **Geodesic** curves are analogous to straight lines in Euclidean Space.
- Geodesics in Lie Groups are given by the exponent map:

\[ g(t) = \exp(tA) \]

- Geodesics are one parameter sub-groups analogous to 1-dimensional subspaces in \( \mathbb{R}^N \)
Principal Geodesics

- Since the set of all medial atoms is a curved manifold, linear PCA is not defined as well.

- Principal Geodesics are defined as the geodesics that minimize residual distance.
  - No closed form solution: Needs non-linear optimization.