COMP 555 Bioalgorithms

Fall 2014

Lecture 3: Algorithms and Complexity

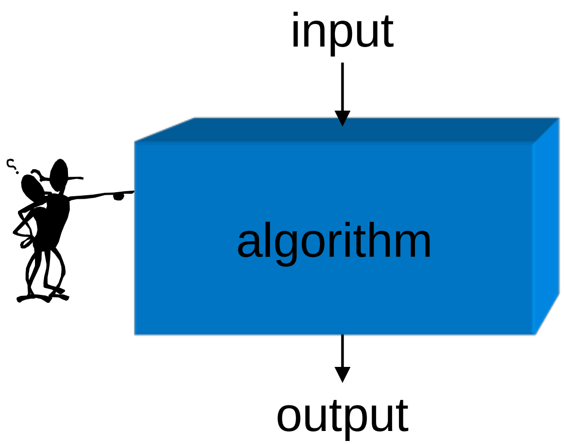
Study Chapter 2.1-2.8

Topics

- Algorithms
  - Correctness
  - Complexity
- Some algorithm design strategies
  - Exhaustive
  - Greedy
  - Recursion
- Asymptotic complexity measures

What is an algorithm?

- An algorithm is a sequence of instructions that one must perform in order to solve a well-formulated problem.



Problem: Complexity

Algorithm: Correctness
Complexity

Problem: US Coin Change

- Input
  - an amount of money $0 \leq M < 100$ in cents
- Output:
  - M cents in US coins using the minimal number of coins

- Example

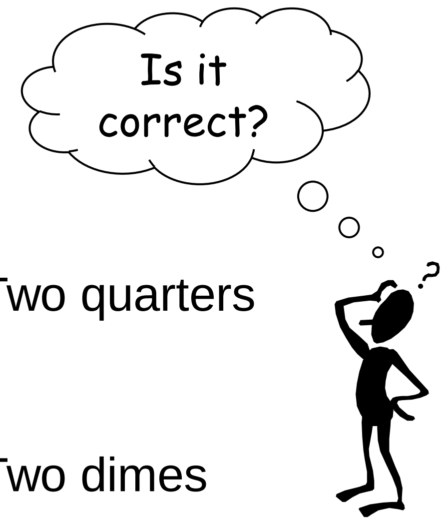
72 cents

↓

Two quarters

Two dimes

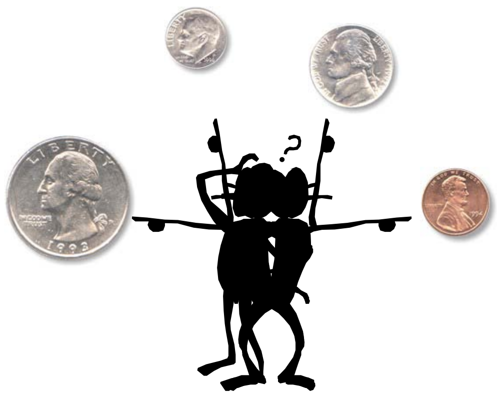
Two pennies



Algorithm 1: Greedy strategy

72 cents

Greedy coin alg



Use large denominations as long as possible

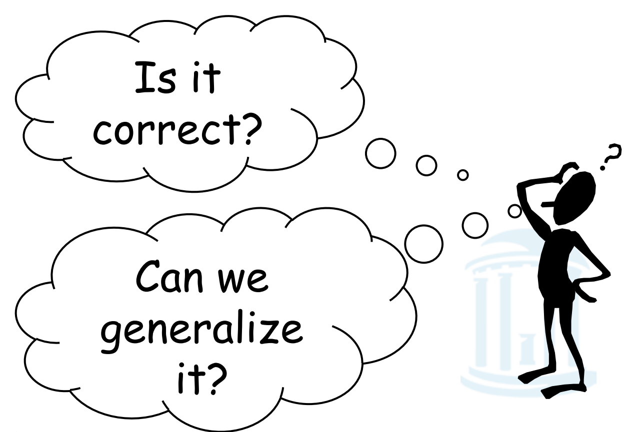
Algorithm description

$$r \leftarrow M$$
$$q \leftarrow \lfloor r/25 \rfloor$$
$$r \leftarrow r - 25 \cdot q$$
$$d \leftarrow \lfloor r/10 \rfloor$$
$$r \leftarrow r - 10 \cdot d$$
$$n \leftarrow \lfloor r/5 \rfloor$$
$$r \leftarrow r - 5 \cdot n$$
$$p \leftarrow r$$

Two quarters, 22 cents left

Two dimes, 2 cents left

Two pennies



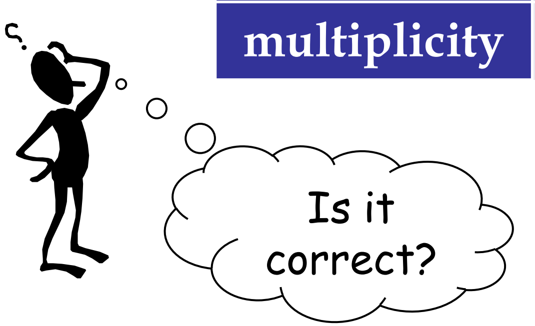
Algorithm 2: Exhaustive strategy

- Enumerate *all* combinations of coins. Record the combination totaling to *M* with fewest coins
  - *All* is impossible. Limit the multiplicity of each coin!
  - First try (80,000 combinations)

coin	Quarter	Dime	Nickel	Penny
multiplicity	0..3	0..9	0..19	0..99

  - Better (200 combinations)

coin	Quarter	Dime	Nickel	Penny
multiplicity	0 .. 3	0 .. 4	0 .. 1	0 .. 4



Correctness

- An algorithm is correct only if it produces correct result for all input instances.
  - If the algorithm gives an incorrect answer for one or more input instances, it is an incorrect algorithm.

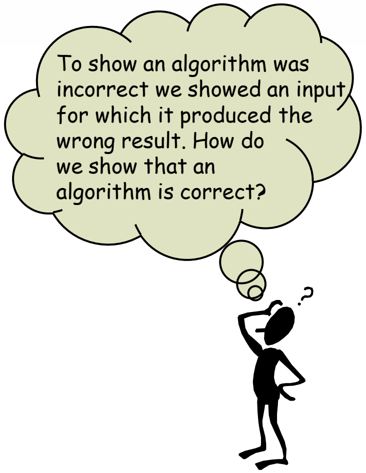
- US coin change problem
  - It is easy to show that the exhaustive algorithm is correct
  - The greedy algorithm is correct but we didn't really show it

Observations

- Given a problem, there may be many correct algorithms.
  - They give identical outputs for the same inputs
  - They give the expected outputs for any valid input
- The costs to perform different algorithms may be different.
- US coin change problem
  - The exhaustive algorithm checks 200 combinations
  - The greedy algorithm performs just a few arithmetic operations

Change Problem: generalization

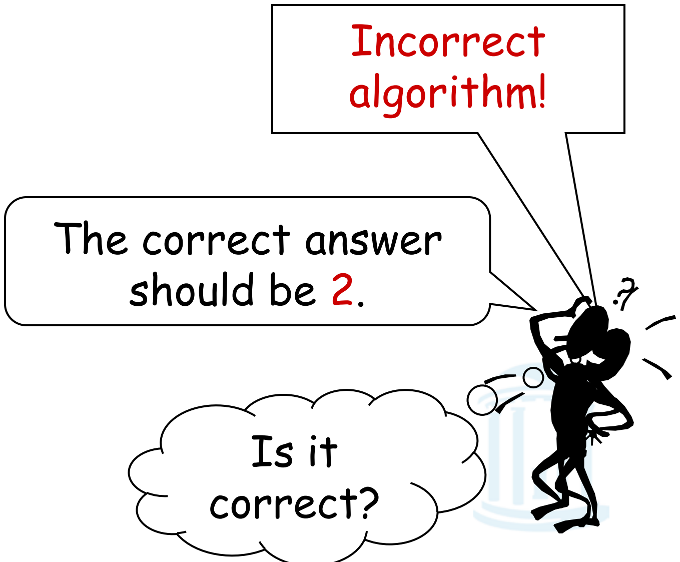
- Input:
  - an amount of money M
  - an array of denominations $c = (c_1, c_2, \ldots, c_d)$ in order of decreasing value
- Output: the smallest number of coins



$M = 40$
$c = (25, 20, 10, 5, 1)$ →

$r \leftarrow M$
$n \leftarrow 0$
for $k \leftarrow 1$ to d
  $i_k \leftarrow \lfloor r / c_k \rfloor$
  $n \leftarrow n + i_k$
  $r \leftarrow r - i_k c_k$
return n

→ 3



How to Compare Algorithms?

- Complexity — the cost of an algorithm can be measured in either time and space
  - Correct algorithms may have different complexities.
- How do we assign "cost" for time?
  - Roughly proportional to number of instructions performed by computer
  - Exact cost is difficult to determine and not very useful
    - Varies with computer, particular input, etc.
- How to analyze an algorithm's complexity
  - Depends on algorithm design

Recursive Algorithms

- Recursion is an algorithm design technique for solving problems in terms of simpler subproblems
  - The simplest versions, called base cases, are merely declared.

Recursive definition: $\text{factorial}(n) = n \times \text{factorial}(n-1)$

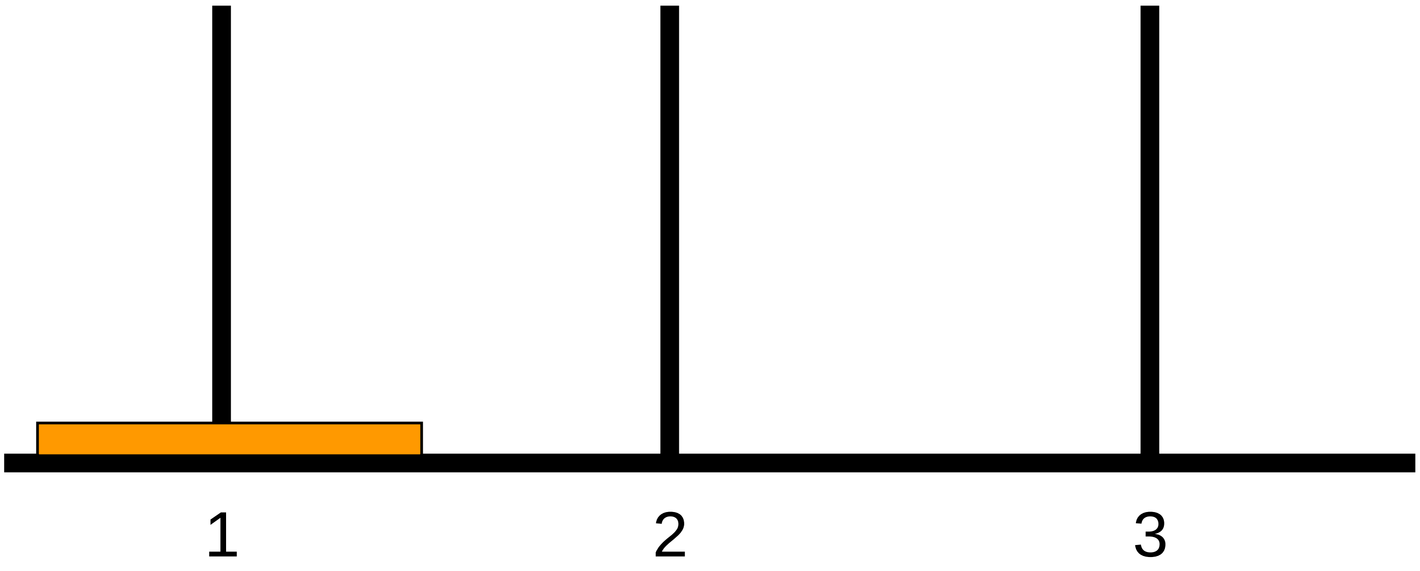
Base case: $\text{factorial}(1) = 1$

  - Easy to analyze
- Thinking recursively…

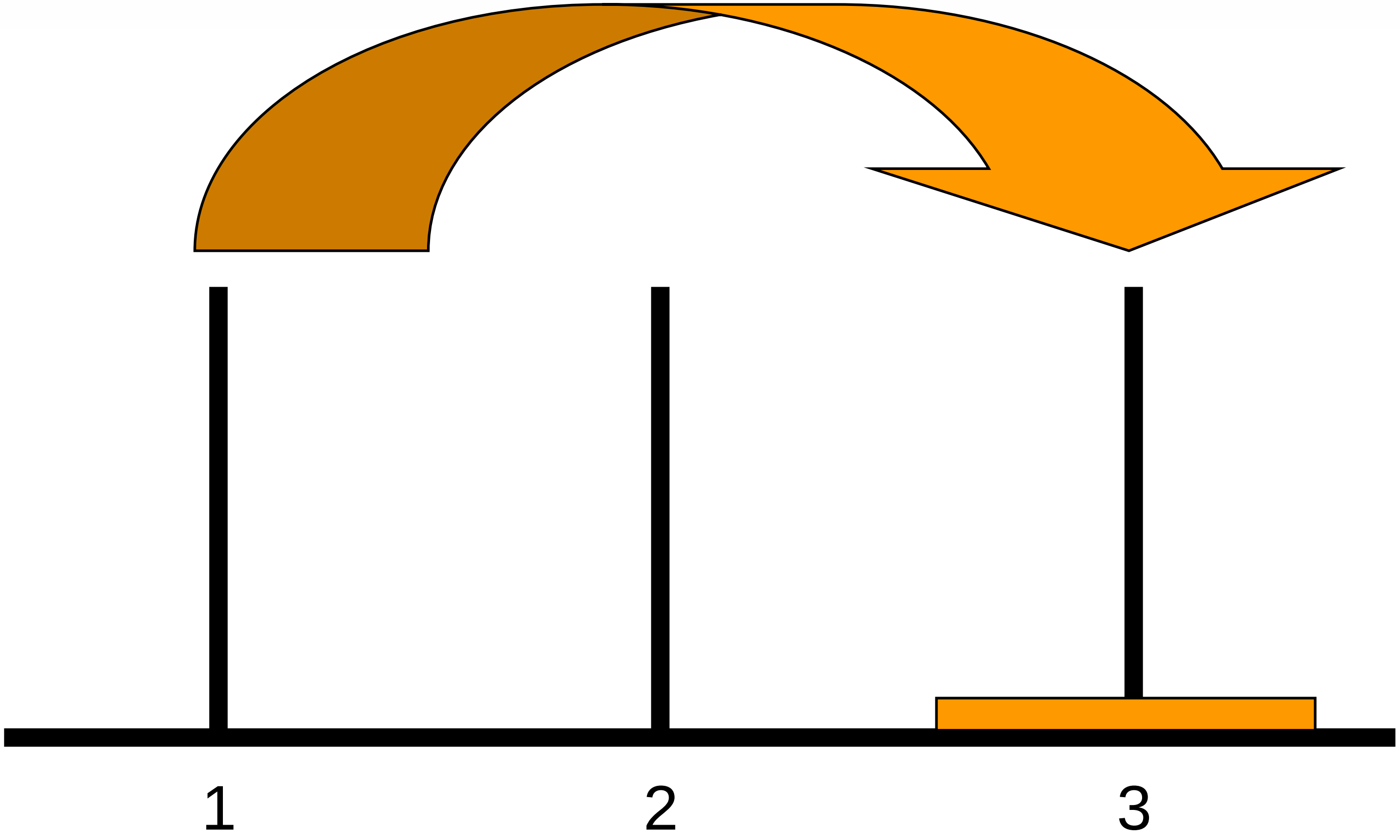
Towers of Hanoi

- There are three pegs and a number of disks with decreasing radii (smaller ones on top of larger ones) stacked on Peg 1.
- Goal: move all disks to Peg 3.
- Rules:
  - When a disk is moved from one peg it must be placed on another peg.
  - Only one disk may be moved at a time, and it must be the top disk on a tower.
  - A larger disk may never be placed upon a smaller disk.

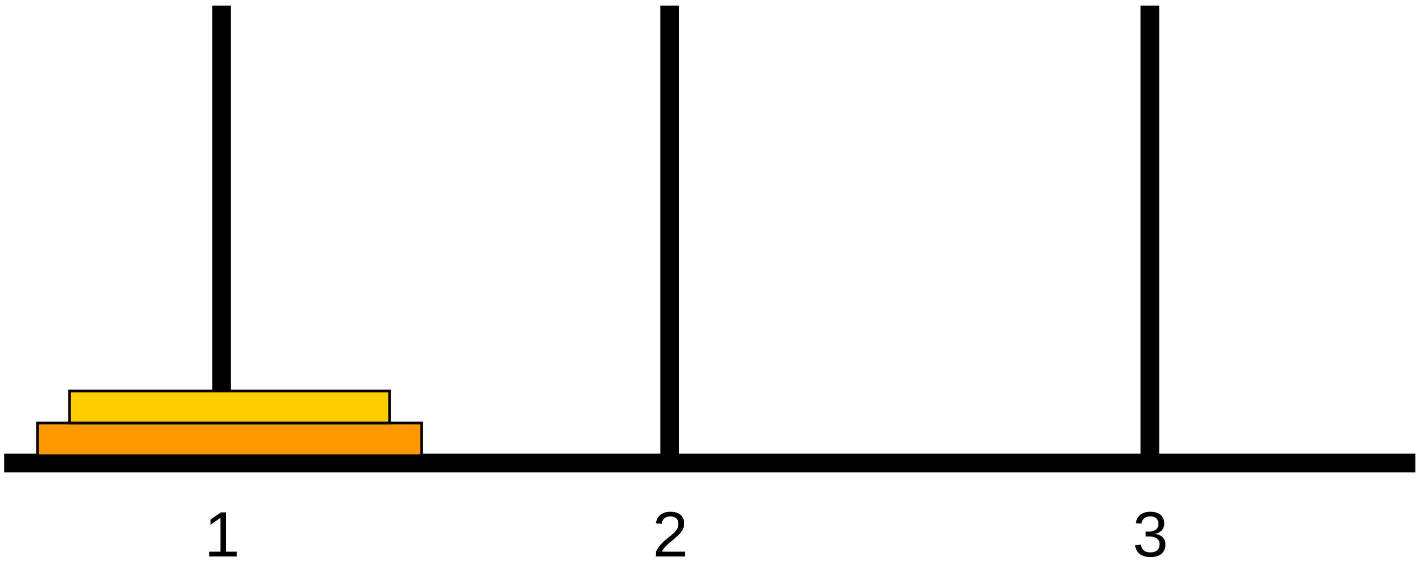
A single disk tower



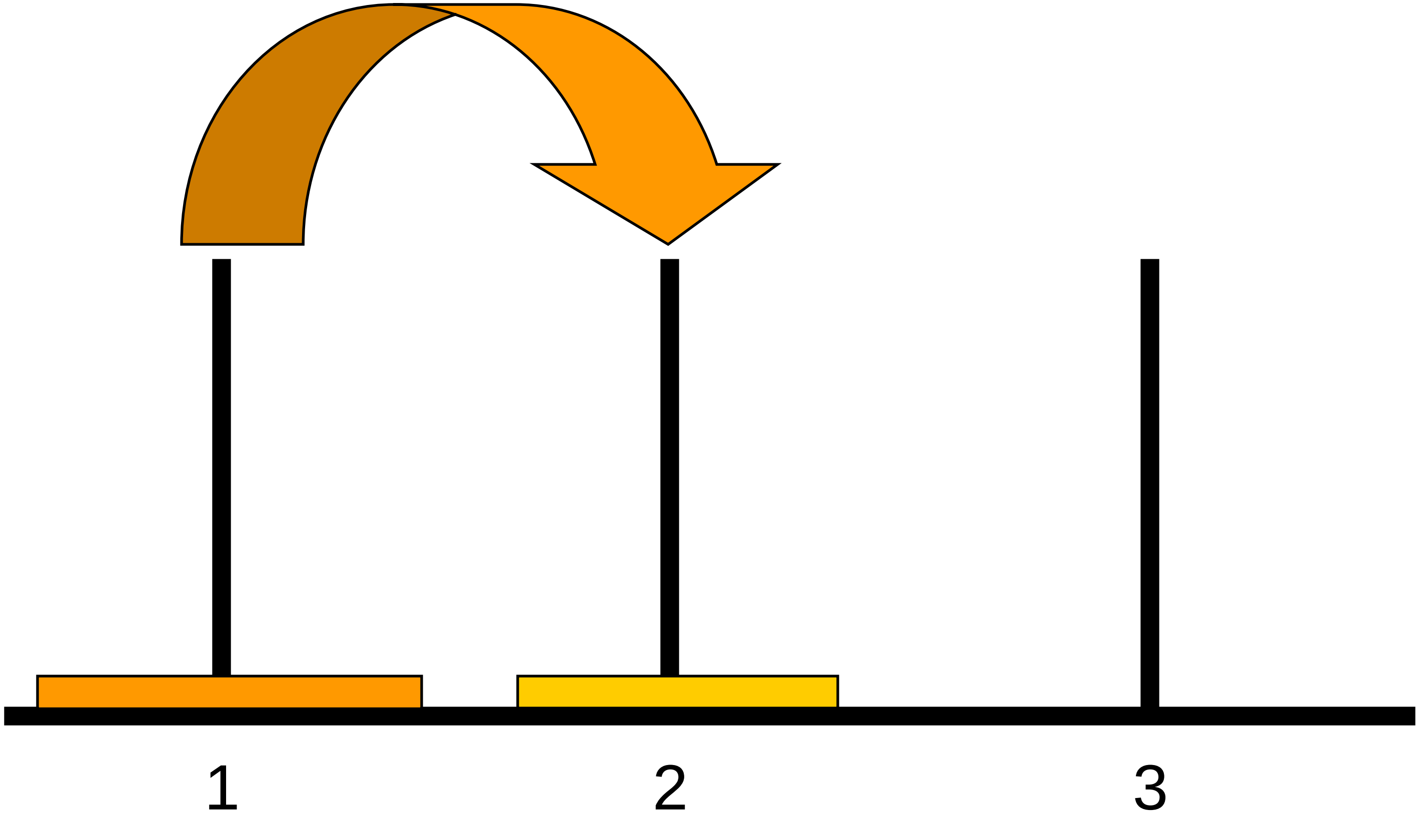
A single disk tower



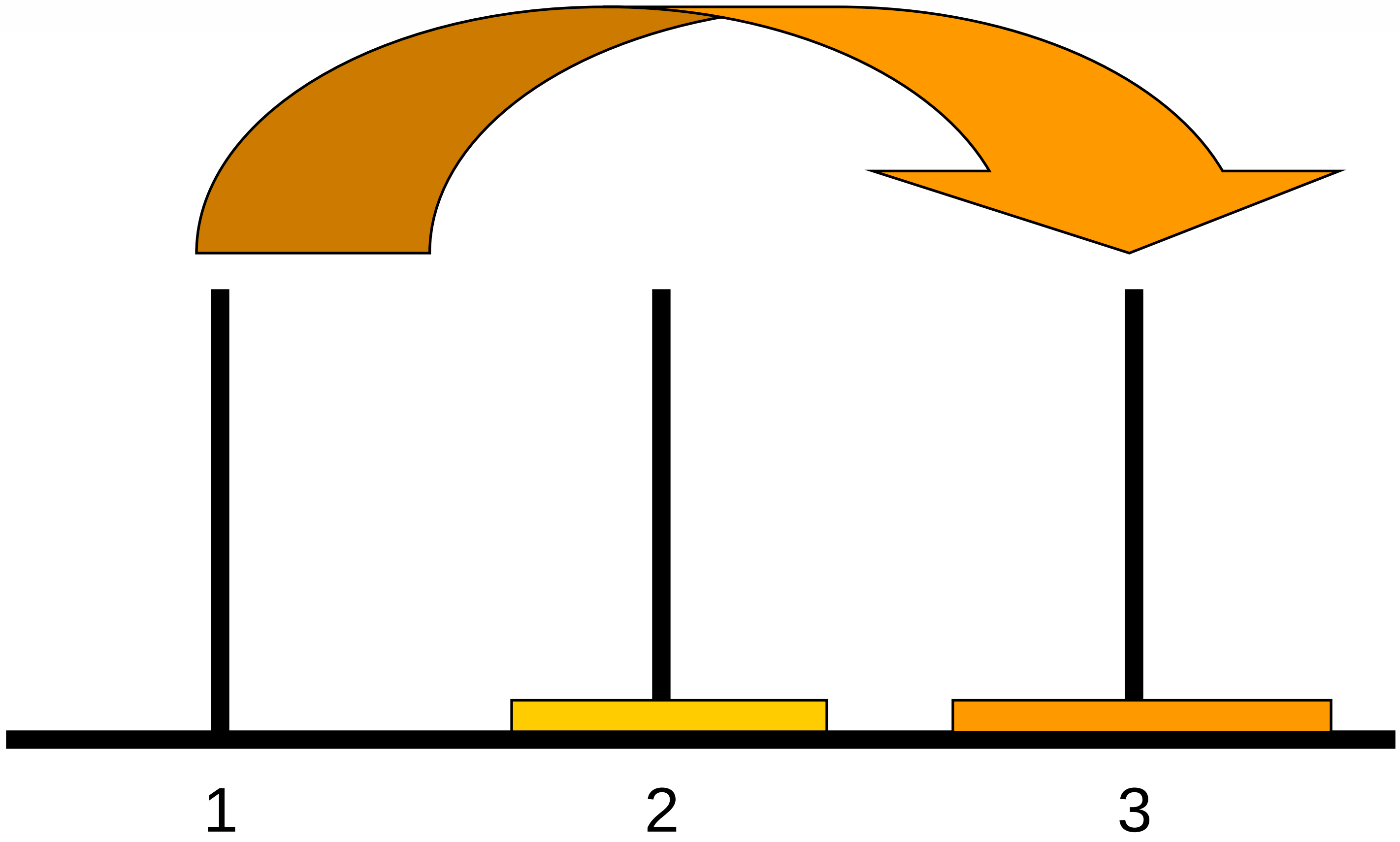
A two disk tower



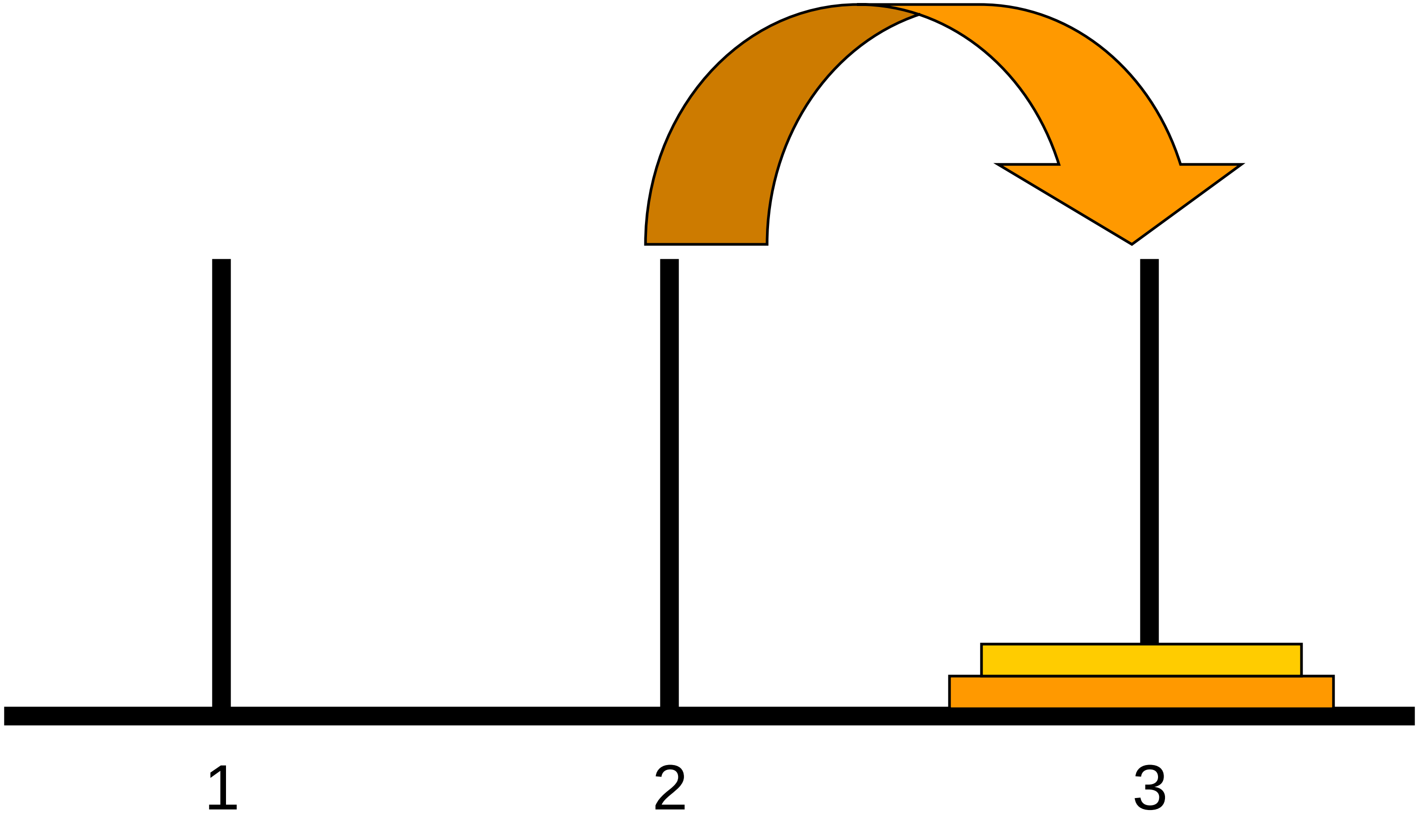
Move 1



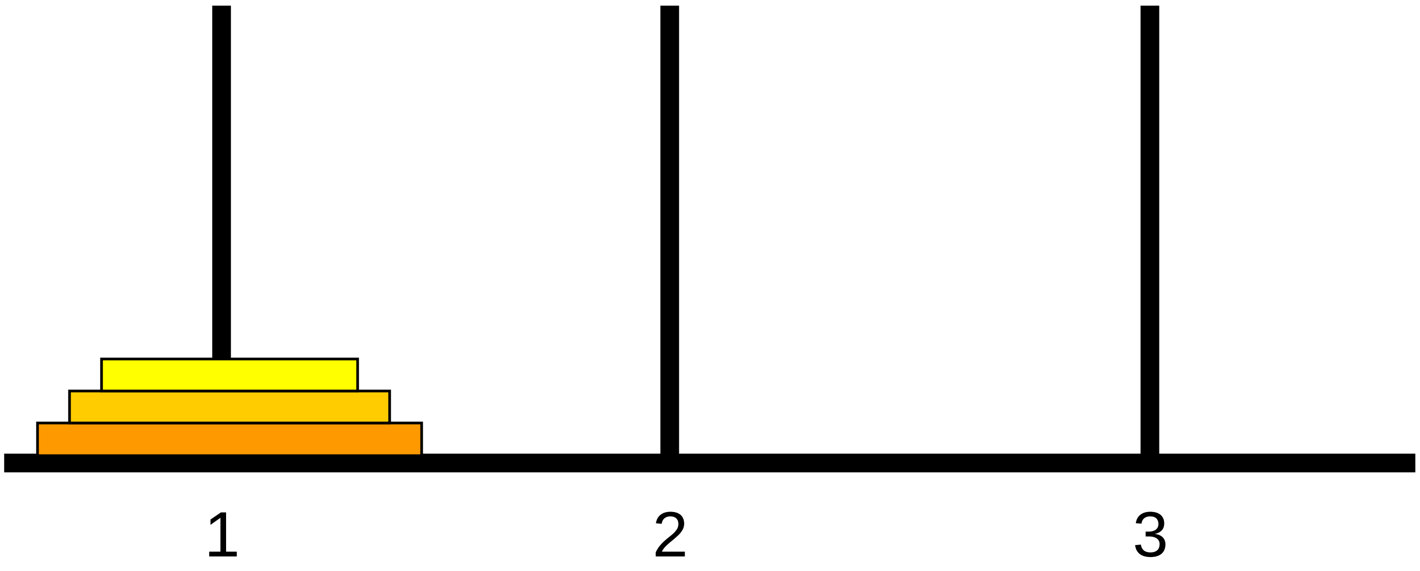
Move 2



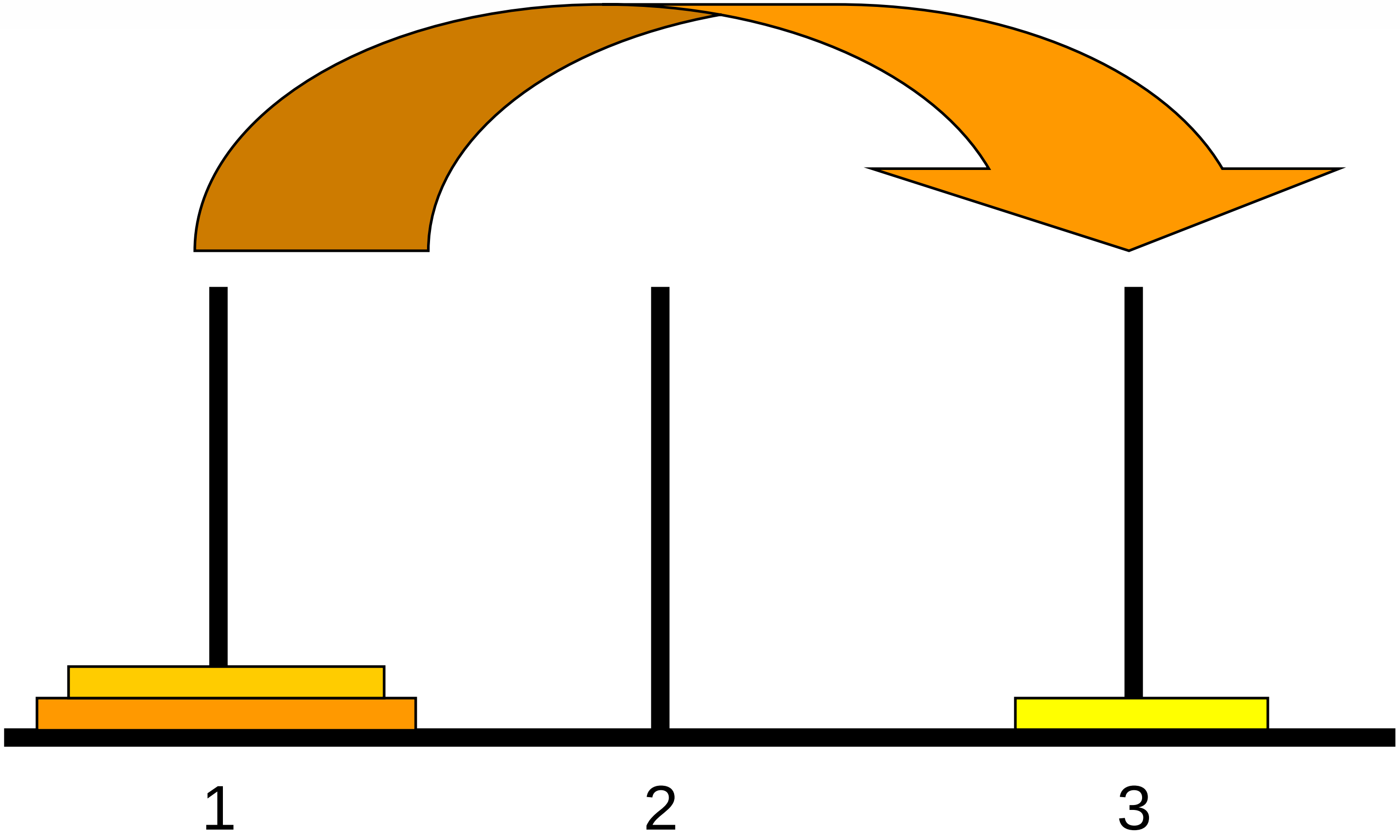
Move 3



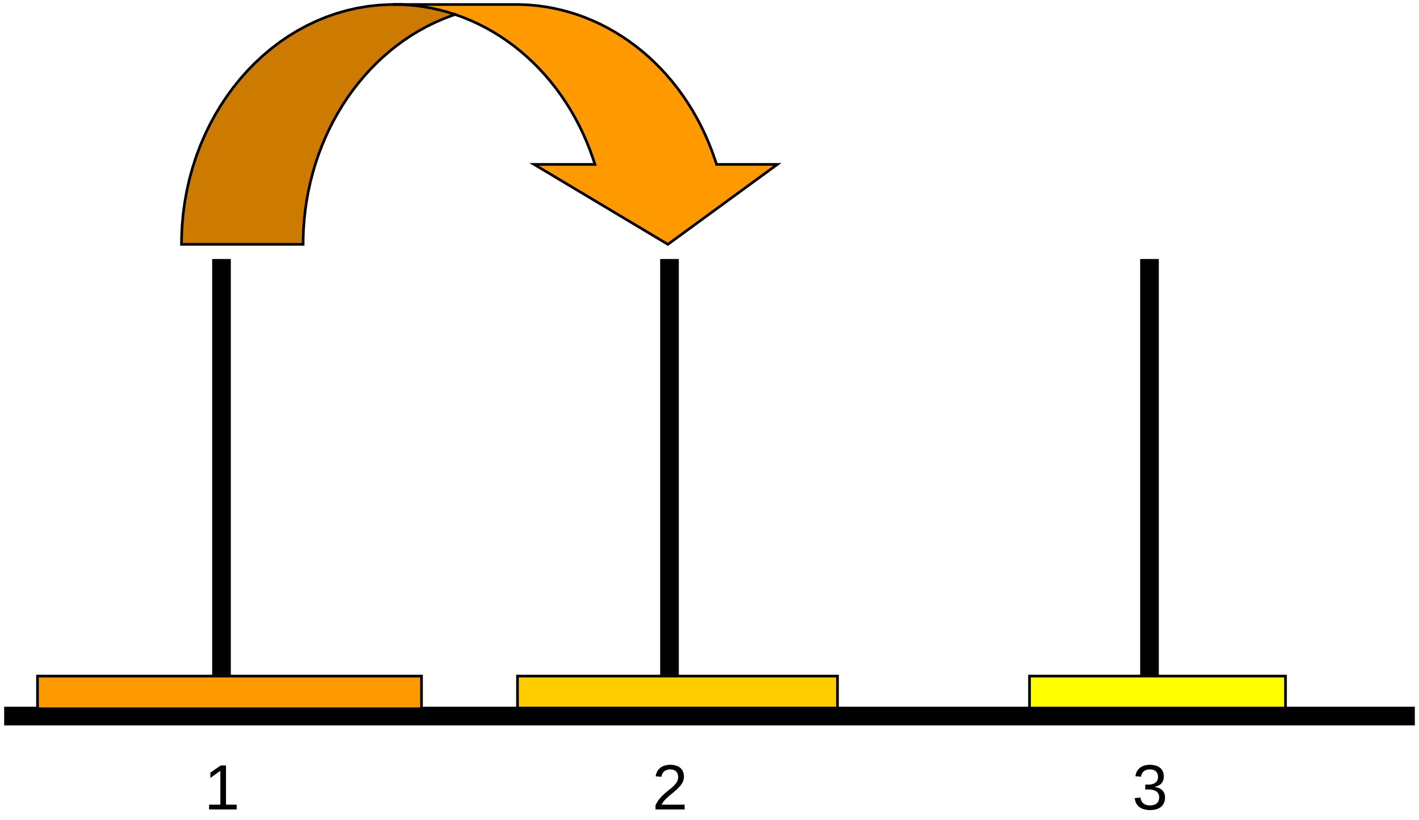
A three disk tower



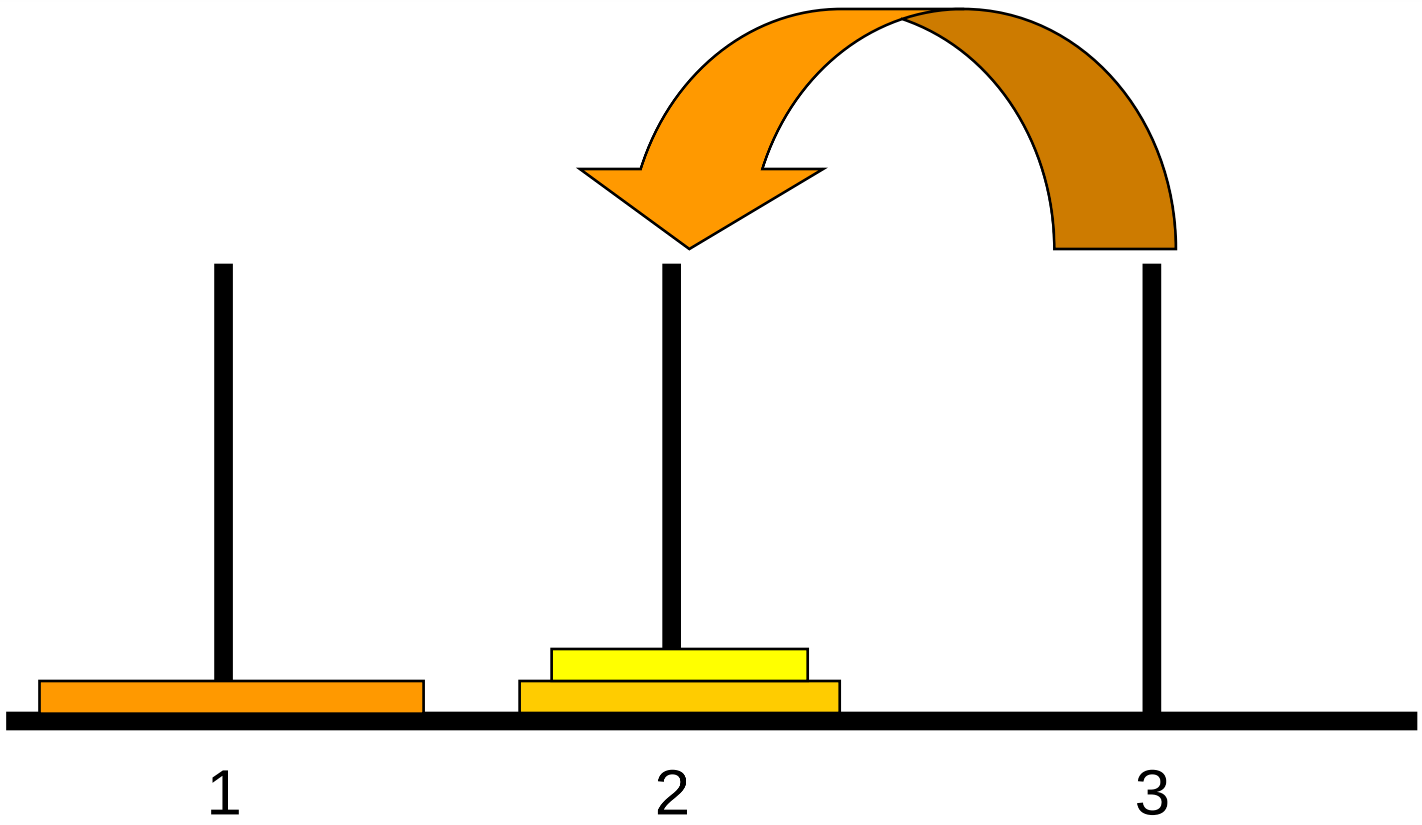
Move 1



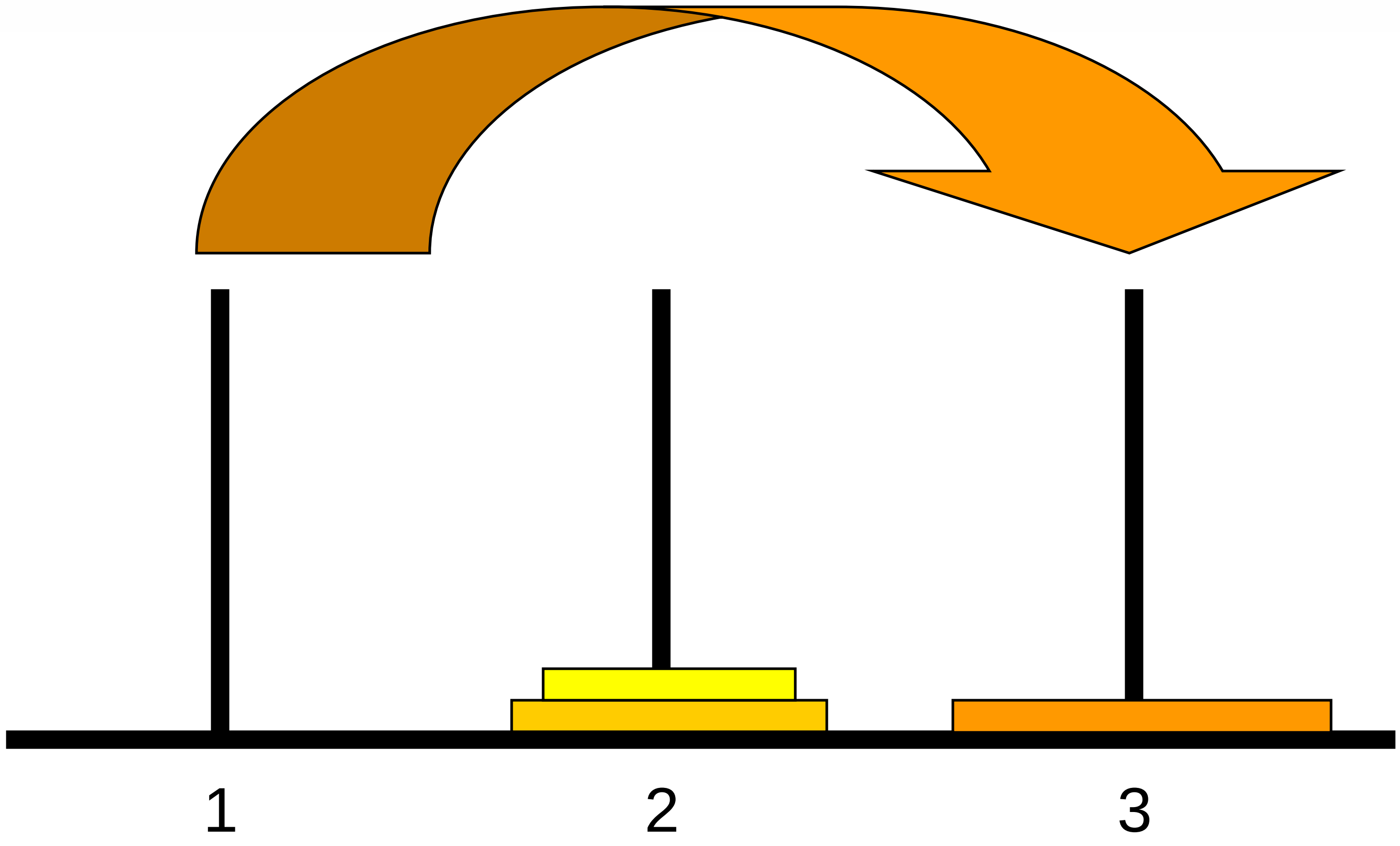
Move 2



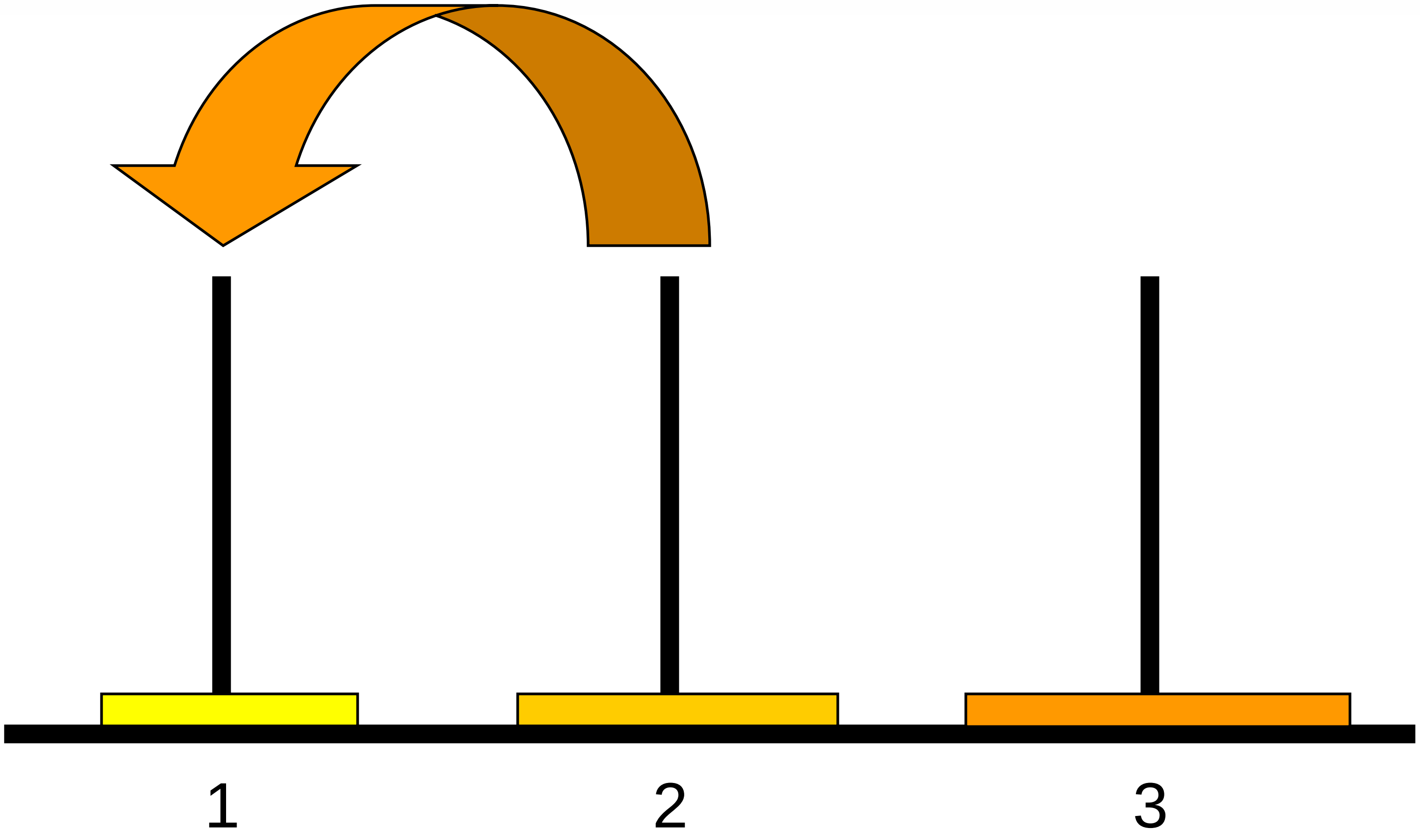
Move 3



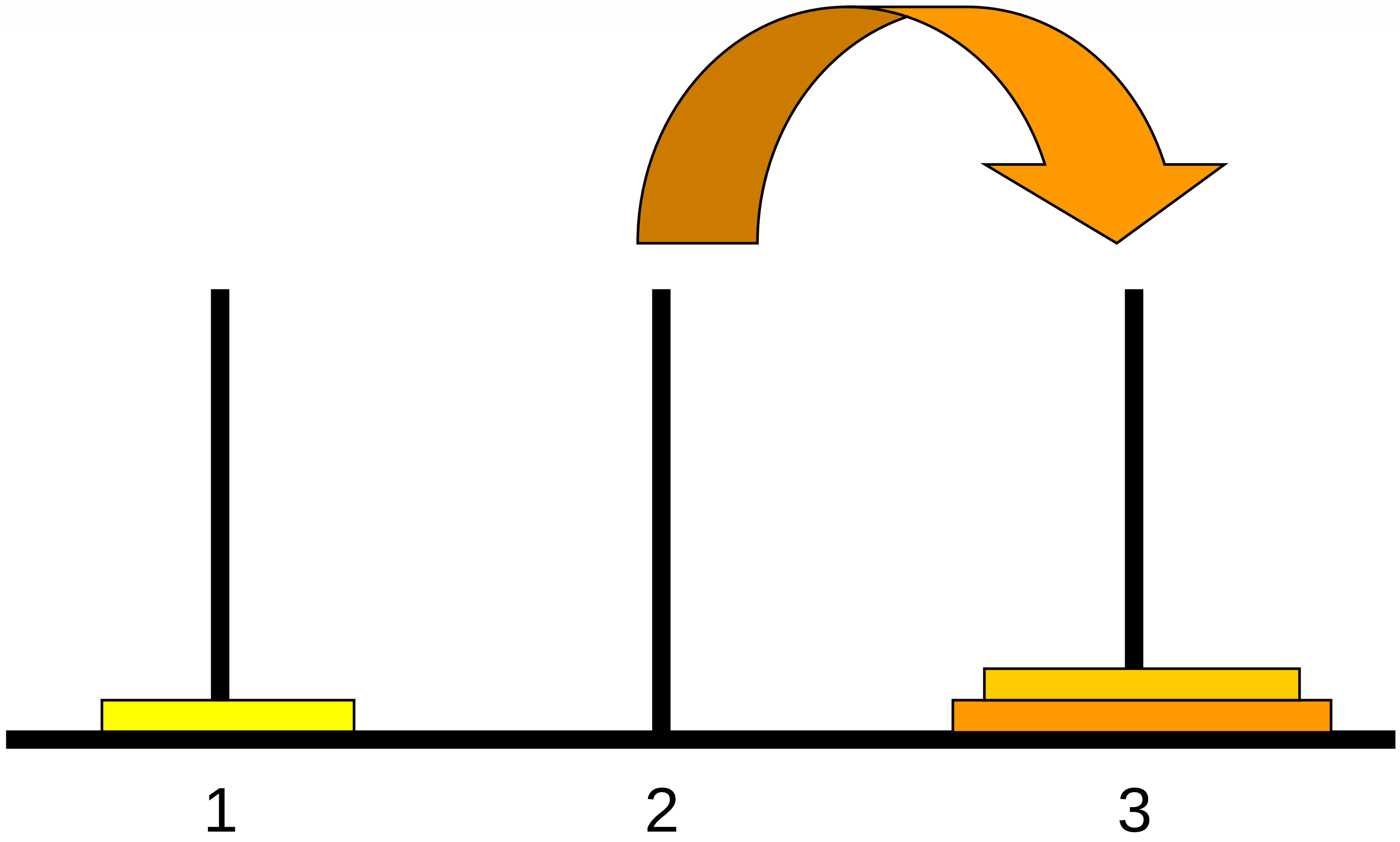
Move 4



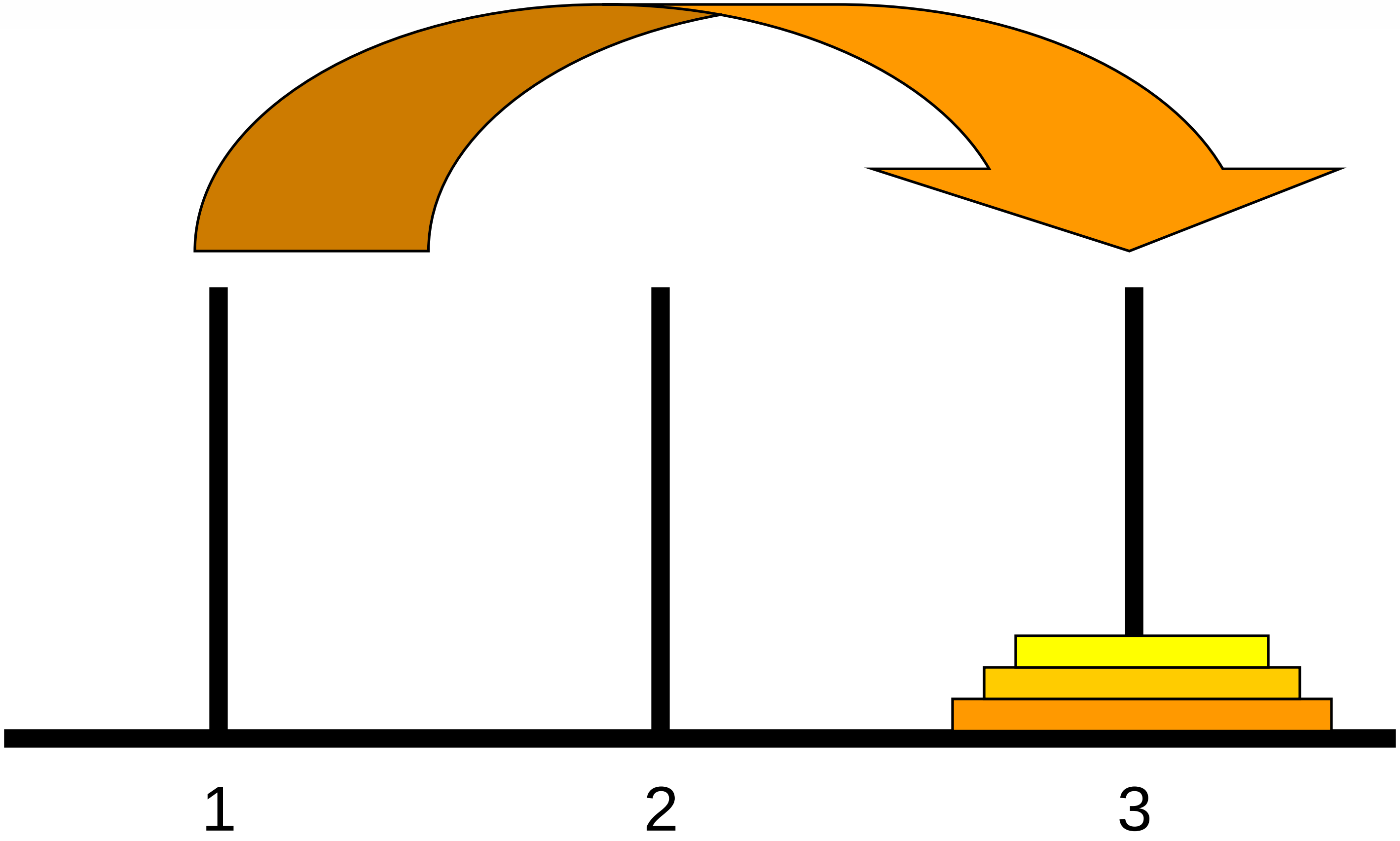
Move 5



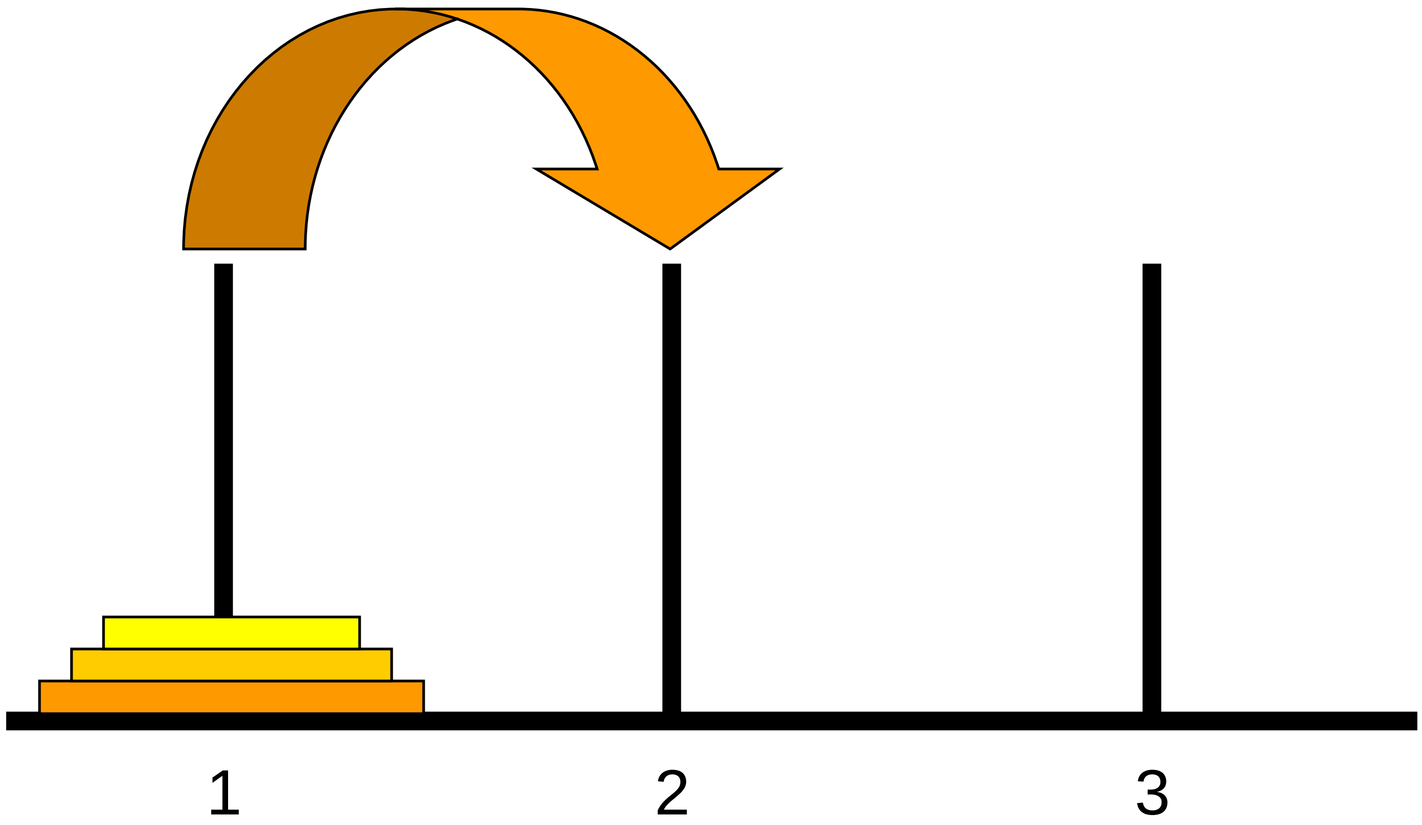
Move 6



Move 7

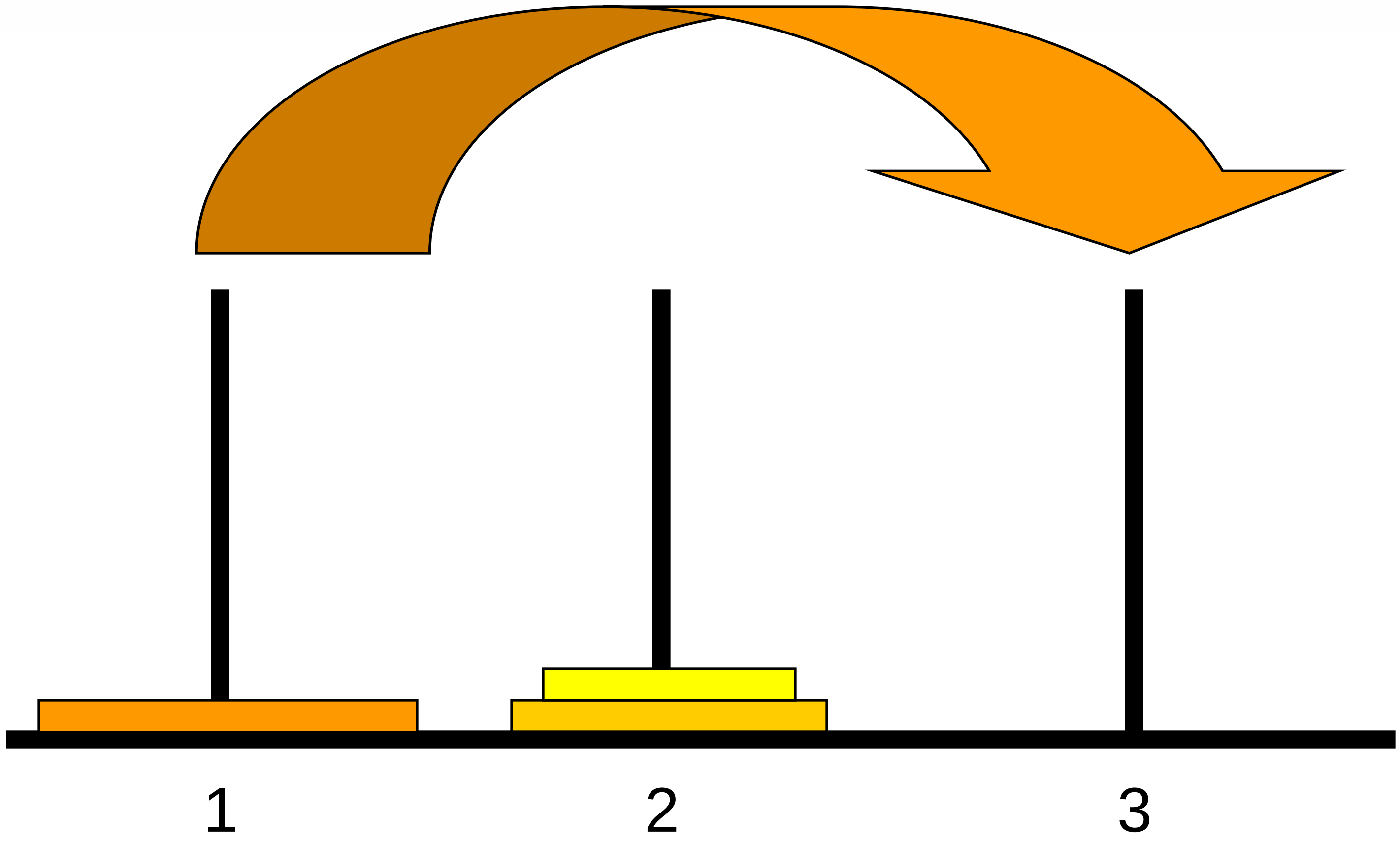


Simplifying the algorithm for 3 disks



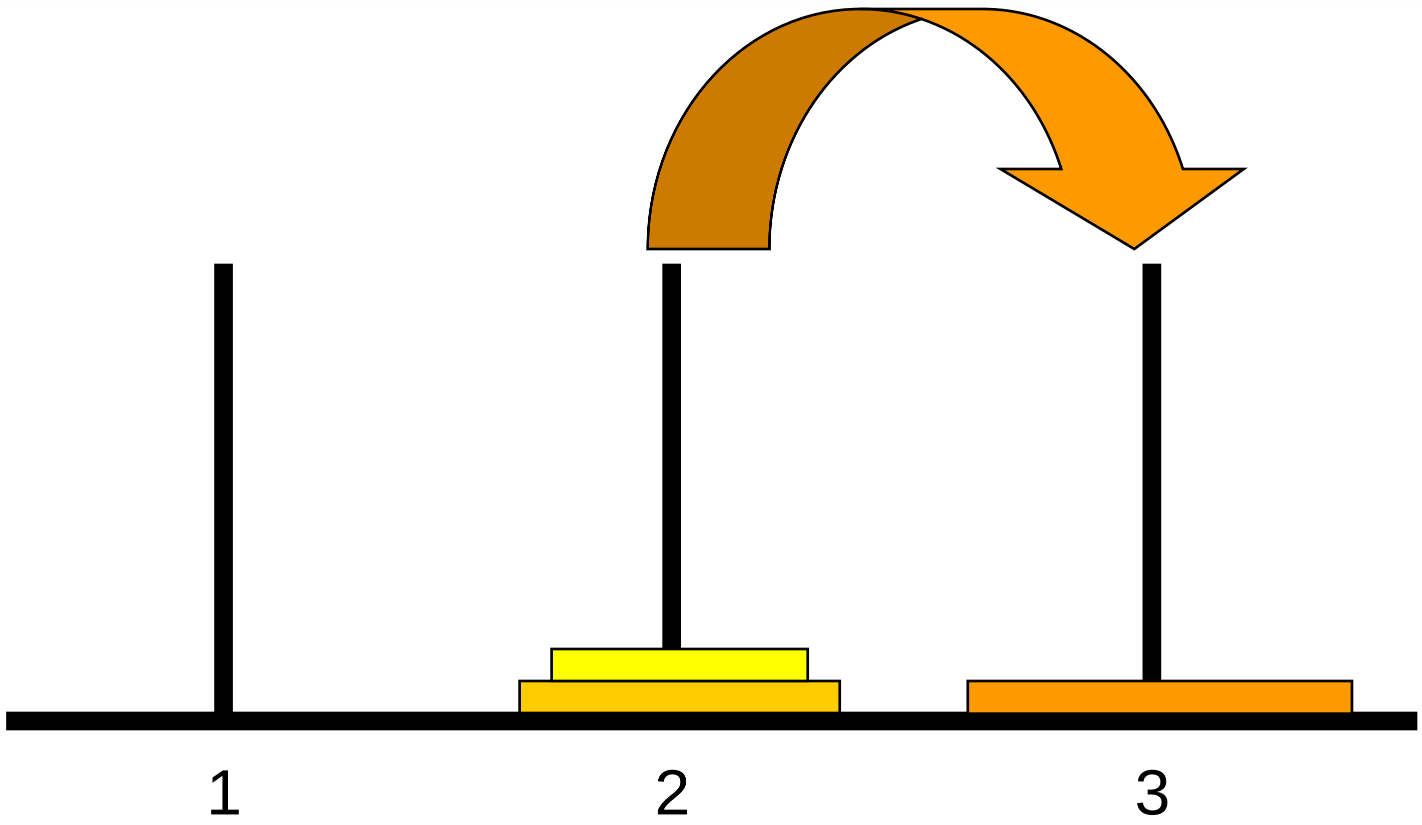
- Step 1. Move the top 2 disks from 1 to 2 using 3 as intermediate

Simplifying the algorithm for 3 disks



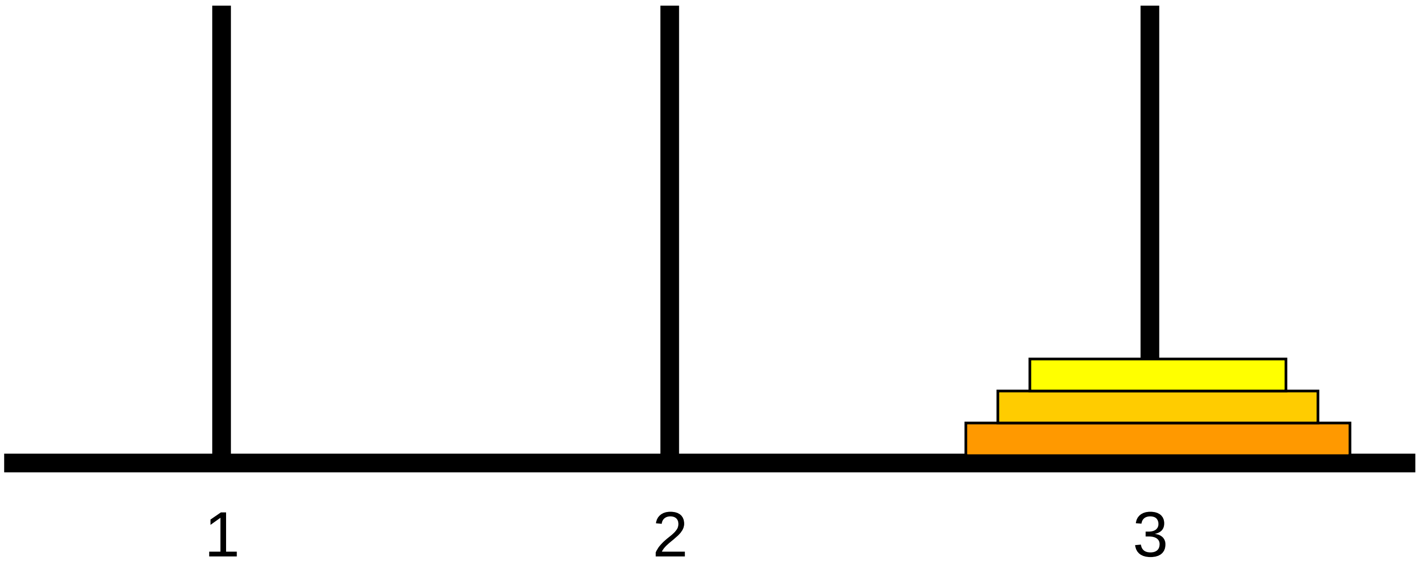
- Step 2. Move the remaining disk from 1 to 3

Simplifying the algorithm for 3 disks



- Step 3. Move 2 disks from 2 to 3 using 1 as intermediate

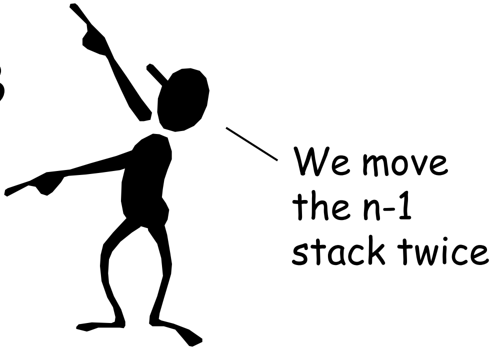
Simplifying the algorithm for 3 disks



The problem for N disks becomes

- A base case of a one-disk move.
- A recursive step for moving n-1 disks.

- To move n disks from Peg 1 to Peg 3, we need to
  - Move (n-1) disks from Peg 1 to Peg 2
  - Move the n^{th} disk from Peg 1 to Peg 3
  - Move (n-1) disks from Peg 2 to Peg 3
  - The number of disk moves is



$$T(1) = 1$$

$$T(n) = 2T(n-1) + 1 = 2^n - 1$$

Exponential algorithm

Towers of Hanoi

- If you play HanoiTowers with . . . it takes . . .
  - 1 disk … 1 move
  - 2 disks … 3 moves
  - 3 disks … 7 moves
  - 4 disks … 15 moves
  - 5 disks … 31 moves
  - .
  - .
  - .
  - 20 disks . . . 1,048,575 moves
  - 32 disks . . . 4,294,967,295 moves

Sorting

- A very common problem is to arrange data into either ascending or descending order
  - **Viewing, printing**
  - **Faster to search, find min/max, compute median/mode, etc.**
- Lots of sorting algorithms
  - **From the simple to very complex**
  - **Some optimized for certain situations (lots of duplicates, almost sorted, etc.)**

Selection Sort

Find the smallest element and swap it with the first:

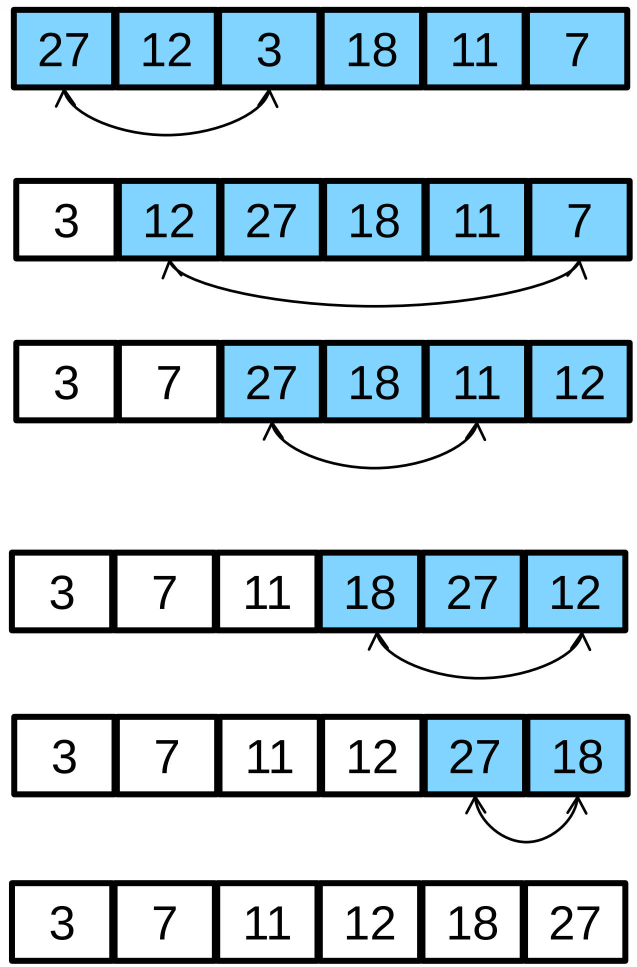
Find the next smallest element and swap it with the second:

Do the same for the third element:

And the fourth:

Finally, the fifth:

Completely sorted:



"In-place" sort

Selection sort

```
def selectionSortRecursive(a,first,last):
    if (first < last):
        index = indexOfMin(a,first,last)
        temp = a[index]
        a[index] = a[first]
        a[first] = temp
        a = selectionSortRecursive(a,first+1,last)
    return a
```

$(n - 1)$ swaps

Quadratic in time

$$\frac{n(n-1)}{2} - 1$$ comparisons

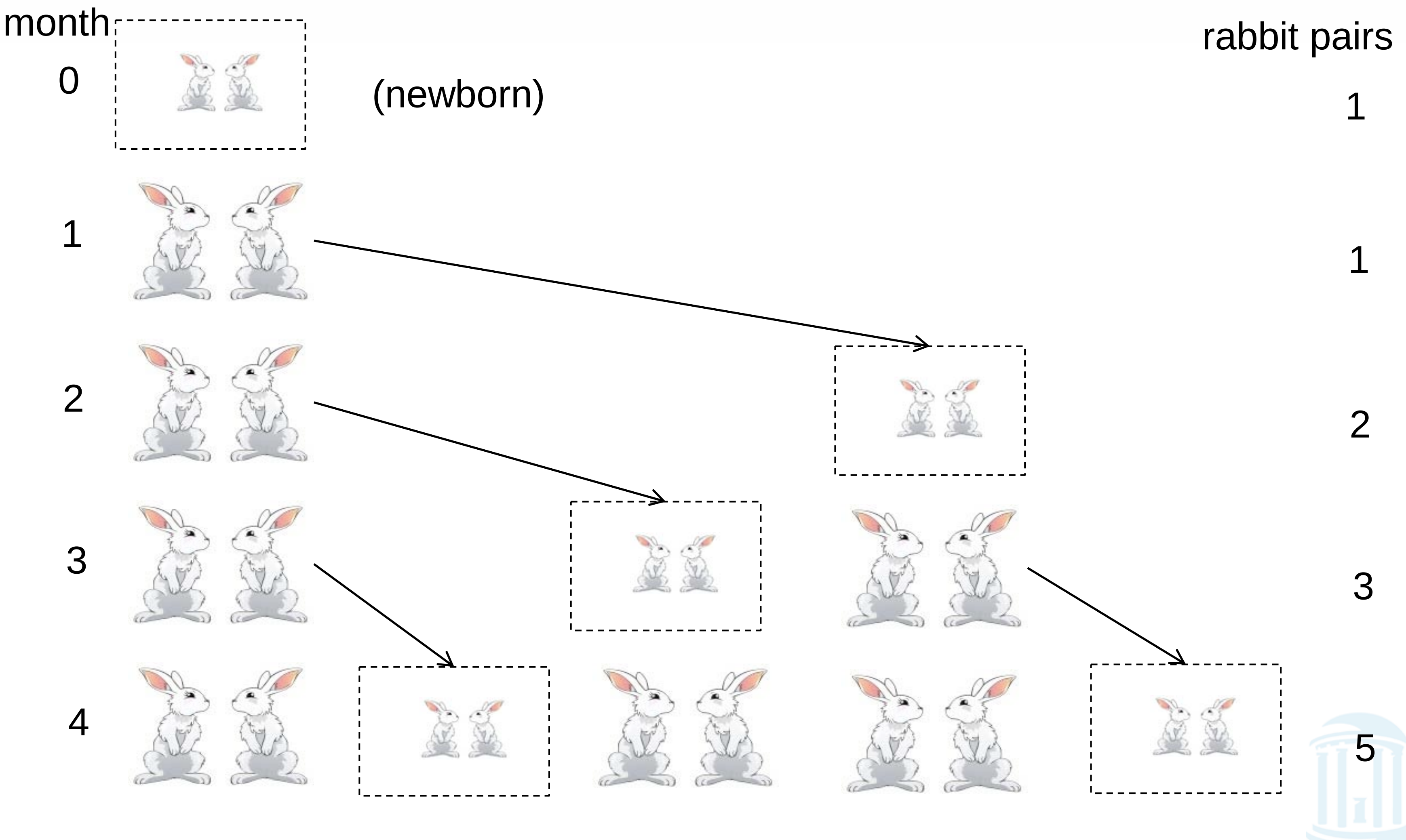
```
def indexOfMin(arr,first,last):
    index = first
    for k in xrange(index+1,last):
        if (arr[k] < arr[index]):
            index = k
    return index
```

Year 1202: Leonardo Fibonacci

- He asked the following question:
  - How many pairs of rabbits are produced from a single pair in *n* months if every month each pair of rabbits more than 1 month old produces a new pair?

  - Here we assume that each pair born has one male and one female and breeds indefinitely
  - The initial pair at month 0 are newborns
  - Let $f(n)$ be the number of rabbit pairs present at the beginning of month n

Fibonacci Number



Fibonacci Number

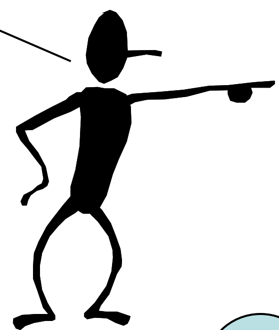
- Clearly, we have:
  - $f(0) = 1$ (the original pair, as newborns)
  - $f(1) = 1$ (still the original pair because newborns need to mature a month before they reproduce)
  - $f(n) = f(n-1) + f(n-2)$ in month n we have
    - the $f(n-1)$ rabbit pairs present in the previous month, and
    - newborns from the $f(n-2)$ rabbit pairs present 2 months earlier
  - f: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …
  - The solution for this recurrence is ($n > 0$):

$$f(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

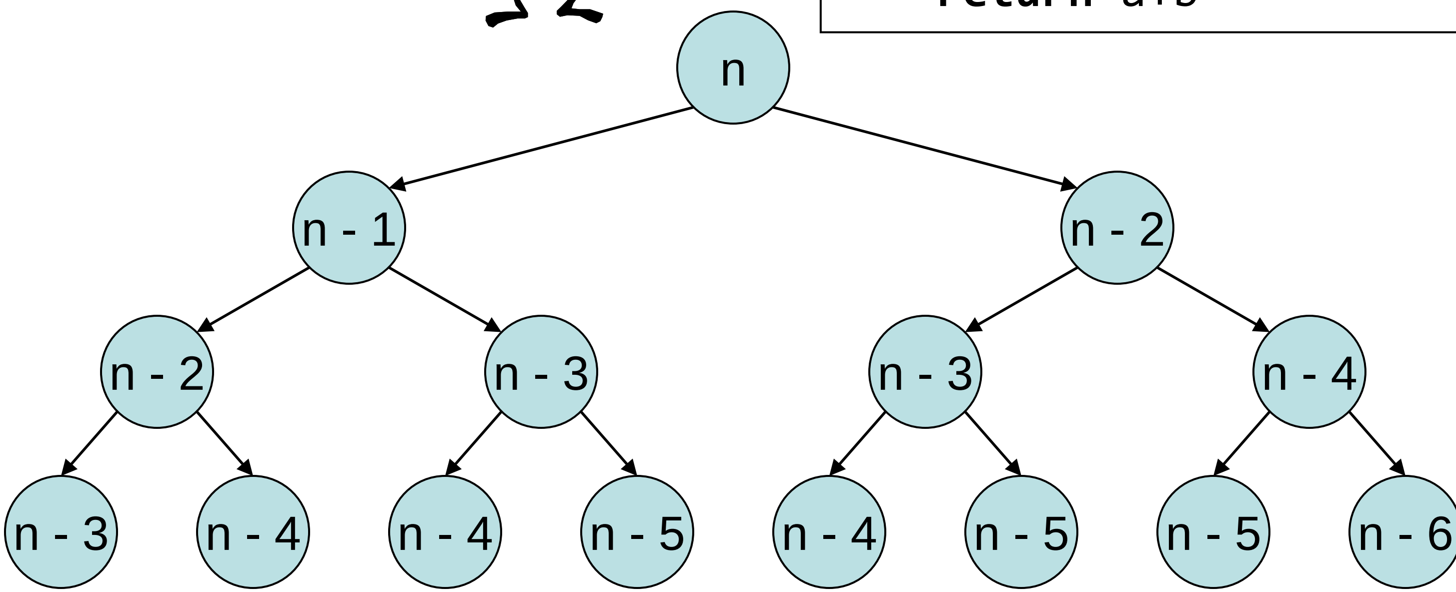
Fibonacci Number

Recursive Algorithm

Exponential time!



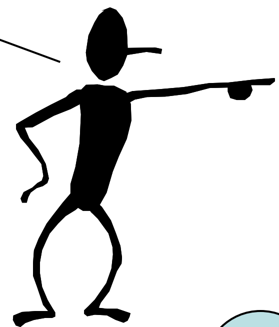
```
def fibonacciRecursive(n):
    if (n <= 1):
        return 1
    else:
        a = fibonacciRecursive(n-1)
        b = fibonacciRecursive(n-2)
    return a+b
```



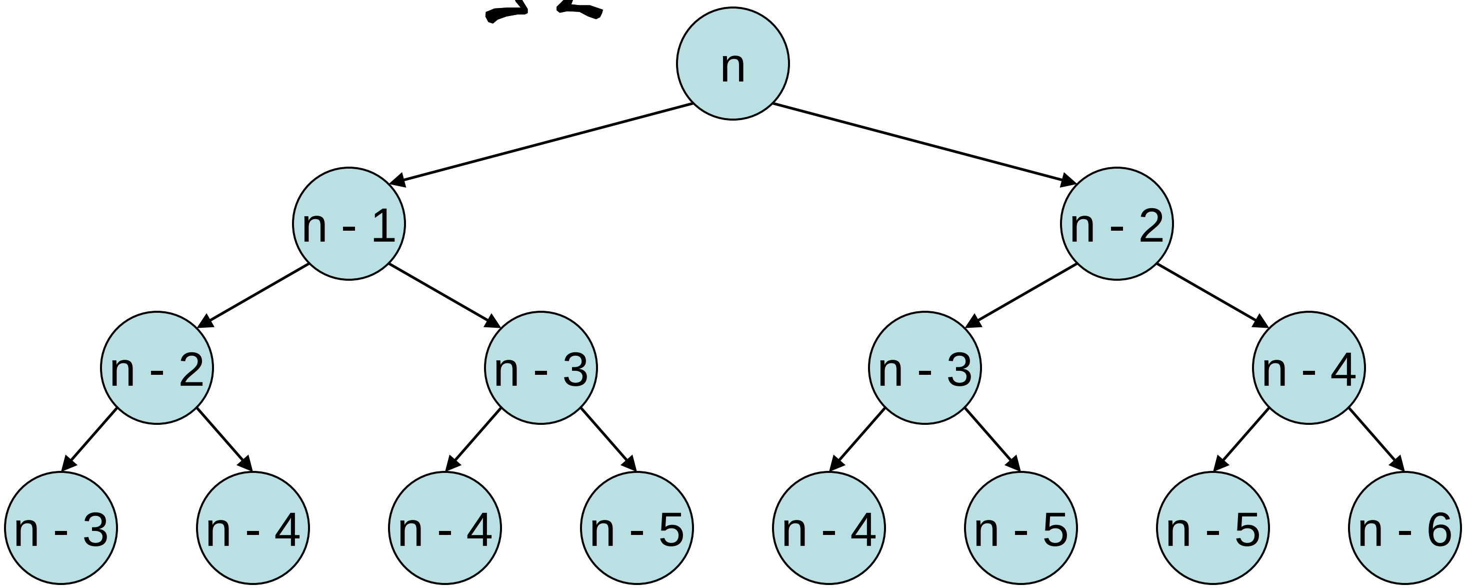
Fibonacci Number

Linear time!

Iterative Algorithm



```
def fibonacciIterative(n):
    f0, f1 = 1, 1
    for i in xrange(0,n):
        f0, f1 = f1, f0 + f1
    return f0
```



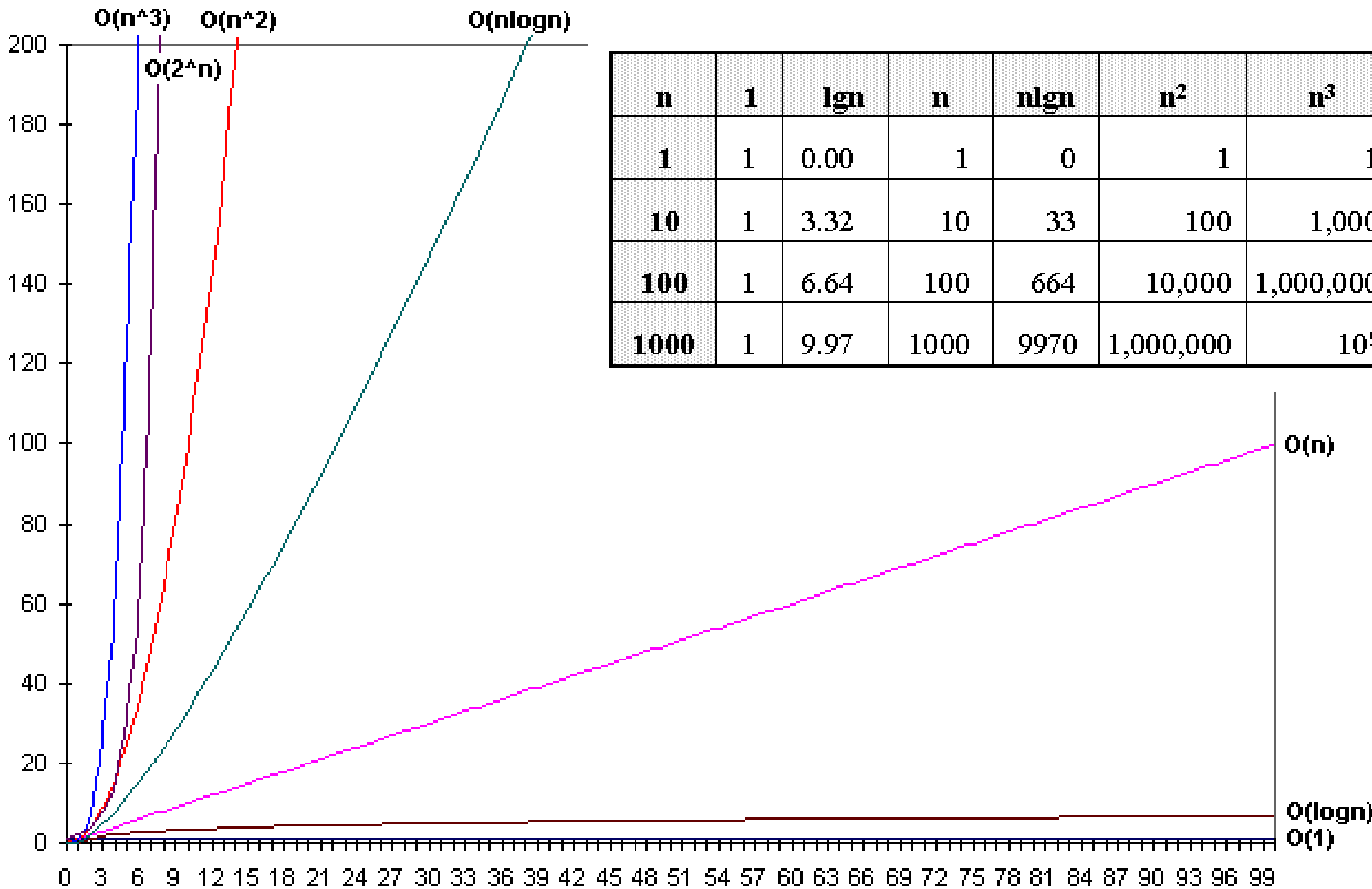
Orders of magnitude

- 10^1
- 10^2 Number of students in computer science department
- 10^3 Number of students in the college of art and science
- 10^4 Number of students enrolled at UNC
- …
- …
- 10^10 Number of stars in the galaxy
- 10^20 Total number of all stars in the universe
- 10^80 Total number of particles in the universe
- 10^100 << Number of moves needed for 400 disks in the Towers of Hanoi puzzle

- Towers of Hanoi puzzle is *computable* but it is NOT feasible.

Is there a "real" difference?

- Growth of functions



n	1	lgn	n	nlgn	n^2	n^3	2^n
1	1	0.00	1	0	1	1	2
10	1	3.32	10	33	100	1,000	1024
100	1	6.64	100	664	10,000	1,000,000	1.2×10^{30}
1000	1	9.97	1000	9970	1,000,000	10^9	1.1×10^{301}

Asymptotic Notation

- *Order of growth* is the interesting measure:
  - Highest-order term is what counts
    - As the input size grows larger it is the high order term that dominates
- Θ notation: $\Theta(n^2)$ = "this function grows similarly to n^2".
- Big-O notation: $O(n^2)$ = "this function grows no faster than n^2".
  - Describes an *upper bound.*

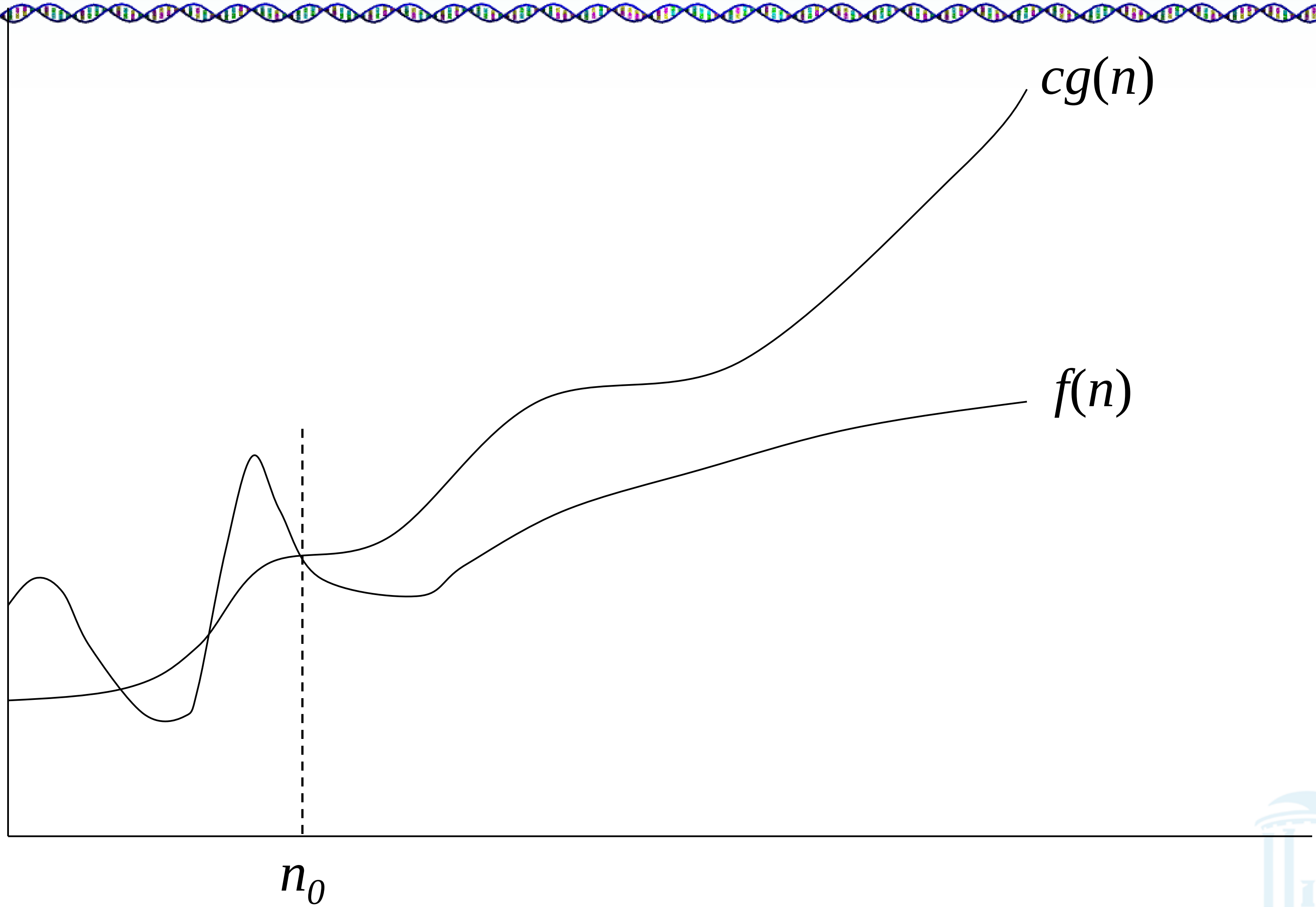
Big-O Notation

$f(n) = O(g(n))$: there exist positive constants c and n_0 such that

$$0 \le f(n) \le cg(n) \text{ for all } n \ge n_0$$

- What does it mean?
  - If $f(n) = O(n^2)$, then:
    - $f(n)$ can be larger than n^2 sometimes, **but…**
    - We can choose some constant c and some value n_0 such that for **every** value of n larger than n_0 : $f(n) < cn^2$
    - That is, for values larger than n_0, $f(n)$ is never more than a constant multiplier greater than n^2
    - Or, in other words, $f(n)$ does not grow more than a constant factor faster than n^2.

Visualization of $O(g(n))$



Big-O Notation

$$2n^2 = O(n^2)$$

$$1{,}000{,}000n^2 + 150{,}000 = O(n^2)$$

$$n^2 + 1{,}000{,}000n + 20 = O(n^2)$$

$$3n + 4 = O(n^2)$$

$$2n^3 + 2 \neq O(n^2)$$

$$n^{2.1} \neq O(n^2)$$

Big-O Notation

- Prove that: $20n^2 + 2n + 5 = O(n^2)$
- Let $c = 21$ and $n_0 = 4$
- $21n^2 > 20n^2 + 2n + 5$ for all $n > 4$

  $n^2 > 2n + 5$ for all $n > 4$

  TRUE

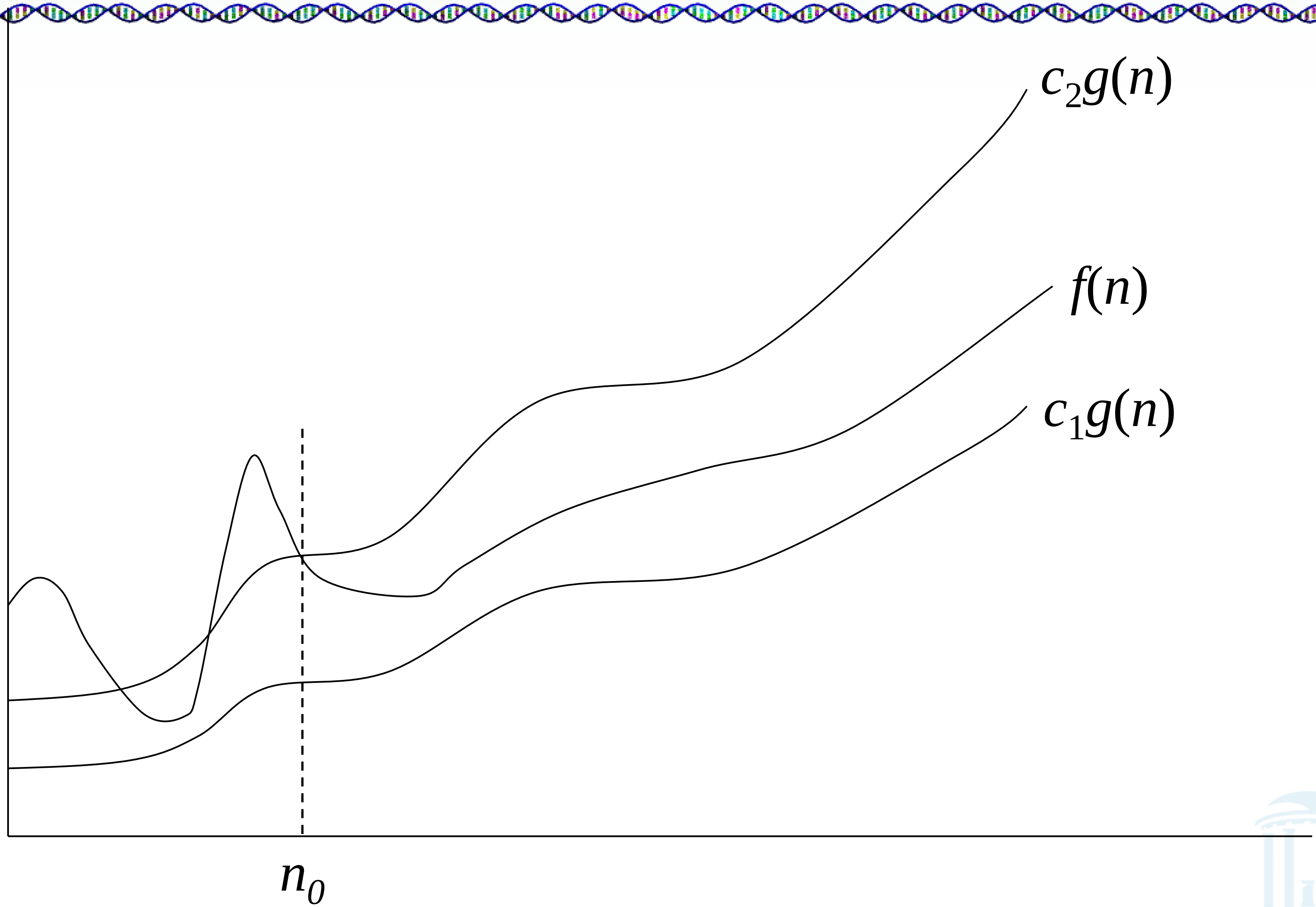
Θ-Notation

- Big-O is not a tight upper bound. In other words $n = O(n^2)$
- Θ provides a tight bound

$$f(n) = \Theta(g(n)) : \text{there exist positive constants } c_1, c_2, \text{and } n_0 \text{ such that}$$
$$0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0$$

- $n = O(n^2) \neq \Theta(n^2)$
- $200n^2 = O(n^2) = \Theta(n^2)$
- $n^{2.5} \neq O(n^2) \neq \Theta(n^2)$

Visualization of $\Theta(g(n))$



Some Other Asymptotic Functions

- Little *o* – A **non-tight** asymptotic upper bound
  - $n = o(n^2)$, $n = O(n^2)$
  - $3n^2 \neq o(n^2)$, $3n^2 = O(n^2)$

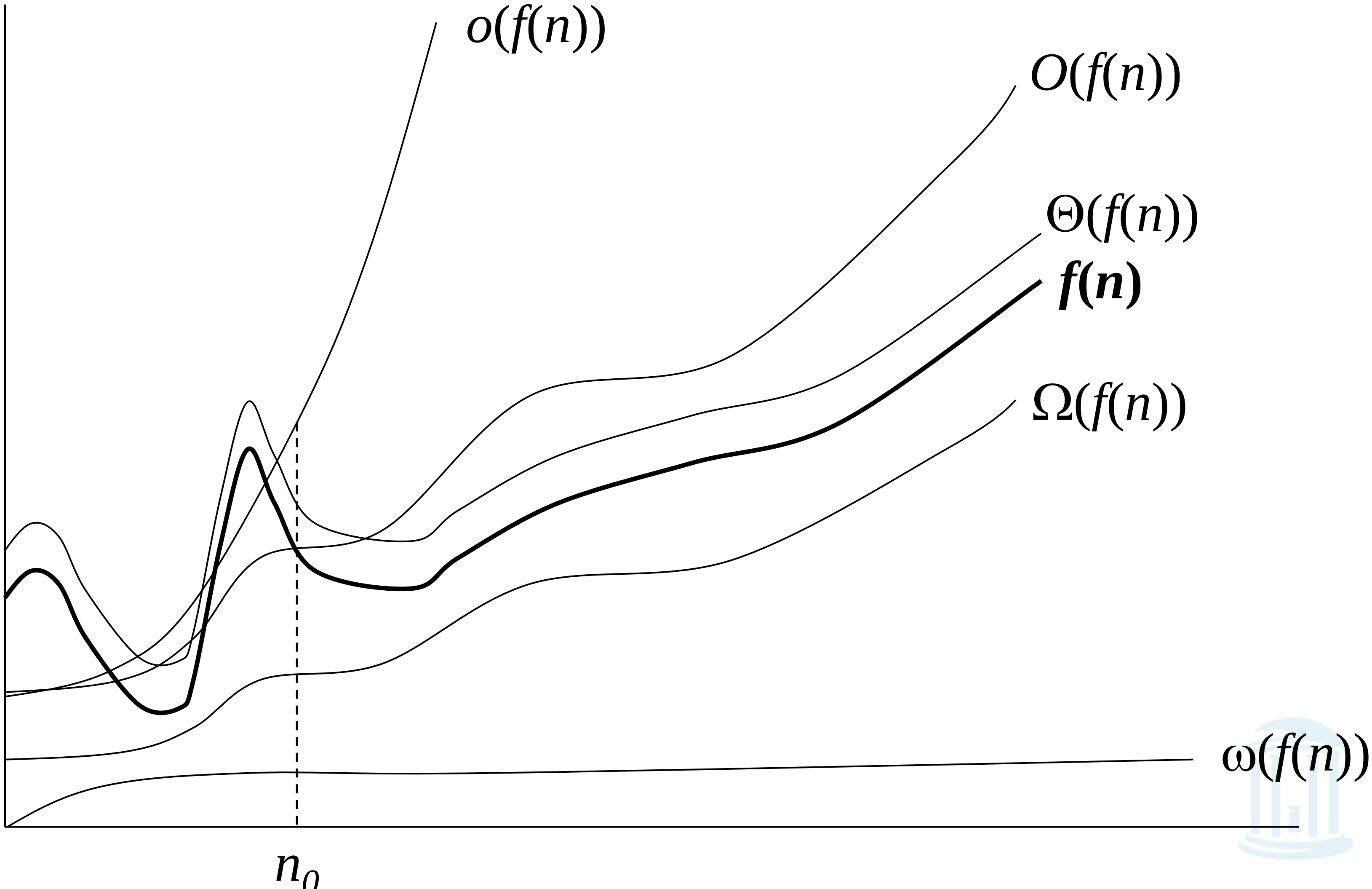
The difference between "big-O" and "little-o" is subtle. For f(n) = O(g(n)) the bound $0 \leq f(n) \leq c\, g(n)$, $n > n_0$ holds for *any* c. For f(n) = o(g(n)) the bound $0 \leq f(n) < c\, g(n)$, $n > n_0$ holds for *all* c.

- Ω – A **lower** bound

$$f(n) = \Omega(g(n)): \text{there exist positive constants } c \text{ and } n_0 \text{ such that } f(n) \geq cg(n) \text{ for all } n \geq n_0$$

  - $n^2 = \Omega(n)$
- ω – A **non-tight** asymptotic lower bound
- $f(n) = \Theta(n) \Leftrightarrow f(n) = O(n)$ **and** $f(n) = \Omega(n)$

Visualization of Asymptotic Growth



Analogy to Arithmetic Operators

$$f(n) = O(g(n)) \quad \approx \quad f \le g$$

$$f(n) = \Omega(g(n)) \quad \approx \quad f \ge g$$

$$f(n) = \Theta(g(n)) \quad \approx \quad f = g$$

$$f(n) = o(g(n)) \quad \approx \quad f < g$$

$$f(n) = \omega(g(n)) \quad \approx \quad f > g$$

Measures of complexity

- Best case
  - **Super-fast in some limited situation is not very valuable information**
- Worst case
  - **Good upper-bound on behavior**
  - **Never gets worse than this**
- Average case
  - **Averaged over all possible inputs**
  - **Most useful information about overall performance**
  - **Can be hard to compute precisely**

Complexity

- Space Complexity Sp(n) : how much memory an algorithm needs (as a function of *n*)
- Space complexity Sp(n) is not necessarily the same as the time complexity T(n)
  - $T(n) \geq Sp(n)$

Next Time

- Our first "bio" algorithm
- Read book 4.1 – 4.3