Flexible Tardiness Bounds for Sporadic Real-Time Task Systems on Multiprocessors *

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Abstract

The earliest-deadline-first (EDF) scheduling of a sporadic real-time task system on a multiprocessor may require that the total utilization of the task system, U_{sum}, not exceed $(m+1)/2$ on m processors if every deadline needs to be met. In recent work, we considered the alleviation of this under-utilization for task systems that can tolerate deadline misses by bounded amounts (*i.e.*, bounded tardiness). We showed that if $U_{sum} \leq m$ and tasks are not pinned to processors, then the tardiness of each task is bounded under both preemptive and non-preemptive EDF. The tardiness bounds that we derived are dependent upon the utilizations and execution costs of the constituent tasks, but are independent of U_{sum}. Furthermore, any task may incur maximum tardiness. In this paper, we address the issue of supporting tasks whose tolerance to tardiness is less than that known to be possible under EDF. We propose a new scheduling policy, called EDF-hl, that is a variant of EDF, and show that under EDF-hl, any tardiness, including zero tardiness, can be ensured for a limited number of *privileged* tasks, and that bounded tardiness can be guaranteed to the remaining tasks if their utilizations are restricted. EDF-hl reduces to EDF in the absence of privileged tasks. The tardiness bound that we derive is a function of U_{sum}, in addition to individual task parameters. Hence, tardiness for all tasks can be lowered by lowering U_{sum}. An experimental evaluation of the tardiness bounds that are possible is provided.

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1 Introduction

A real-time system has to meet certain *timing constraints* to be correct. Such timing constraints are typically specified as deadline requirements. Tasks in a real-time system are often recurrent in nature. The sporadic task model is one of the most widely-studied notions of recurrent real-time task execution. In this model, each task is a sequential program that is invoked repeatedly; each such invocation is called a *job*. The i^{th} task is denoted $T_i(e_i, p_i)$, where $p_i > 0$ is the minimum *inter-arrival separation* for its successive jobs (*i.e.*, successive job invocations of T_i must be spaced apart by at least p_i time units), and $e_i \leq p_i$ is its *per-job execution cost*. p_i is also referred to as the *period* of T_i. In the variant of the sporadic model considered here, p_i is also the *relative deadline* of T_i, *i.e.*, each job of T_i must complete execution within p_i time units of its invocation. The quantity e_i/p_i denotes the *utilization* of T_i. This quantity corresponds to the share of a single processor that T_i requires in the long run.

It is highly desirable that jobs be scheduled so that they do not miss their deadlines. However, in a *soft real-time system*, deadline misses can sometimes be tolerated, if the amount by which a deadline is missed is within a specified per-task *tardiness threshold*: If δ is the tardiness threshold of task T_i, then a job of T_i with a deadline at time d should be guaranteed to complete execution by time $d + \delta$. Such a guarantee would ensure that each task receives a processor share close to its utilization.

In work on real-time systems, multiprocessor platforms (SMPs) are of growing importance. This is due to both hardware trends such as the emergence of multicore technologies, and also to the prevalence of computationally-intensive applications for which single-processor designs are not sufficient. Examples of such applications include systems that track people and machines, many computer-vision systems, and signal-processing applications such as synthetic aperture imaging (to name a few). Timing constraints in several of these applications are predominantly soft. Given these observations, designing efficient scheduling algorithms for multiprocessor-based soft real-time systems and extending the analysis of traditional algorithms to soft real-time systems are goals of considerable value and interest.

Sporadic task systems can be scheduled on a multiprocessor using either a *partitioning* or a *global-scheduling* approach. Under partitioning, tasks are statically assigned to processors, and a uniprocessor scheduling algorithm is used on each processor to schedule its assigned tasks. In contrast, under global scheduling, a task may execute on any processor and may migrate across processors. Each approach can be differentiated further based on the scheduling algorithm that is used. For instance, the *earliest-deadline-first* (EDF) [7] or the *rate-monotonic* (RM)* [9] algorithm could be used as the per-processor scheduler under partitioning, or as the system-wide global scheduler.

Pfair scheduling [4], when deployed in a global setting, is currently the only known way of *optimally* scheduling sporadic task systems on a multiprocessor. (The term "optimal" means that such algorithms are capable of scheduling on m processors any task system with total utilization at most m.) However, Pfair algorithms schedule tasks one quantum at a time, and as a result, jobs may be preempted and migrate across processors frequently. Such preemption and migration overheads can lower the amount of useful work that is actually accomplished. On the other hand, no known non-Pfair-based scheduling algorithm is optimal, and in the worst case, every such algorithm requires that total utilization not exceed $(m + 1)/2$ (*i.e.*, the underlying platform is underutilized by roughly 50%), if every deadline is to be met [5, 10, 3, 2].

Prior work has shown that such restrictions on overall utilization can be eliminated for soft real-time systems. In [1], Anderson *et al*. presented a variant of partitioned-EDF that ensures bounded tardiness with no such restrictions, provided per-task utilizations are capped at 1/2. In addition, in a recent paper [8], we derived tardiness bounds for

*Under RM scheduling, priorities for jobs are fixed offline and are inversely proportional to the periods of their tasks: the jobs of a task with a smaller period have higher priority than those of another task with a larger period.

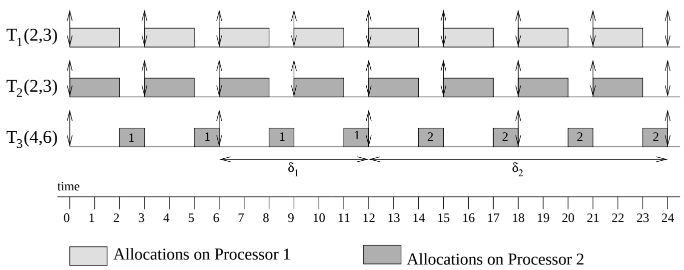


Figure 1: A global-RM schedule for an example task set. Numbers within shaded rectangles indicate job numbers. δ_i indicates the tardiness of the i^{th} job of T_3.

both preemptive and non-preemptive global EDF. For EDF, we established a bound of $((m-1)e_{\max}-e_{\min})/(m-(m-2)u_{\max})+e_{\min}$, where m is the number of processors and $e_{\max}$ (resp., $e_{\min}$) is the maximum (resp., minimum) execution cost of any task in the task system. For non-preemptive EDF, we established a bound of $(me_{\max}-e_{\min})/(m-(m-1)u_{\max})+e_{\min}$. Tardiness bounds under EDF have also been presented by Valente and Lipari [12]. A precursor to all of the work mentioned here is a paper by Srinivasan and Anderson [11] in which tardiness bounds are presented for the earliest-pseudo-deadline first (EPDF) Pfair scheduling algorithm, which is sub-optimal but more efficient than optimal algorithms.

While it may be reasonable to expect that most scheduling algorithms should be capable of guaranteeing bounded tardiness in the absence of overutilization, in reality this is not the case for some well-known algorithms. Partitioning algorithms and global RM are examples. Under partitioned scheduling, if a task set cannot be partitioned without overutilizing some processor, then tardiness for the tasks on that processor will increase with time. An example of a task system with unbounded tardiness under global RM is shown in Fig. 1. Assume here that each job of every task is invoked as early as permissible. Then, the first job of T_3 does not complete executing until time 12, for a tardiness of 6 time units, and the second job suffers from a tardiness of 12 time units. It is easy to see that the i^{th} job of T_3 does not complete until time $12i$, for all i, for a tardiness of $6i$ time units. This tardiness increases with time and thus is unbounded.

Contributions. The tardiness bounds derived by us previously for preemptive and non-preemptive EDF [8] are dependent on per-task utilizations and execution costs but independent of total system utilization, U_{sum}. Furthermore, any task may incur maximum tardiness. This may not be acceptable to applications that are comprised of hard and soft real-time tasks or those comprised of soft tasks with different tardiness tolerances. In this paper, we make an attempt to address this limitation.

Our contributions are twofold. First, we consider guaranteeing lower tardiness to some tasks at the expense of others. To this end, we propose a new scheduling policy, called EDF-hl, which is a variant of global EDF. We show that under EDF-hl, on m processors, up to m tasks can be accorded preferential treatment and thereby guaranteed any tardiness, including zero, and that bounded tardiness can be guaranteed to the remaining tasks if their utilizations are capped. In the absence of tasks that require lower tardiness, EDF-hl reduces to EDF. Simulations involving randomly-generated task sets presented herein suggest that for many systems, the tardiness bounds that can be ensured for tasks that do not receive preferential treatment are acceptable.

Unlike in [8], the tardiness bound derived here is a function of U_{sum}, in addition to individual task parameters. Thus, as a second contribution, our bound offers the possibility of lowering tardiness for all tasks by lowering U_{sum}. To assess the tardiness-utilization trade-off for EDF (*i.e.*, without special tasks), we again conducted experiments involving randomly-generated task sets. We found that, with the improved analysis, considerable reductions in tardiness are possible even for reasonable reductions in total system utilization. For instance, in the simulation results for eight processors shown in Fig. 6(b) in Sec. 4, lowering U_{sum} by around 10% results in a reduction in maximum tardiness by over 35%, and lowering U_{sum} by 25% lowers maximum tardiness by close to 50%.

Organization. The rest of this paper is organized as follows. Our system model and algorithm EDF-hl are described in Sec. 2. Tardiness bounds are derived in Sec. 3. An experimental evaluation of the tardiness-total utilization trade-off is provided in Sec. 4. Finally, Sec. 5 concludes.

2 Definitions

In this section, our task model is described and algorithm EDF-hl is presented.

Task model. A sporadic task system comprised of $n \geq 1$ sporadic tasks is to be scheduled on $m \geq 2$ processors. Each sporadic task $T_i(e_i, p_i)$ is as described in the introduction. The *utilization* of T_i is given by $u_i = e_i/p_i$. $u_i \leq 1$ holds for all i. The *total utilization* of τ is defined as $U_{sum}(\tau) = \sum_{i=1}^{n} u_i$. It is required that $U_{sum}(\tau) \leq m$ hold. The maximum utilization (resp., execution cost) of any task in τ is denoted $u_{\max}(\tau)$ (resp., $e_{\max}(\tau)$). The minimum execution cost of any task is denoted $e_{\min}(\tau)$. $\mathcal{U}_{\max}(\tau, k)$, where $k \leq n$, denotes the k tasks with highest utilizations in τ. More formally, $\mathcal{U}_{\max}(\tau, k)$ denotes a subset of k tasks of τ, where the utilization of each task in the subset is at least as high as that of every task in $\tau \setminus \mathcal{U}_{\max}(\tau, k)$. $\mathcal{E}_{\max}(\tau, k)$ is defined analogously with respect to execution costs. (In all of these $\max$ and $\min$ terms, τ will be omitted when it is unambiguous.)

The k^{th} job of T_i, where $k \geq 1$, is denoted $T_{i,k}$, and its *release time* and *absolute deadline* (or simply *deadline* for short) are denoted $r_{i,k}$ and $d_{i,k}(= r_{i,k} + p_i)$, respectively. $r_{i,k}$ denotes the invocation time of $T_{i,k}$ and is the time at or after which $T_{i,k}$ can be executed. $r_{i,k+1} - r_{i,k} \geq p_i$ holds for all $k \geq 1$. Each task is sequential, and hence no job of any task may execute in parallel. Furthermore, no two jobs of any task may execute in parallel.

A sporadic task system τ is said to be *concrete* if the release time of every job of each of its tasks is specified, and *non-concrete*, otherwise. Note that an infinite number of concrete task systems can be specified for every non-concrete task system. We omit specifying the type of the task system unless it is necessary. The results in this paper are for non-concrete task systems, and hence hold for every concrete task system.

The *tardiness of a job* $T_{i,j}$ in a schedule $\mathcal{S}$ is defined as $tardiness(T_{i,j}, \mathcal{S}) = \max(0, t - d_{i,j})$, where t is the time at which $T_{i,j}$ completes executing in $\mathcal{S}$. The *tardiness of a task system* τ under scheduling algorithm $\mathcal{A}$ is defined as the maximum tardiness of any job of a task in τ in any schedule under $\mathcal{A}$. If κ is the maximum tardiness of any task system under $\mathcal{A}$, then $\mathcal{A}$ is said to *ensure a tardiness bound of* κ. Though tasks in a soft real-time system are allowed to have nonzero tardiness, we assume that *missed deadlines do not delay future job releases*. That is, even if a job of a task misses its deadline, the release time of the next job of that task remains unaltered. Since consecutive jobs of the same task cannot be scheduled in parallel, a missed deadline effectively reduces the interval over which the next job should be scheduled in order to meet its deadline.

The sporadic task model is augmented as follows for EDF-hl (described below). Each task in τ is classified as either a *privileged task* or an *unprivileged task*. The set of all privileged (resp., unprivileged) tasks is denoted τ_H (resp., τ_L). (H and L stand for high and low privilege, respectively.) $|\tau_H| \leq m$ holds. Each privileged task T_h has a maximum tardiness parameter $\Delta_h \geq 0$, which denotes the maximum tardiness that any of its jobs can tolerate. $d_{h,j} + \Delta_h$ is referred to as the *effective deadline* of job $T_{h,j}$ and is denoted $\rho_{h,j}$.

Algorithm EDF-hl. Our goal is to design an algorithm that can guarantee a tardiness of Δ_h to each privileged task T_h while guaranteeing bounded tardiness to the remaining tasks. Let the *slack* of a job $T_{h,j}$ of a privileged task T_h at time t be defined as $d_{h,j} + \Delta_h - t - (e_h - \delta_{h,j})$, where $\delta_{h,j}$ denotes the amount of time that $T_{h,j}$ executed before t. Informally, the slack of job $T_{h,j}$ at t is the amount of time it can afford *not* to execute after t until completion for its tardiness to be at most Δ_h. A tardiness of at most Δ_h can be guaranteed for task T_h if each job $T_{h,j}$ is scheduled based on its deadline until time $d_{h,j} + \Delta_h - e_h$, but is guaranteed continuous execution from $d_{h,j} + \Delta_h - e_h$ onward. (This is somewhat similar to the behavior of the earliest-deadline-until-zero-laxity algorithm described in [6].) Job $T_{h,j}$ is said to be *urgent* at time t, if $t \geq d_{h,j} + \Delta_h - e_h$ and $T_{h,j}$ has not completed execution by t. Note that $T_{h,j}$ is flagged as urgent from $d_{h,j} + \Delta_h - e_h$ until completion even if its slack is positive. This eliminates the overhead of updating the urgency for each privileged job at runtime and may also result in fewer preemptions and migrations.

With the above definitions in place, Algorithm EDF-hl can be described as follows. At any time t, each of the urgent jobs, if any, of tasks in τ_H is assigned a unique processor. If not every processor is assigned to an urgent job, then the non-urgent jobs of τ_H and jobs of tasks in τ_L are scheduled on the remaining processors on an earliest-deadline-first basis, where ties are resolved arbitrarily. A job may be preempted at any time by a higher priority job and may later resume execution on a different processor.

Note that EDF-hl reduces to EDF if $\tau_H = \emptyset$. Since $|\tau_H| \leq m$ holds, EDF-hl clearly ensures the required tardiness for each privileged task. Hence, the question to be addressed is whether bounded tardiness can be guaranteed for the remaining tasks. The answer turns out to be yes if there is a cap on the utilizations of the remaining tasks. This cap depends on the number of privileged tasks and their utilizations. To see that such a cap is necessary, at least in some cases, consider a task system comprised of four tasks $T_1(3,4), \ldots, T_3(3,4)$, and $T_4(3i, 4i)$, where $i \geq 1$. If tasks $T_1, \ldots, T_3$ require a tardiness of zero, then tardiness for T_4 can grow unboundedly.

Discussion. Though the tardiness bounds derived in [8] guarantee that tardiness for each task in the above example (with $i = 1$) is at most 4.33 time units under EDF, no task is immune from incurring maximum tardiness. The bound for EDF-hl derived here would enable one of the four tasks to be guaranteed zero tardiness if the remaining tasks can tolerate a tardiness of 6 time units (which is only slightly higher than 4.33). However, if two tasks have a tardiness requirement of zero, then tardiness for the remaining tasks may be as high as 21.0 (which is still bounded). Lower tardiness can be guaranteed if the utilizations of the unprivileged tasks are lower. For instance, with two privileged tasks $T_1(3,4)$ and $T_2(3,4)$ and three unprivileged tasks $T_3(3,6), \ldots, T_5(3,6)$, the unprivileged tasks would have a tardiness of at most 12.0.

3 Tardiness under EDF-hl

In this section, we determine a tardiness bound for τ_L. The approach for doing this is the same as that used in [8]. This involves comparing the allocations to a concrete task system τ in a processor sharing (PS) schedule for τ and an actual EDF-hl schedule of interest for τ, and quantifying the difference between the two. In a PS schedule, each job of T_i is allocated a fraction u_i of a processor at each instant (or equivalently, a fraction u_i of each instant) in the interval between its release time and its deadline. Because $U_{sum} \leq m$ holds, the total demand at any instant will not exceed m in a PS schedule, and hence no deadlines will be missed; in fact, every job will complete executing exactly at its deadline. We begin by setting the required machinery in place.

3.1 Definitions and Notation

A time interval $[t_1, t_2)$, where $t_2 \geq t_1$, consists of all times t, where $t_1 \leq t < t_2$, and is of length $t_2 - t_1$. The system start time is assumed to be zero. For any time $t > 0$, t^- denotes the time $t - \epsilon$ in the limit $\epsilon \to 0+$.

Definition 1 (active tasks and active jobs): A task T_i is said to be *active* at time t, if there exists a job $T_{i,j}$ (called T_i's *active job* at t) such that $r_{i,j} \leq t < d_{i,j}$. By our task model, every task can have at most one active job at any time.

Definition 2 (pending jobs): $T_{i,j}$ is said to be *pending* at t in a schedule $\mathcal{S}$ if $r_{i,j} \leq t$ and $T_{i,j}$ has not completed execution by t in $\mathcal{S}$. Note that a job with a deadline at or before t is not considered to be active at t even if it is pending at t.

Definition 3 (ready jobs): A pending job $T_{i,j}$ is said to be *ready* at t in a schedule $\mathcal{S}$ if $t \geq r_{i,j}$ and all prior jobs of T_i have completed execution by t in $\mathcal{S}$.

We now quantify the total allocation to τ in an interval $[t_1, t_2)$ in a PS schedule for τ, PS_τ. Let $\mathsf{A}(\mathcal{S}, T_i, t_1, t_2)$ denote the total time allocated to T_i in an arbitrary schedule $\mathcal{S}$ for τ in $[t_1, t_2)$. Then, since T_i is allocated in PS_τ a fraction u_i of each instant at which it is active in $[t_1, t_2)$, we have

$$\mathsf{A}(\mathsf{PS}_\tau, T_i, t_1, t_2) \leq (t_2 - t_1)u_i. \quad (1)$$

The total allocation to τ in the same interval in PS_τ is

$$\begin{aligned} &\mathsf{A}(\mathsf{PS}_\tau, \tau, t_1, t_2) \\ &\quad \leq \sum_{T_i \in \tau} (t_2 - t_1)u_i = U_{sum}(\tau) \cdot (t_2 - t_1). \quad (2) \end{aligned}$$

We are now ready to define lag and LAG, which play a pivotal role in this paper. The *lag of task T_i at time t in schedule $\mathcal{S}$*, denoted $\mathsf{lag}(T_i, t, \mathcal{S})$, is given by

$$\mathsf{lag}(T_i, t, \mathcal{S}) = \mathsf{A}(\mathsf{PS}_\tau, T_i, 0, t) - \mathsf{A}(\mathcal{S}, T_i, 0, t). \quad (3)$$

In $\mathcal{S}$, less work than in PS_τ on the jobs of T_i has been completed by time t if $\mathsf{lag}(T_i, t, \mathcal{S})$ is positive (*i.e.*, T_i is under-allocated in $\mathcal{S}$), and more work, if $\mathsf{lag}(T_i, t, \mathcal{S})$ is negative (*i.e.*, T_i is over-allocated in $\mathcal{S}$). The *total lag of a task system τ at t*, denoted $\mathsf{LAG}(\tau, t, \mathcal{S})$, is given by

$$\begin{aligned} \mathsf{LAG}(\tau, t, \mathcal{S}) &= \sum_{T_i \in \tau} \mathsf{lag}(T_i, t, \mathcal{S}) \\ &= \mathsf{A}(\mathsf{PS}_\tau, \tau, 0, t) - \mathsf{A}(\mathcal{S}, \tau, 0, t). \quad (4) \end{aligned}$$

Note that $\mathsf{LAG}(\tau, 0, \mathcal{S})$ and $\mathsf{lag}(T_i, 0, \mathcal{S})$ are both zero, and that by (3) and (4), we have the following for $t_2 > t_1$.

$$\begin{aligned} \mathsf{lag}(T_i, t_2, \mathcal{S}) &= \mathsf{lag}(T_i, t_1, \mathcal{S}) + \\ &\quad \mathsf{A}(\mathsf{PS}_\tau, T_i, t_1, t_2) - \mathsf{A}(\mathcal{S}, T_i, t_1, t_2) \quad (5) \\ \mathsf{LAG}(\tau, t_2, \mathcal{S}) &= \mathsf{LAG}(\tau, t_1, \mathcal{S}) + \\ &\quad \mathsf{A}(\mathsf{PS}_\tau, \tau, t_1, t_2) - \mathsf{A}(\mathcal{S}, \tau, t_1, t_2) \quad (6) \end{aligned}$$

Lag for jobs. The notion of lag defined above for tasks and task sets can be applied to jobs and job sets in an obvious manner. Let τ denote a concrete task system, and Ψ a subset of jobs in τ. Let $\mathsf{A}(\mathsf{PS}_\tau, T_{i,j}, t_1, t_2)$ and $\mathsf{A}(\mathcal{S}, T_{i,j}, t_1, t_2)$ denote the allocations to $T_{i,j}$ in $[t_1, t_2)$ in PS_τ and $\mathcal{S}$, respectively. Then, $\mathsf{lag}(T_{i,j}, t, \mathcal{S}) = \mathsf{A}(\mathsf{PS}_\tau, T_{i,j}, r_{i,j}, t) - \mathsf{A}(\mathcal{S}, T_{i,j}, r_{i,j}, t)$, and $\mathsf{LAG}(\Psi, t, \mathcal{S}) = \sum_{T_{i,j} \in \Psi} \mathsf{lag}(T_{i,j}, t, \mathcal{S})$. The total allocation in $[0, t)$, where $t > 0$, to a job that is neither pending at t^- in $\mathcal{S}$ nor is active at t^- is the same in both $\mathcal{S}$ and PS_τ, and hence, its lag at t is zero. Therefore, for $t > 0$, we have

$$\mathsf{LAG}(\Psi, t, \mathcal{S}) = \sum_{\substack{\{T_{i,j} \text{ is in } \Psi, \text{ and is} \\ \text{pending or active at} \\ t^-\}}} \mathsf{lag}(T_{i,j}, t, \mathcal{S}).$$

The above expression can be rewritten using task lags as follows (since no job can be scheduled before its release time).

$$\mathsf{LAG}(\Psi, t, \mathcal{S}) \leq \sum_{\substack{\{T_i \in \tau \,:\, T_{i,j} \text{ is in } \Psi, \\ \text{and is pending or active} \\ \text{at } t^-\}}} \mathsf{lag}(T_i, t, \mathcal{S}) \quad (7)$$

Similarly, the total utilization of Ψ at time t is given by the sum of the utilizations of tasks with an active job at t in Ψ:

$$U_{sum}(\Psi, t) = \sum_{\substack{\{T_i \in \tau \,:\, T_{i,j} \text{ is in } \Psi \text{ and} \\ \text{is active at } t\}}} u_i. \quad (8)$$

Definition 4 (busy interval): A time interval $[t_1, t_2)$, where $t_2 > t_1$, is said to be *busy* for τ if all m processors are executing jobs of tasks in τ throughout the interval, *i.e.*, no processor is ever idle in the interval or executes a job of a task not in τ. An interval $[t_1, t_2)$ that is not busy for τ is said to be *non-busy* for τ, and is *maximally non-busy* if every time instant in $[t_1, t_2)$ is non-busy, and either $t_1 = 0$ or t_1- is busy.

If at least $U_{sum}(\tau)$ tasks are executing at any instant in $[t_1, t_2)$ in a schedule $\mathcal{S}$ for τ, then the tasks in τ receive a total allocation of $U_{sum}(\tau) \cdot (t_2 - t_1)$ time in $\mathcal{S}$ in that interval. By (2), the total allocation to τ in $[t_1, t_2)$ cannot exceed $U_{sum}(\tau) \cdot (t_2 - t_1)$ in PS_τ. Therefore, by (6), the LAG of τ at t_2 cannot exceed that at t_1, and we have the following lemma.

Lemma 1 *If* $\mathsf{LAG}(\tau, t + \delta, \mathcal{S}) > \mathsf{LAG}(\tau, t, \mathcal{S})$*, where* $\delta > 0$ *and* $\mathcal{S}$ *is a schedule for* τ*, then* $[t, t+\delta)$ *is a non-busy interval for* τ*. Furthermore, there exists at least one instant in* $[t, t + \delta)$ *at which fewer than* $U_{sum}(\tau)$ *tasks are executing.*

The busy interval in Def. 4 is defined with respect to τ. With respect to Ψ, $[t_1, t_2)$ is said to be *busy* only if every processor is executing some job of Ψ throughout $[t_1, t_2)$. The job-set counterpart of Lemma 1 is as follows.

Lemma 2 *If* $\mathsf{LAG}(\Psi, t + \delta, \mathcal{S}) > \mathsf{LAG}(\Psi, t, \mathcal{S})$*, where* $\delta > 0$ *and* $\mathcal{S}$ *is a schedule for* Ψ*, then* $[t, t+\delta)$ *is a non-busy interval for* Ψ*. Furthermore, there exists at least one instant* t' *in* $[t, t+\delta)$ *at which fewer than* $U_{sum}(\Psi, t')$ *tasks are executing jobs in* Ψ*.*

3.2 Deriving a Tardiness Bound

Given an arbitrary non-concrete task system τ^N, we are interested in determining the highest tardiness of any job of any task in τ_L^N in any concrete instantiation of τ^N. Let τ (resp., τ_H and τ_L) be a concrete instantiation of τ^N (resp., τ_H^N and τ_L^N), $T_{\ell,j}$ a job in τ_L, $t_d = d_{\ell,j}$, and $\mathcal{S}$ an EDF-hl schedule for τ with the following property.

(P) The tardiness of every job of every task T_k in τ_L with deadline less than t_d is at most $x + e_k$, where $x \geq 0$.

Then, determining the smallest x, independent of the parameters of T_ℓ, such that the tardiness of $T_{\ell,j}$ remains at most $x + e_\ell$ would by induction imply a tardiness of at most $x + e_k$ for all jobs of tasks in τ_L. Because τ is arbitrary, the tardiness bound will hold for every concrete instantiation of τ^N.

Our proof obligation is easily met if $T_{\ell,j}$ completes by its deadline, t_d, so assume otherwise. The completion time of $T_{\ell,j}$ depends on the amount of work that can compete with $T_{\ell,j}$ after t_d. We follow the steps below to determine x.

(S1) Compute an upper bound (UB) on the amount of work (including that due to $T_{\ell,j}$) that can compete with $T_{\ell,j}$ after t_d.

(S2) Determine a lower bound (LB) on the amount of such work required for the tardiness of $T_{\ell,j}$ to exceed $x + e_\ell$.

(S3) Determine the smallest x such that the tardiness of $T_{\ell,j}$ is at most $x + e_\ell$ using UB and LB.

Let Ψ denote the set of all jobs with deadlines at most t_d of all tasks in τ. Under EDF-hl, no job of a task in τ_L with a deadline after t_d can compete with $T_{\ell,j}$. Therefore, competing work for $T_{\ell,j}$ is given by **(i)** the amount of work pending at t_d for jobs in Ψ, *i.e.*, $\mathsf{LAG}(\Psi, t_d, \mathcal{S})$, plus **(ii)** the amount of work demanded by jobs of tasks in τ_H that are not in Ψ but can compete with jobs in Ψ in $[t_d, t_d + x + e_\ell)$. We now determine an upper bound on these two components (step (S1) described above).

(In the analysis that follows, we assume that $\Delta_h \ll x$ holds for all T_h in τ_H. The analysis has to be extended slightly, otherwise. We have refrained from presenting a more general analysis in the interest of clarity.)

3.2.1 Upper Bound on $\mathsf{LAG}(\Psi, t_d, \mathcal{S})$

Let the *carry-in* job of a task T_h in τ_H be defined as that job of T_h, if any, with a release time before t_d and an absolute deadline afterward. Clearly, at most one such job exists for each T_h. Similarly, let the job of T_h, if any, with a release time before $t_d + x+e_\ell$ and an effective deadline afterward be defined as its *carry-out* job. This is illustrated in Fig. 4. The carry-in job of T_h is its only job with an *absolute deadline* after t_d that may preempt (*i.e.*, compete with) jobs in Ψ before t_d (*i.e.*, become urgent before t_d). Let Ψ_H be the set of all carry-in jobs of tasks in T_h. (For ease of reference, descriptions for these task sets and job sets are repeated in Fig. 2.)

By Lemma 2, the LAG of Ψ can increase only across a non-busy interval for Ψ. Recall that in a non-busy interval for Ψ fewer than m jobs from Ψ execute. In the case of an EDF-hl schedule, such a non-busy interval for Ψ can be classified into two types depending on whether a job from Ψ_H is executing in the interval while a ready job from Ψ is waiting. At the risk of slightly abusing terms,

τ_H	$\stackrel{\text{def}}{=}$	Set of all privileged tasks in τ
τ_L	$\stackrel{\text{def}}{=}$	Set of all unprivileged tasks in τ
Ψ	$\stackrel{\text{def}}{=}$	Set of all jobs of all tasks in τ with deadlines at most t_d
Ψ_H	$\stackrel{\text{def}}{=}$	Set of carry-in jobs of tasks in τ_H

Figure 2: Task and job sets heavily referred to.

we will refer to the two types as *blocking* and *non-blocking* non-busy intervals. A *blocking non-busy interval* is one in which a job from Ψ_H is executing while a ready job from Ψ is waiting, whereas a *non-blocking, non-busy interval* is one in which fewer than m jobs from Ψ are executing, but there does not exist a ready job in Ψ that is waiting. Note that it is immaterial whether a job from Ψ_H is executing in a non-blocking, non-busy interval.

Before determining an upper bound on LAG, we state some needed properties. In [8], we showed that if a task does not execute continuously within a non-busy interval in an EDF schedule, then its lag at the end of the interval is at most zero. This property can be extended to a non-blocking, non-busy interval of an EDF-hl schedule, as follows.

Lemma 3 (from [8]) *Let $[t, t')$ be a maximally non-blocking, non-busy interval in $[0, t_d)$ in $\mathcal{S}$ and let T_k be a task in τ with a job in Ψ that is active or pending at t'^{-}. If T_k does not execute continuously in $[t, t')$, then* $\mathsf{lag}(T_k, t', \mathcal{S}) \leq 0$.

To see why this lemma holds, note that, because $[t, t')$ is maximally non-busy and is non-blocking, at least one processor is idle throughout this interval, or a job from Ψ_H is executing while no job in Ψ is waiting. Recall that the absolute deadline of a job in Ψ_H is after t_d. Hence, if T_k is not executing at t'^{-}, then it has no pending work at t', and hence, its lag at t' is at most zero. On the other hand, if T_k is executing at t'^{-}, but was not executing some time earlier in $[t, t')$, then it must have had no pending work when its most-recent job was released and must have executed continuously since then. In this case too, its lag cannot exceed zero.

The two lemmas that follow are proved in an appendix. The first lemma bounds the lag of a task in τ_L at any arbitrary time at or before t_d. The second concerns the lags of tasks in τ_H.

Lemma 4 *Let v be an arbitrary time instant at or before t_d. Let T_k be a task in τ_L and $T_{k,q}$ its earliest pending job at v, and let $\delta_{k,q} < e_k$ be the amount of time that $T_{k,q}$ executed for before v. Then,* $\mathsf{lag}(T_k, v, \mathcal{S}) \leq (v - d_{k,q}) \cdot u_k + e_k - \delta_{k,q}$. *Furthermore, $v - d_{k,q} \leq x + \delta_{k,q}$. Hence,* $\mathsf{lag}(T_k, v, \mathcal{S}) \leq x \cdot u_k + e_k$.

Lemma 5 *Let T_k be a task in τ_H and $T_{k,q}$ its earliest pending job at any arbitrary time v. Then,* $\mathsf{lag}(T_k, v, \mathcal{S}) \leq \min(d_{k,q} + \Delta_k - v, e_k) + (v - d_{k,q}) \cdot u_k \leq e_k + \Delta_k \cdot u_k$.

We now turn to determining an upper bound on the LAG of Ψ at t_d. By Lemma 2, the LAG of Ψ can increase only across a non-busy interval for Ψ. Hence, an upper bound on LAG at the end of the latest non-busy interval before t_d across which LAG increases will serve as an upper bound for that at t_d. As discussed earlier, a non-busy interval in an EDF-hl schedule can be either blocking or non-blocking. We will consider these two cases separately. Let f be defined as follows.

$$f = \begin{cases} U_{sum}(\tau) - 1, & U_{sum}(\tau) \text{ is integral} \\ \lfloor U_{sum}(\tau) \rfloor, & \text{otherwise} \end{cases} \quad (9)$$

Expressions that occur frequently in the analysis are provided in Fig. 3. The lemma that follows shows how to bound LAG at the end of a non-blocking, non-busy interval.

Lemma 6 *Let $[t, t')$ be a maximally non-blocking, non-busy interval in $[0, t_d)$ in $\mathcal{S}$ and let* $\mathsf{LAG}(\Psi, t', \mathcal{S}) > \mathsf{LAG}(\Psi, t, \mathcal{S})$. *Then,* $\mathsf{LAG}(\Psi, t', \mathcal{S}) \leq x \cdot U_L + U_H + E_L$.

Proof: By (7), the LAG of Ψ at t' is given by the sum of the lags at t' of all tasks in τ with at least one job in Ψ that is active or pending at t'^{-}. By Lemma 3, the lag of such a task that does not execute continuously in $[t, t')$ is at most zero. Hence,

$$
\begin{aligned}
E_L &= \sum_{T_k \in \mathcal{E}_{\max}(\tau,f)} e_k \\
U_H &= \sum_{T_h \in \mathcal{U}_{\max}(\tau_H, \max(0, f-1-|\tau_L|))} \Delta_h \cdot u_h \\
E_H &= \sum_{T_h \in \tau_H} e_h(1-u_h) \\
U_L &= \sum_{T_k \in \mathcal{U}_{\max}(\tau_L, \min(f-1, |\tau_L|))} u_k \\
E'_H &= \sum_{T_h \in \tau_H} ((e_h(1-u_h) + u_h(e_\ell - \Delta_h) + \\
&\quad \min(e_h \cdot u_h, \Delta_h)) + \max(u_h(e_h - e_\ell), 0)) \\
U'_H &= \sum_{T_h \in \tau_H} u_h
\end{aligned}
$$

Figure 3: Some expressions used in the paper.

to determine an upper bound on LAG at t', it is sufficient to determine an upper bound on the lags of such tasks that are executing continuously in $[t, t')$. Let f' denote the number of such tasks. Then, by Lemma 2,

$$f' < \max_{t \le \hat{t} < t'} \{U_{sum}(\Psi, \hat{t})\} \le U_{sum}(\tau). \tag{10}$$

Let α_H (resp., α_L) denote the subset of all tasks in τ_H (resp., τ_L) that are executing continuously in $[t, t')$ and have a job in Ψ that is active or pending at t'^-. Then,

$$|\alpha_H| + |\alpha_L| = f', \tag{11}$$

and by the above discussion on bounding LAG,

$$
\begin{aligned}
&\mathsf{LAG}(\Psi, t', \mathcal{S}) \\
&\le \sum_{T_h \in \alpha_H} \mathsf{lag}(T_h, t', \mathcal{S}) + \sum_{T_k \in \alpha_L} \mathsf{lag}(T_k, t', \mathcal{S}) \\
&\le \sum_{T_h \in \alpha_H} (\Delta_h \cdot u_h + e_h) + \sum_{T_k \in \alpha_L} (x \cdot u_k + e_k) \\
&\qquad \{\text{by Lemmas 5 and 4}\} \\
&= \sum_{T_k \in \alpha_L \cup \alpha_H} e_k + \sum_{T_k \in \alpha_L} x \cdot u_k + \sum_{T_h \in \alpha_H} \Delta_h \cdot u_h \\
&\le \sum_{T_k \in \mathcal{E}_{\max}(\tau, f')} e_k + \sum_{T_k \in \alpha_L} x \cdot u_k + \sum_{T_h \in \alpha_H} \Delta_h \cdot u_h \\
&\qquad \{\text{By (11)}\} \\
&\le \sum_{T_k \in \mathcal{E}_{\max}(\tau, f')} e_k + \sum_{T_k \in \mathcal{U}_{\max}(\tau_L, \min(f', |\tau_L|))} x \cdot u_k \\
&\quad + \sum_{T_h \in \mathcal{U}_{\max}(\tau_H, \max(0, f'-|\tau_L|))} \Delta_h \cdot u_h. \\
&\qquad \{\text{By (11) and assuming } \Delta_h \ll x \text{ so that} \\
&\qquad \Delta_h \cdot u_h < x \cdot u_k \text{ for all } T_h \in \tau_H,\, T_k \in \tau_L\}
\end{aligned}
$$

Finally, as in [8], it can be shown that for LAG to increase across $[t, t')$, at least one job of Ψ with a deadline at or after t' should have completed execution before t and that at least one job executing at t should have a deadline at or after t'. Hence, the lag for its task T_k at t' is at most e_k. By this argument, the upper bound on LAG derived above reduces to

$$
\begin{aligned}
&\mathsf{LAG}(\Psi, t', \mathcal{S}) \\
&\le \sum_{T_k \in \mathcal{E}_{\max}(\tau, f')} e_k + \sum_{T_k \in \mathcal{U}_{\max}(\tau_L, \min(f'-1, |\tau_L|))} x \cdot u_k \\
&\quad + \sum_{T_h \in \mathcal{U}_{\max}(\tau_H, \max(0, f'-1-|\tau_L|))} \Delta_h \cdot u_h.
\end{aligned}
$$

The lemma follows because, by (9) and (10), $f' \le f$. ■

The next lemma shows how to bound LAG at the end of a blocking, non-busy interval.

Lemma 7 *Let $[t, t')$ be a blocking, non-busy interval in $[0, t_d)$ in $\mathcal{S}$ such that every instant in $[t, t')$ is a blocking instant and any job of Ψ_H that executes in $[t, t')$ executes continuously in $[t, t')$. Then,* $\mathsf{LAG}(\Psi, t', \mathcal{S}) \le \mathsf{LAG}(\Psi, t, \mathcal{S}) + \sum_{T_h \in \alpha_H} (t' - t) \cdot (1 - u_h)$, *where α_H is the subset of all tasks in τ_H whose jobs in Ψ_H execute continuously in $[t, t')$.*

Proof: Let T_h be a task in α_H, where α_H is as defined in the statement of the lemma. Then, because the job of T_h that is executing in $[t, t')$ is in Ψ_H, T_h does not have a job in Ψ that is either active or pending anywhere in $[t, t')$. Thus, by (8),

$$(\forall \hat{t} : t \le \hat{t} < t' :: U_{sum}(\Psi, \hat{t}) \le U_{sum}(\tau) - \sum_{T_h \in \alpha_H} u_h), \tag{12}$$

and since the cumulative allocation at each instant in $[t, t')$ in PS_τ to jobs in Ψ is at most $U_{sum}(\tau) - \sum_{T_h \in \alpha_H} u_h$, the following holds.

$$\mathsf{A}(\mathsf{PS}_\tau, \Psi, t, t') \le (t' - t) \left(U_{sum}(\tau) - \sum_{T_h \in \alpha_H} u_h \right) \tag{13}$$

Because $[t, t')$ is continuously blocking, at every instant in $[t, t')$, there exists at least one job in Ψ that is ready, but does not execute. This in turn implies that no processor is idle in the interval. Hence, we have the following.

$$\mathsf{A}(\mathcal{S}, \Psi, t, t') = (t' - t)(m - |\alpha_H|) \quad (14)$$

By (13) and (14), and (6) (with $t_1 = t$ and $t_2 = t'$), we have

$$\begin{aligned} &\mathsf{LAG}(\Psi, t', \mathcal{S}) \\ &\leq \mathsf{LAG}(\Psi, t, \mathcal{S}) + \\ &\quad (t' - t)\left((U_{sum}(\tau) - \sum_{T_h \in \alpha_H} u_h) - (m - |\alpha_H|)\right) \\ &= \mathsf{LAG}(\Psi, t, \mathcal{S}) + \\ &\quad (t' - t)\left((U_{sum}(\tau) - m) + |\alpha_H| - \sum_{T_h \in \alpha_H} u_h\right). \end{aligned}$$

Because $U_{sum}(\tau) \leq m$, the above implies $\mathsf{LAG}(\Psi, t', \mathcal{S}) \leq \mathsf{LAG}(\Psi, t, \mathcal{S}) + (t' - t) \cdot \sum_{T_h \in \alpha_H}(1 - u_h)$. ■

An upper bound on the LAG of Ψ at t_d can now be determined by combining Lemmas 6 and 7 as follows. (This lemma is proved in an appendix.)

Lemma 8 *Let $\delta_h \leq e_h$ denote the amount of time that the carry-in job (*i.e.*, job in Ψ_H), if any, of task T_h in τ_H executes for before t_d. Then, $\mathsf{LAG}(\Psi, t_d, \mathcal{S}) \leq x \cdot U_L + U_H + E_L + \sum_{T_h \in \tau_H} \delta_h \cdot (1 - u_h)$.*

To complete step (S1), we need to determine an upper bound on the work due to tasks in τ_H that can compete with jobs in Ψ in $[t_d, t_d + x + e_\ell)$. We do this next.

3.2.2 Competing Demand by Jobs of Tasks in τ_H not in Ψ

Let $\mathsf{D}(T_h)$ denote the amount of work due to the jobs of a task T_h in τ_H that are not in Ψ and that can compete with jobs of other tasks in Ψ in $[t_d, t_d + x + e_\ell)$. Then, $\mathsf{D}(T_h)$ is composed of three parts: **(i)** Work that needs to be done on a carry-in job, if any, **(ii)** mandatory work that needs to be done on a carry-out job, if any, and **(iii)** work to be done on all jobs that lie between the carry-in and carry-out jobs, which is e_h times the number of such jobs. This is illustrated in Fig. 4. (Note that because the effective deadlines of any two consecutive jobs of T_h are separated by at least p_h time units, the latter of the two jobs does not become urgent until after the effective deadline of the former job has elapsed. Hence, no job released after the carry out job can compete with a job in Ψ.)

We now derive a bound on $\mathsf{D}(T_h)$.

Lemma 9 *Let T_h be any task in τ_H. Then, $\mathsf{D}(T_h) \leq e_h - \delta_h + u_h \cdot (x + e_\ell - \Delta_h) + \min(0, \Delta_h - (e_h - \delta_h)u_h) + \max(0, u_h(e_h - e_\ell))$, where $\delta_h \leq e_h$ is the amount of time that the carry-in job, if any, of T_h executes before t_d.*

Proof: If no job of T_h has its effective deadline in $[t_d, t_d + x + e_\ell)$, then at most one job of T_h executes in the interval, and the maximum amount of time it executes for cannot exceed $e_h - \delta_h$. Therefore, $\mathsf{D}(T_h) \leq e_h - \delta_h$ holds. Assuming $\Delta_h \ll x$, it can be shown that $u_h \cdot (x + e_\ell - \Delta_h) + \min(0, \Delta_h - (e_h - \delta_h)u_h) + \max(0, u_h(e_h - e_\ell)) \geq 0$ holds. Hence, $\mathsf{D}(T_h) \leq e_h - \delta_h + u_h \cdot (x + e_\ell - \Delta_h) + \min(0, \Delta_h - (e_h - \delta_h)u_h) + \max(0, u_h(e_h - e_\ell))$, which proves the lemma.

Therefore, for the rest of the proof assume that at least one job of T_h has its effective deadline in $[t_d, t_d + x + e_\ell)$. Let T_{h,c_i} and T_{h,c_o} denote the carry-in and carry-out jobs of T_h, if any.

Let $\xi_h = \rho_{h,c_i} - t_d$ and let ϕ_h denote the offset from $t_d + x + e_\ell$ of the last effective deadline in $[t_d, t_d + x + e_\ell)$ of a job of T_h. Refer to Fig. 4. We now determine the three components of $\mathsf{D}(T_h)$ mentioned above.

Work due to T_{h,c_i}. Since T_{h,c_i} completes executing by ρ_{h,c_i}, the amount of time that T_{h,c_i} can execute for after t_d is at most $\rho_{h,c_i} - t_d = \xi_h$ time units. Because T_{h,c_i} executes for δ_h time units before t_d, it cannot execute for more than $e_h - \delta_h$ time units after t_d. Thus, the amount of work to be done on T_{h,c_i} after t_d is at most $\min(e_h - \delta_h, \xi_h)$.

Work due to T_{h,c_o}. The effective deadline of T_{h,c_o} is separated from the previous effective deadline of T_h by at least p_h time units. Since the last

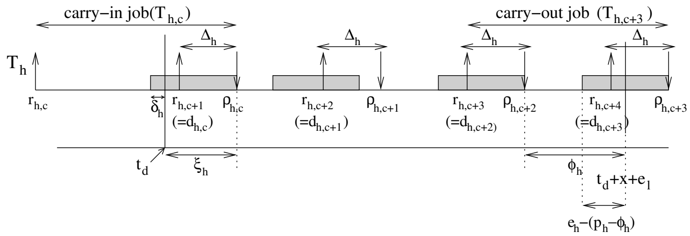


Figure 4: Competing demand due to task T_h in τ_H in the interval $[t_d, t_d + x + e_\ell)$. Competing demand due to the carry-in job T_{h,c_i} ($T_{h,c}$ here) is at most $\min(e_h - \delta_h, \xi_h)$ and that due to the carry-out job T_{h,c_o} ($T_{h,c+3}$ here) is at most $\max(0, e_h - (p_h - \phi_h))$.

effective deadline within $[t_d, t_d + x + e_\ell)$ is ϕ_h time units before $t_d + x + e_\ell$, ρ_{h,c_o} is at least $p_h - \phi_h$ time units after t_d. Therefore, $\min(e_h, p_h - \phi_h)$ units of work due to T_{h,c_o} does not compete with jobs in Ψ before $t_d + x + e_\ell$. Hence, the competing work in $[t_d, t_d + x + e_\ell)$ due to the carry-out job is at most $\max(0, e_h - (p_h - \phi_h))$.

Work due to jobs between T_{h,c_i} and T_{h,c_o}. The effective deadlines of successive jobs of T_h are separated by at least p_h time units. Therefore, the number of jobs of T_h that lie between ρ_{h,c_i} and $t_d + x + e_\ell - \phi_h$ is at most $\lfloor \frac{x+e_\ell-\xi_h-\phi_h}{p_h} \rfloor \leq \frac{x+e_\ell-\xi_h-\phi_h}{p_h}$. Combining the three components above, we have

$$
\begin{aligned}
&\mathsf{D}(T_h) \\
&\leq \left(\frac{x + e_\ell - \xi_h - \phi_h}{p_h}\right) \cdot e_h + \\
&\quad \max(0, e_h - (p_h - \phi_h)) + \min(e_h - \delta_h, \xi_h) \\
&= \max((x + e_\ell - \xi_h - \phi_h)u_h, (x + e_\ell - \xi_h - \phi_h)u_h \\
&\quad + e_h - (p_h - \phi_h)) + \min(e_h - \delta_h, \xi_h) \\
&\qquad\qquad \{\text{Because } e_h/p_h = u_h\} \\
&\leq u_h(x + e_\ell - \xi_h) + \min(e_h - \delta_h, \xi_h) \\
&\qquad\qquad \{\text{Because } 0 \leq \phi_h \leq p_h \text{ and } u_h \leq 1\} \\
&= u_h(x + e_\ell - \Delta_h - \chi_h) + \min(e_h - \delta_h, \Delta_h + \chi_h) \\
&\quad \{\text{Letting } \xi_h = \Delta_h + \chi_h\text{; because } d_{h,c_i} > t_d, \chi_h > 0\} \\
&= \min(e_h - \delta_h + u_h(x + e_\ell - \Delta_h - \chi_h), \\
&\quad \chi_h(1 - u_h) + \Delta_h + u_h(x + e_\ell - \Delta_h)) \\
&= \min(e_h - \delta_h + u_h(x + e_\ell - \Delta_h - \chi_h), \\
&\quad (e_h - \delta_h)(1 - u_h) + \Delta_h + u_h(x + e_\ell - \Delta_h)) \\
&\quad \{\text{Because } \chi_h + \Delta_h < e_h - \delta_h \Rightarrow \chi_h < e_h - \delta_h\} \\
&\leq \min(e_h - \delta_h + u_h(x + e_\ell - \Delta_h), \\
&\quad (e_h - \delta_h)(1 - u_h) + \Delta_h + u_h(x + e_\ell - \Delta_h)) \\
&\qquad\qquad \{\text{Because } \chi_h > 0\} \\
&\leq e_h - \delta_h + u_h(x + e_\ell - \Delta_h) + \\
&\quad \min(0, \Delta_h - (e_h - \delta_h)u_h). \\
&\leq e_h - \delta_h + u_h(x + e_\ell - \Delta_h) + \\
&\quad \min(0, \Delta_h - (e_h - \delta_h)u_h) + \max(0, u_h(e_h - e_\ell)).
\end{aligned}
$$

■

The next lemma gives a bound on the sum of the LAG of Ψ and the competing work due to tasks in τ_H.

Lemma 10 $\mathsf{LAG}(\Psi, t_d, \mathcal{S}) + \sum_{T_h \in \tau_H} \mathsf{D}(T_h) \leq L + \sum_{T_h \in \tau_H} (e_h(1 - u_h) + u_h \cdot (x + e_\ell - \Delta_h) + \min(e_h \cdot u_h, \Delta_h) + \max(0, u_h(e_h - e_\ell)))$*, where* $L = x \cdot U_L + U_H + E_L$.

Proof: By Lemma 8,

$$\mathsf{LAG}(\Psi, t_d, \mathcal{S}) \leq L + \sum_{T_h \in \tau_H} \delta_h \cdot (1 - u_h), \quad (15)$$

where $L = x \cdot U_L + U_H + E_L$ and δ_h is the amount of time the carry-in job of T_h in Ψ_H executed before t_d. By Lemma 9,

$$
\begin{aligned}
&\sum_{T_h \in \tau_H} \mathsf{D}(T_h) \\
&\quad \leq \sum_{T_h \in \tau_H} (e_h - \delta_h + u_h \cdot (x + e_\ell - \Delta_h) \\
&\quad + \min(0, \Delta_h - (e_h - \delta_h)u_h) + \max(0, u_h(e_h - e_\ell)))
\end{aligned}
\quad (16)
$$

By (15) and (16), we have

$$
\begin{aligned}
&\mathsf{LAG}(\Psi, t_d, \mathcal{S}) + \sum_{T_h \in \tau_H} \mathsf{D}(T_h) \\
&\leq L + \sum_{T_h \in \alpha} (\delta_h(1 - u_h) + e_h - \delta_h + u_h(x + e_\ell - \Delta_h) \\
&\quad + \min(0, \Delta_h - (e_h - \delta_h)u_h) + \max(0, u_h(e_h - e_\ell))) \\
&= L + \sum_{T_h \in \tau_H} ((e_h - \delta_h \cdot u_h) \\
&\qquad + u_h(x + e_\ell - \Delta_h) - (e_h - \delta_h)u_h \\
&\qquad + \min((e_h - \delta_h)u_h, \Delta_h) + \max(0, u_h(e_h - e_\ell))) \\
&\leq L + \sum_{T_h \in \tau_H} (e_h(1 - u_h) + u_h(x + e_\ell - \Delta_h) \\
&\qquad + \min(e_h \cdot u_h, \Delta_h) + \max(0, u_h(e_h - e_\ell))).
\end{aligned}
$$

■

That completes step (S1). The next step is to determine a lower bound on the amount or such work required for tardiness of $T_{\ell,j}$ to exceed a certain amount, which we do next.

3.3 Lower Bound on $\mathsf{LAG} + \mathsf{D}$

Lemma 11 *If* $\mathsf{LAG}(\Psi, t_d, \mathcal{S}) \leq (m - |\tau_H|) \cdot x + e_\ell$ *and* $|\tau_H| < m$ *or* $\mathsf{LAG}(\Psi, t_d, \mathcal{S}) + \sum_{T_h \in \tau_H} \mathsf{D}(T_h) \leq (m - \max(|\tau_H| - 1, 0) \cdot u_\ell) \cdot x + e_\ell$ *and* $|\tau_H| > 0$, *then the tardiness of* $T_{\ell,j}$ *in* $\mathcal{S}$ *is at most* $x + e_\ell$.

Proof: To prove the lemma, we show that $T_{\ell,j}$ completes executing by $t_d + x + e_\ell$. If $j > 1$, then $d_{\ell,j-1} \leq t_d - p_\ell$ holds, and by (P), we have the following.

(R) $T_{\ell,j-1}$ completes executing by $t_d + x + e_\ell - p_\ell$, for $j > 1$.

We consider the two conditions stated in the lemma in two separate cases below. In what follows, let $H = |\tau_H|$.

Case 1: $\mathsf{LAG}(\Psi, t_d, \mathcal{S}) \leq (m - |\tau_H|) \cdot x + e_\ell$ and $|\tau_H| < m$. Let $\delta_{\ell,j}$ denote the amount of time that $T_{\ell,j}$ executed before t_d. By the conditions of this case, the amount of work pending at t_d for jobs in Ψ, and hence for those of τ_L in Ψ is at most $(m - H) \cdot x + e_\ell$. Without loss of generality, assume that the jobs in Ψ are the only jobs of τ_L (or, equivalently, jobs with deadlines beyond t_d have been discarded). Hence, if $m - H$ tasks of τ_L are executing at any instant in $[t_d, t_d + x + \frac{\delta_{\ell,j}}{m-H})$, then the amount of work done in the interval on jobs of τ_L in Ψ is at least $(m-H)(x+\frac{\delta_{\ell,j}}{m-H})$. Therefore, the amount of work pending at $t_d + x + \frac{\delta_{\ell,j}}{m-H}$ for those jobs is at most $(m-H) \cdot x + e_\ell - (m-H)(x + \frac{\delta_{\ell,j}}{m-H}) = e_\ell - \delta_{\ell,j}$. Since $T_{\ell,j}$ has executed for $\delta_{\ell,j}$ time before t_d, the latest time that $T_{\ell,j}$ completes executing is $t_d + x + \frac{\delta_{\ell,j}}{m-H} + e_\ell - \delta_{\ell,j} < t_d + x + e_\ell$. If fewer than $m - H$ tasks are executing at some time, say $t' < t_d + x + \frac{\delta_{\ell,j}}{m-H}$, then fewer than $m - H$ tasks of τ_L have pending work at t'. Because tasks of τ_H can execute on at most H processors at any instant, T_ℓ can execute uninterruptedly from t' until $T_{\ell,j}$ completes execution. Suppose the job of T_ℓ executing at t' is $T_{\ell,j}$. Then, since $t' < t_d + x + \frac{\delta_{\ell,j}}{m-H}$ holds, and the amount of work pending for $T_{\ell,j}$ is at most $e_\ell - \delta_{\ell,j}$, $T_{\ell,j}$ completes executing before $t_d + x + e_\ell$. So, assume that a prior job of T_ℓ is executing at t'. In this case, $T_{\ell,j}$ could not have executed before t_d, and hence, $\delta_{\ell,j} = 0$, which implies (from the definition of t') that $t' < t_d + x$. Furthermore, $j \geq 2$ holds, and by (R), $T_{\ell,j-1}$ completes executing by $t_d + x$, and hence, the latest time that $T_{\ell,j}$ commences execution is at or before $t_d + x$, and so the latest time that $T_{\ell,j}$ completes execution is $t_d + x + e_\ell$.

Case 2: $\mathsf{LAG}(\Psi, t_d, \mathcal{S}) + \sum_{T_h \in \tau_H} \mathsf{D}(T_h) \leq (m - |\tau_H| \cdot u_\ell) \cdot x + e_\ell$. At the risk of some notational abuse, let a time interval (resp., instant) in $[t_d, t_d + x + e_\ell)$ in which all m processors are executing a job of Ψ or that part of a task in τ_H that can compete with Ψ be referred to as a *busy interval* (resp., *instant*). Then, if pending, task T_ℓ can execute in every non-busy instant. If the latest busy instant in $[t_d, t_d + x + e_\ell)$ is at or before $t_d + x$, then because, by (R), $T_{\ell,j-1}$, if it exists, completes execution at or before $t_d + x$, the latest time that $T_{\ell,j}$ completes execution is $t_d + x + e_\ell$.

So, for the rest of this proof we assume that the latest busy instant is after $t_d + x$. Let the total lengths of the busy intervals in $[t_d + x, t_d + x + e_\ell)$ be $\delta \leq e_\ell$. (Refer to Fig. 5.) Therefore, T_ℓ can execute for at least $e_\ell - \delta$ time in that interval. If fewer than $m - H + 1$ tasks are executing at any non-busy instant t_n at or before $t_d + x$, then at most $m - H$ tasks of τ_L have pending work at or after t_n. Hence, since tasks in τ_H can execute on at most H processors at any instant, T_ℓ is guaranteed uninterrupted execution from t_n until $T_{\ell,j}$ completes. Hence, since by (R), $T_{\ell,j-1}$ (if it exists) completes execution by $t_d + x$, $T_{\ell,j}$ would complete execution no later than $t_d + x + e_\ell$. Therefore, for the rest of this case, assume the following.

(N) At least $\min(m-H+1, m)$ tasks are executing

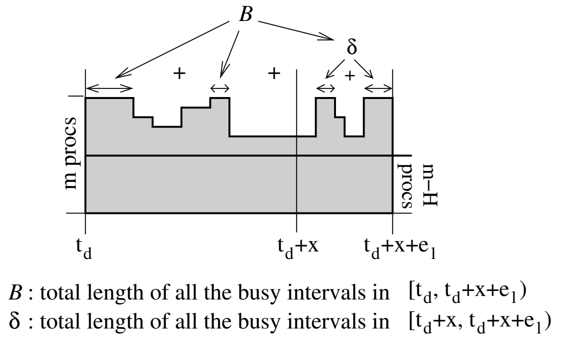


Figure 5: Case 2 of Lemma 11. Sample schedule in $[t_d, t_d + x + e_\ell)$.

at every non-busy instant in $[t_d, t_d + x)$.

Let B denote the total length of all the busy intervals in $[t_d, t_d + x + e_\ell)$. (Refer to Fig. 5.) If $B \leq x - x \cdot u_\ell$, then T_ℓ can execute for at least $x \cdot u_\ell + e_\ell$ time in $[t_d, t_d + x + e_\ell)$. By Lemma 4, $\mathsf{lag}(T_\ell, t_d, \mathcal{S}) \leq x \cdot u_\ell + e_\ell$, and hence, $T_{\ell,j}$ would complete executing at or before $t_d + x + e_\ell$. So, assume $B = x - x \cdot u_\ell + \delta_1$, where $\delta_1 > 0$. With this assumption, we now compute the total amount of work done in $[t_d, t_d + x + e_\ell)$. The total amount of work done in all busy intervals in $[t_d, t_d + x + e_\ell)$ is $m \cdot B$. By (N), at least $\min(m - H + 1, m)$ tasks are executing at every non-busy instant in $[t_d, t_d + x)$. The total length of all non-busy intervals in $[t_d, t_d + x)$ is $x - (B - \delta)$. Therefore, the amount of work done in all non-busy intervals in $[t_d, t_d + x)$ is at least $\min(m - H + 1, m) \cdot (x - B + \delta)$. The total length of all non-busy intervals in $[t_d + x, t_d + x + e_\ell)$ is $e_\ell - \delta$, and at least task T_ℓ of τ_L has pending jobs in Ψ until $t_d + x + e_\ell$, and hence, executes in every non-busy instant in $[t_d + x, t_d + x + e_\ell)$. (Otherwise, it would imply that $T_{\ell,j}$ has completed executing before $t_d + x + e_\ell$, completing the proof, as well). Hence, the total amount of work done in $[t_d, t_d + x + e_\ell)$ is at least $mB + \min(m - H + 1, m) \cdot (x - B + \delta) + (e_\ell - \delta)$, which, on substituting $x - x \cdot u_\ell + \delta_1$ for B, simplifies to $mx - H \cdot x \cdot u_\ell + (m - H) \cdot \delta + x \cdot u_\ell + H \cdot \delta_1 + (e_\ell - \delta)$ for $H > 0$ and $m \cdot (x + \delta) + e_\ell - \delta$ for $H = 0$.

By the condition of this case, the amount of work that needs to be done in $[t_d, t_d + x + e_\ell)$ for jobs in Ψ and of tasks in τ_H that can compete with Ψ is at most $mx - \max(H - 1, 0) \cdot x \cdot u_\ell + e_\ell$. Therefore, the amount of work pending at $t_d + x + e_\ell$ is at most $-(H - 1) \cdot \delta_1 - (m - H) \cdot \delta$, for $1 \leq H \leq m$, and is at most $-(m - 1) \cdot \delta$, for $H = 0$. Because δ and δ_1 are positive, both the above bounds are negative. Thus, no work of jobs in Ψ, and in particular, that of $T_{\ell,j}$, can be pending at $t_d + x + e_\ell$. ■

This completes step (S2). We are left with determining a value for x for which the tardiness of $T_{\ell,j}$ is at most $x + e_\ell$.

3.4 Finishing Up

Solving for x using Lemma 8 and the first condition in Lemma 11, *i.e.*, solving for x in $x \cdot U_L + U_H + E_L + E_H \leq (m - |\tau_H|)x + e_\ell$, yields

$$x \geq \frac{E_L + U_H + E_H - e_\ell}{(m - |\tau_H|) - U_L}, \tag{17}$$

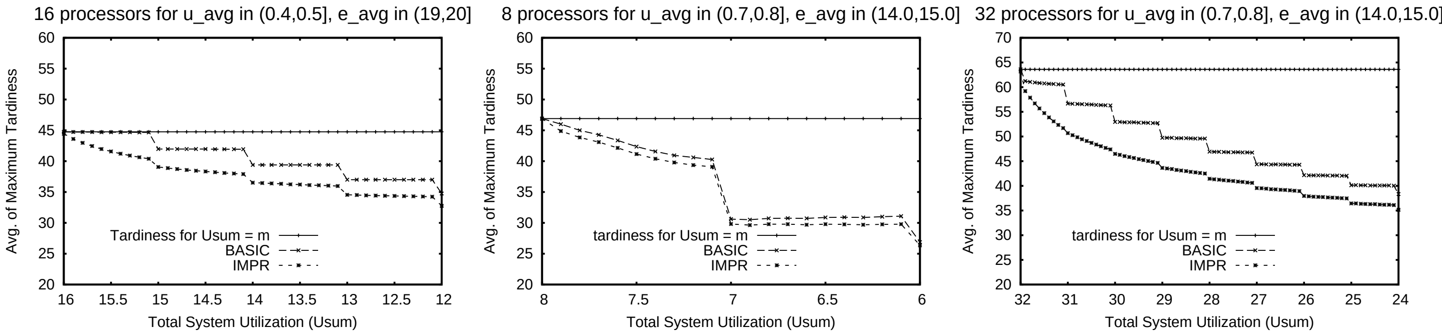
where E_H is as in Fig. 3. Solving using Lemma 10 and the second condition of Lemma 11, *i.e.*, using $\sum_{T_h \in \tau_H}(e_h \cdot (1 - u_h) + u_h \cdot (x + e_\ell - \Delta_h) + \min(e_h \cdot u_h, \Delta_h) + \max(0, u_h \cdot (e_h - e_\ell))) + x \cdot U_L + U_H + E_L \leq mx - \max(|\tau_H| - 1, 0) \cdot x \cdot u_\ell + e_\ell$, we have

$$x \geq \frac{E_L + U_H + E'_H - e_\ell}{m - \max(|\tau_H| - 1, 0) \cdot u_\ell - U_L - U'_H}, \tag{18}$$

where E'_H and U'_H are as in Fig. 3. Hence, if x is smaller of the two values that are on the right-hand sides of (17) and (18), then the tardiness of $T_{\ell,j}$ would not exceed $x + e_\ell$. A value of x that is independent of the parameters of T_ℓ is obtained by replacing e_ℓ by $e_{\min}$ and u_ℓ by $u_{\max}(\tau_L)$ in (17) and (18). Similarly, the e_ℓ term in the expression for E'_H has to be replaced by $e_{\max}(\tau_L)$. By inducting on the jobs of τ_L in the non-decreasing order of their deadlines, we have the following theorem.

Theorem 1 EDF-hl *ensures a tardiness of at most* $\min(X_1, X_2) + e_k$ *to every task* T_k *of* τ_L *if* $|\tau_H| \leq m$ *and* $U_{sum}(\tau) \leq m$, *where* $X_1 = \frac{E_L + U_H + E_H - e_{\min}(\tau_L)}{(m - |\tau_H|) - U_L}$ *and* $X_2 = \frac{E_L + U_H + E'_H - e_{\min}(\tau_L)}{m - \max(|\tau_H| - 1, 0) \cdot u_{\max}(\tau_L) - U_L - U'_H}$.

Conditions for bounded tardiness. Since the derivation was based on the assumption that $x \geq 0$, X_1 and X_2 are valid only if their denominators are



(a) **(b)** **(c)**

Figure 6: Tardiness by U_{sum} using the BASIC and IMPR bounds for **(a)** $m = 16$, $0.4 < u_{avg} \leq 0.5$, and $19.0 < e_{avg} \leq 20.0$, **(b)** $m = 8$, $0.7 < u_{avg} \leq 0.8$, and $14.0 < e_{avg} \leq 15.0$, and **(c)** $m = 32$, $0.7 < u_{avg} \leq 0.8$, and $14.0 < e_{avg} \leq 15.0$.

non-negative. X_1 and X_2 are bounded only if their denominators are greater than zero. Hence, if the sum of the utilizations of the $f-1$ heaviest tasks in τ_L is less than $m - |\tau_H|$, then X_1 is bounded. Similarly, X_2 is bounded only if the sum of the utilizations of the heaviest $f-1$ tasks in τ_L is less than $m - |\tau_H| \cdot u_{\max}(\tau_L) - U'_H$. Hence, if either of the above conditions holds, then bounded tardiness is guaranteed to tasks in τ_L.

Computational complexity. Each of the E and U terms in the tardiness bound can be computed in $O(f)$ time. The f tasks with highest utilizations or execution costs can selected from τ, τ_L, or τ_H in $O(n)$ time. Hence, the tardiness bound can be computed in $O(n)$ time.

4 Experimental Evaluation

In this section, we present the results of experiments conducted to **(i)** determine the range of the tardiness bound guaranteed by EDF-hl on an average and **(ii)** evaluate the tardiness-utilization trade-off possible in the absence of high-priority tasks. Due to space constraints, only a subset of the results is presented here.

Tardiness-Utilization trade-off. As mentioned earlier, EDF-hl reduces to EDF in the absence of high-priority tasks. Hence, in this case, the tardiness bound given in Thm. 1 applies to every task in τ. Note that the tardiness bound is expressed in terms of $U_{sum}(\tau)$ in addition to individual task parameters. Hence, an alternative to EDF-hl for guaranteeing lower tardiness is to lower U_{sum}. This approach may be preferable if a majority of the tasks require lower tardiness and the gains are reasonable for slight decreases in U_{sum}.

In the absence of high priority tasks, using a slightly different, but more complicated, analysis than that used in Sec. 3 or in [8], it can be shown that

$$U_L \leq \sum_{T_k \in \mathcal{U}_{\max}(\tau_L, f-1)} \frac{u_k^2(m-f)}{(m - U_{sum}) + u_k(U_{sum} - f)}, \tag{19}$$

which when used in the expression for the tardiness bound in Thm. 1 results in slightly lower values. We will refer to the bound given in Thm. 1 as BASIC and the bound obtained by using (19) as IMPR.

We evaluated the tardiness-utilization trade-off that is possible by generating random task sets with varying values for U_{sum} and computing the BASIC and IMPR bounds for each and comparing these bounds with those obtained from our earlier work, when $U_{sum} = m$ [8]. Simulation experiments were conducted for four, eight, 16, and 32 processors, with U_{sum} varying between $3m/4$ and m in increments of 0.1. 600,000 task sets, with at least $m+1$ tasks in each, were generated for each (U_{sum}, m) pair. The maximum utilization of any task in a task set varied uniformly from 0.5 to 1.0. The task sets generated were grouped based on u_{avg} and e_{avg}, where u_{avg} and e_{avg} are the av-

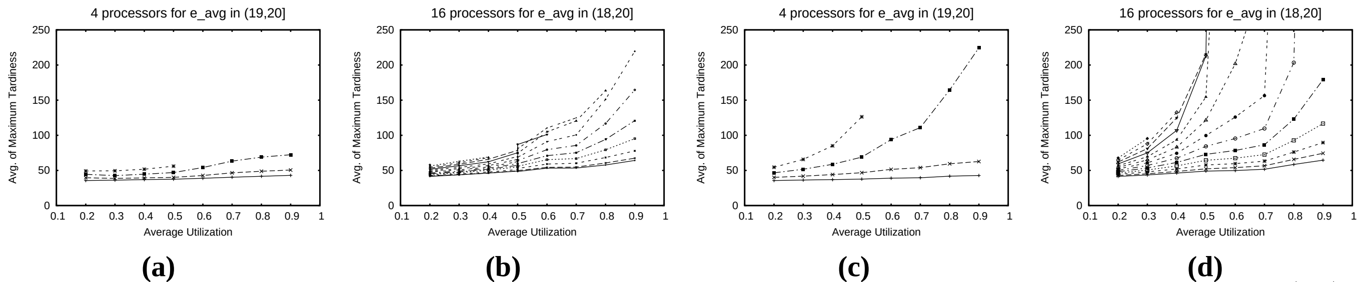


Figure 7: Experimental evaluation of the tardiness bounds guaranteed by Thm. 1 under EDF-hl for tasks in $|\tau_L|$. LU with **(a)** $m = 4$ and **(b)** $m = 16$. HU with **(c)** $m = 4$ and **(d)** $m = 16$. The different curves in each inset correspond to different values of $|\tau_H|$. $|\tau_H| = 0$ for the bottom-most curve and is greater by one for each curve higher up.

erages of the highest $\lfloor U_{sum} \rfloor$ task utilizations and execution costs, respectively. The variation in tardiness (mean of the maximum tardiness for all task sets in a group) with U_{sum} for **(i)** $m = 16$ when $0.4 < u_{avg} \leq 0.5$ and $19.0 < e_{avg} \leq 20.0$ and **(ii)** $m = 8$ and $m = 32$ when $0.7 < u_{avg} \leq 0.8$ and $14.0 < e_{avg} \leq 15.0$ are presented in Fig. 6. Note that the rate at which tardiness drops with decreasing U_{sum} is higher when u_{avg} is higher (in the $(0.7, 0.8]$ range). Furthermore, the rate at which tardiness drops with U_{sum} decreases with decreasing U_{sum}. For instance, in inset (c), reducing U_{sum} to 31.0 (which is 96.8% of $m(= 32)$) lowers tardiness to less than 50.0 from over 60.0, which is a drop of over 20%, whereas to lower tardiness to less than 40.0, U_{sum} has to be decreased to approximately 27.0 (which is 84.3% of m). Hence, setting U_{sum} to a value slightly lower than m may be appropriate when high utilization tasks are present in the task system. At this point, we would like to note that these characteristics should be attributed to the bounds derived (and to the analysis) and not to the algorithm per se.

	u_{avg}						
$\lvert\tau_H\rvert$	0.2	0.3	0.4	0.5	0.6	0.7	0.8
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	1.9	100.0
8	0.0	0.0	0.0	0.0	37.2	98.09	100.0
9	0.0	0.0	0.0	0.0	99.3	100.0	100.0
10	0.0	0.0	0.0	0.0	100.0	100.0	100.0
11	0.0	0.0	0.0	14.76	100.0	100.0	100.0
12	0.0	0.0	25.0	100.0	100.0	100.0	100.0
13	0.0	9.7	99.78	100.0	100.0	100.0	100.0
14	0.0	99.67	100.0	100.0	100.0	100.0	100.0
15	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 1: Percentage of task sets for which unbounded tardiness was computed for $m = 16$ under HU.

Tardiness bounds for EDF-hl. We also experimentally evaluated the tardiness bounds that can be guaranteed to low-priority tasks on an average under EDF-hl for $m = 4$ and $m = 16$, with $U_{sum} = m$. The task sets generated were grouped based on the average of the m highest task utilizations and the utilizations of the tasks in τ_H, denoted u_{avg}. (e_{avg} is with respect to execution costs, analogously.) For each task set generated, the number of tasks in τ_H was varied from zero to m, and for each $|\tau_H|$, the members of τ_H were chosen in two different ways: first, as tasks with the highest $|\tau_H|$ utilizations in the generated task set (denoted HU), and then, as tasks with the lowest $|\tau_H|$ utilizations (denoted LU). The variation in tardiness with u_{avg} as the number of high-priority tasks is increased is plotted in Fig. 7 for both HU and LU. As expected, tardiness increases with $|\tau_H|$ and u_{avg}, and the increase is higher for HU than for LU. The tardiness bounds computed grew to unbounded values for certain task sets at high values of $|\tau_H|$, with the percentage of such task sets increasing with increasing u_{avg}. The percentage of such task sets for HU is tabulated by $|\tau_H|$ and u_{avg} for $m = 16$ in Table 1. The figures for LU are slightly lower.

5 Conclusion

We have addressed the issue of supporting tasks whose tolerance to tardiness is lower than that currently known to be possible under EDF. We have proposed a new scheduling policy called EDF-hl, which is based on EDF, and have shown that under EDF-hl, a limited number of *privileged* tasks can be guaranteed any tardiness, including zero tardiness, and that bounded tardiness can be guaranteed to the remaining tasks if their utilizations are restricted. The tardiness bound derived is a function of U_{sum}, in addition to individual task parameters, and hence, tardiness for all tasks can be lowered by slightly lowering U_{sum}. We have, through simulations, assessed the impact of privileged tasks on the tardiness bounds that can be guaranteed to the remaining tasks, and the tardiness-utilization trade-off that is possible in the absence of privileged tasks.

This problem of supporting sporadic tasks with different tardiness requirements may alternatively be viewed as one of supporting tasks with relative deadlines at least periods. The EDF schedulability tests available for task systems with relative deadlines equal to periods on a multiprocessor, though applicable when deadlines may exceed periods also, are pessimistic and tend to under-utililize the underlying platform. The work presented in this paper is an attempt towards remedying this limitation.

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Appendix: Additional Proofs

Lemma 4 *Let v be an arbitrary time instant at or before t_d. Let T_k be a task in τ_L and $T_{k,q}$ its earliest pending job at v, and let $\delta_{k,q} < e_k$ be the amount of time that $T_{k,q}$ executed for before v. Then, $\mathsf{lag}(T_k, v, \mathcal{S}) \leq (v - d_{k,q}) \cdot u_k + e_k - \delta_{k,q}$. Furthermore, $v - d_{k,q} \leq x + \delta_{k,q}$. Hence, $\mathsf{lag}(T_k, v, \mathcal{S}) \leq x \cdot u_k + e_k$.*

Proof: We prove the lemma for the case $d_{k,q} \leq v$, leaving the case $d_{k,q} > v$ to the reader. The amount of work pending for $T_{k,q}$ at v is $e_k - \delta_{k,q}$. T_k is allocated at most u_k time at every instant after $d_{k,q}$ in PS_τ. The first lag bound indicated follows from these two facts. The bound on $v - d_{k,q}$ follows from (P) and the second lag bound is obtained by substituting the bound for $v - d_{k,q}$ in the first lag bound. ∎

Lemma 5 *Let T_k be a task in τ_H and $T_{k,q}$ its earliest pending job at any arbitrary time v. Then, $\mathsf{lag}(T_k, v, \mathcal{S}) \leq \min(d_{k,q} + \Delta_k - v, e_k) + (v - d_{k,q}) \cdot u_k \leq e_k + \Delta_k \cdot u_k$.*

Proof: Because T_k is in τ_H and $T_{k,q}$ is pending at v, $v \leq d_{k,q} + \Delta_k$ holds. Since $T_{k,q}$ is guaranteed continuous execution until completion after time $d_{k,q} + \Delta_k - e_k$, the amount of work pending for $T_{k,q}$ at v is at most $\min(d_{k,q} + \Delta_k - v, e_k)$, *i.e.*, $\mathsf{A}(\mathcal{S}, T_{k,q}, r_{k,q}, v) \geq e_k - \min(d_{k,q} + \Delta_k - v, e_k)$. If $v \leq d_{k,q}$, then $\mathsf{A}(\mathsf{PS}_\tau, T_{k,q}, r_{k,q}, v) = e_k - (d_{k,q} - v) \cdot u_k$. Thus, $\mathsf{lag}(T_{k,q}, v, \mathcal{S}) = \mathsf{A}(\mathsf{PS}_\tau, T_{k,q}, r_{k,q}, v) - \mathsf{A}(\mathcal{S}, T_{k,q}, r_{k,q}, v) \leq \min(d_{k,q} + \Delta_k - v, e_k) - (d_{k,q} - v) \cdot u_k$. Because $T_{k,q}$ is the earliest pending job of T_k at v and no later job of T_k can be released before $d_{k,q}$, $\mathsf{lag}(T_k, v, \mathcal{S}) = \mathsf{lag}(T_{k,q}, v, \mathcal{S}) \leq \min(d_{k,q} + \Delta_k - v, e_k) - (d_{k,q} - v) \cdot u_k$. On the other hand, if $v > d_{k,q}$, then the lag of T_k at v is given by the sum of the work pending for $T_{k,q}$ (which is at most $\min(d_{k,q} + \Delta_k - v, e_k)$) and the total allocation to T_k in PS_τ in $[d_{k,q}, v)$. In PS_τ, T_k is allocated at most a fraction u_k in every instant in $[d_{k,q}, v)$. Hence, in this case too, $\mathsf{lag}(T_k, v, \mathcal{S}) \leq \min(d_{k,q} + \Delta_k - v, e_k) + (v - d_{k,q}) \cdot u_k$. Finally, because $v \leq d_{k,q} + \Delta_k$, we have $\min(d_{k,q} + \Delta_k - v, e_k) + (v - d_{k,q}) \cdot u_k \leq e_k + \Delta_k \cdot u_k$. ∎

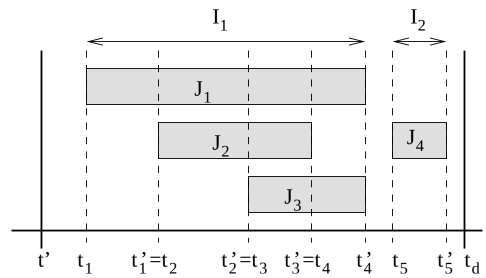

Figure 8: Lemma 8. There does not exist a non-blocking, non-busy interval across which LAG increases in $[t', t_d)$. $J_1, \ldots, J_4$ are jobs in Ψ_H. Their execution in blocking, non-busy intervals I_1 and I_2 is shown, as well as the slicing of the blocking interval I_1 as specified in the proof.

Lemma 8 *Let $\delta_h \leq e_h$ denote the amount of time that the carry-in job, if any, in Ψ_H of task T_h in τ_H executes for before t_d. Then, $\mathsf{LAG}(\Psi, t_d, \mathcal{S}) \leq x \cdot U_L + U_H + E_L + \sum_{T_h \in \tau_H} \delta_h \cdot (1 - u_h)$.*

Proof: Let $[t, t')$ denote the latest non-blocking, non-busy interval before t_d across which LAG increases. If $[t, t')$ exists, then by Lemma 6, $\mathsf{LAG}(\Psi, t', \mathcal{S}) \leq x \cdot U_L + U_H + E_L$. If $[t, t')$ does not exist, then let t' be equal to the first blocking instant in $[0, t_d)$, if any. Otherwise, let $t' = t_d$. Then, $\mathsf{LAG}(\Psi, t', \mathcal{S}) \leq 0$.

If no blocking, non-busy interval follows t' (by our assumption, $[t, t')$ is the latest non-blocking interval before t_d across which LAG increases), then by Lemma 2, $\mathsf{LAG}(\Psi, t_d, \mathcal{S}) \leq \mathsf{LAG}(\Psi, t', \mathcal{S})$, completing the proof. So assume that some blocking, non-busy interval follows t'.

Let $[t_i, t'_i)$, where $1 \leq i \leq b$ and $t_i < t'_{i-1}$

for all $1 < i \leq b$, denote the b disjoint (*i.e.*, non-overlapping) blocking, non-busy subintervals in $[t', t_d)$ such that the following holds: any job of Ψ_H that executes in any of the b non-busy subintervals executes continuously in the interval. It is straightforward to show that such subintervals can be defined—see Fig. 8.

Any increase in LAG for Ψ only occurs across a blocking interval after t'. By Lemma 7, the increase in LAG across the blocking subinterval $[t_i, t'_i)$ is at most $\sum_{T_h \in \alpha_i}(t'_i - t_i) \cdot (1 - u_h)$, where α_i is the subset of all tasks in τ_H whose carry-in jobs are executing continuously in $[t_i, t'_i)$. Let T_h be a task of τ_h whose carry-in job T_{h,c_i} executes in $[t_i, t'_i)$. Then, the increase in LAG due to T_h across $[t_i, t'_i)$ is at most $(t'_i - t_i) \cdot (1 - u_h)$. By the statement of the lemma, T_{h,c_i} does not execute for more than δ_h time units in $[t', t_d)$. Hence, the increase due to T_h over all the blocking subintervals cannot be more than $\delta_h \cdot (1 - u_h)$, and the increase due to all the tasks in τ_H cannot be more than $\sum_{T_h \in \tau_H}(1 - u_h) \cdot \delta_h$. The lemma then follows from the upper bound for LAG at t' determined in the beginning of the proof. ■