Computing the Implicit Voronoi Diagram in Triple Precision

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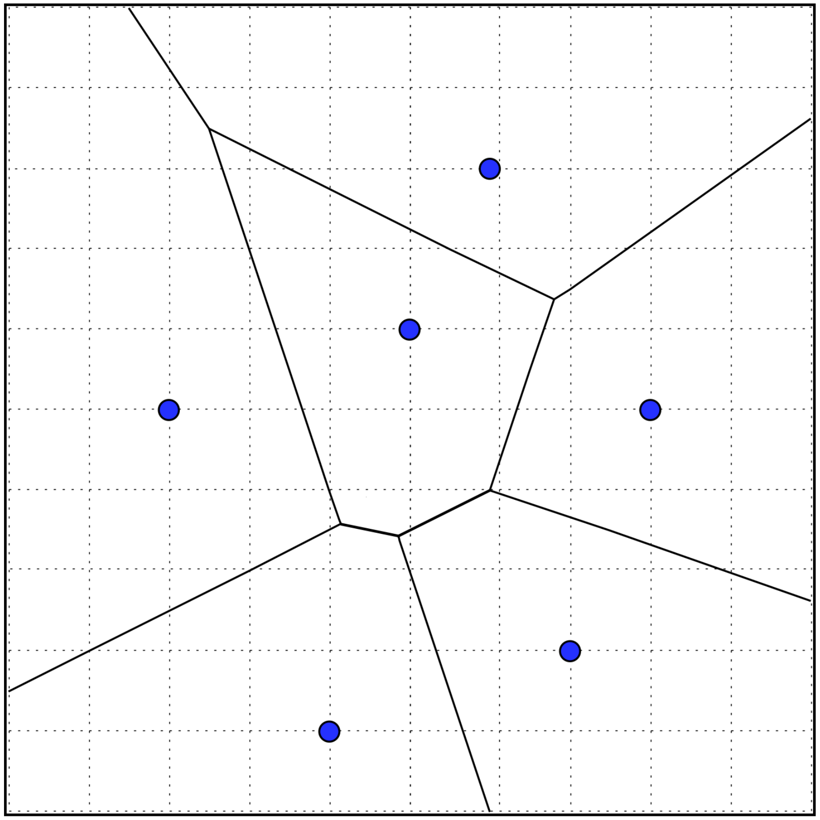
I will…

- define Voronoi, arithmetic degree, implicit Voronoi [LPT99]
- define a more structured *reduced-precision Voronoi diagram*
- describe its incremental construction
- present open problems created by this work

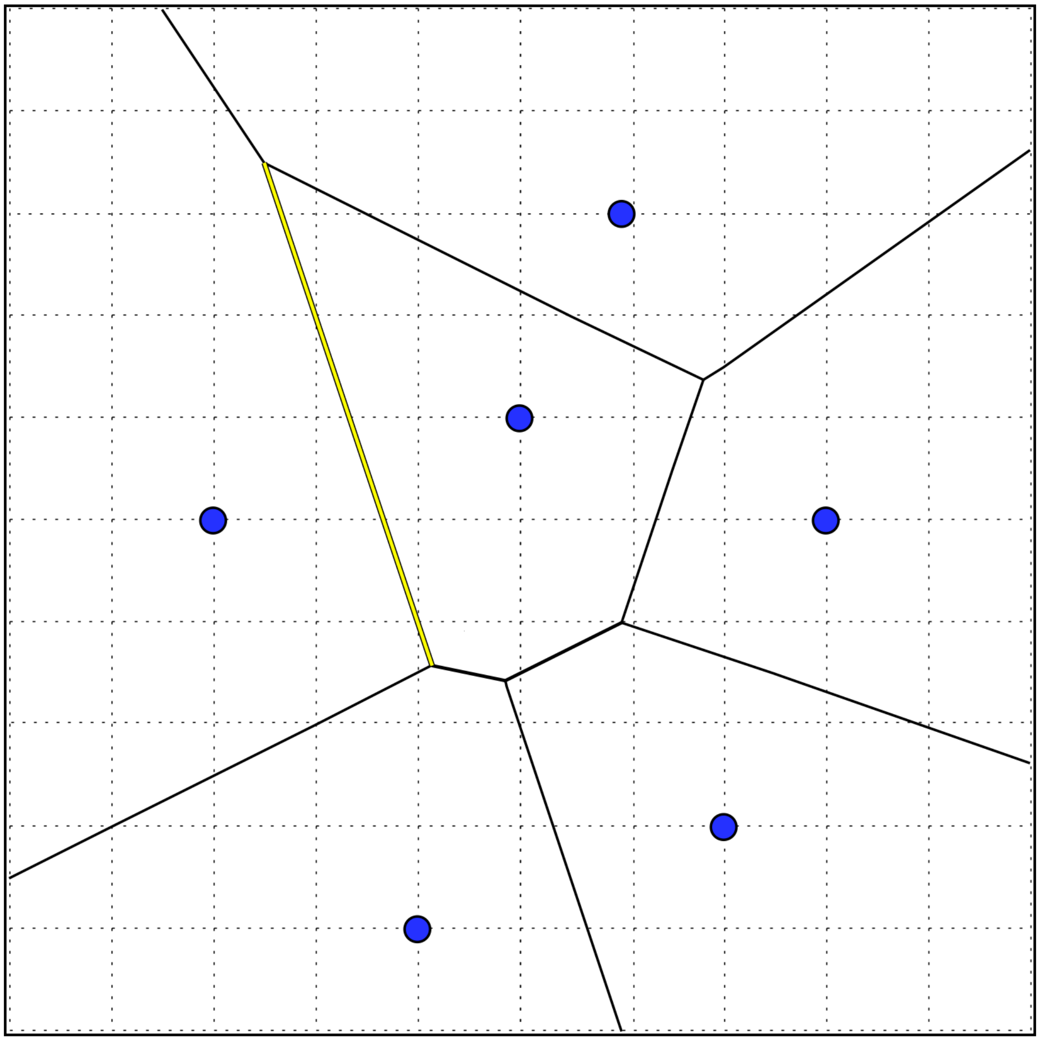
WADS09
August 23, 2009

Voronoi Diagram

Given a finite set of sites on a grid, the *Voronoi diagram* is the partition of the plane into maximally connected regions having the same set of closest sites



Voronoi Diagram



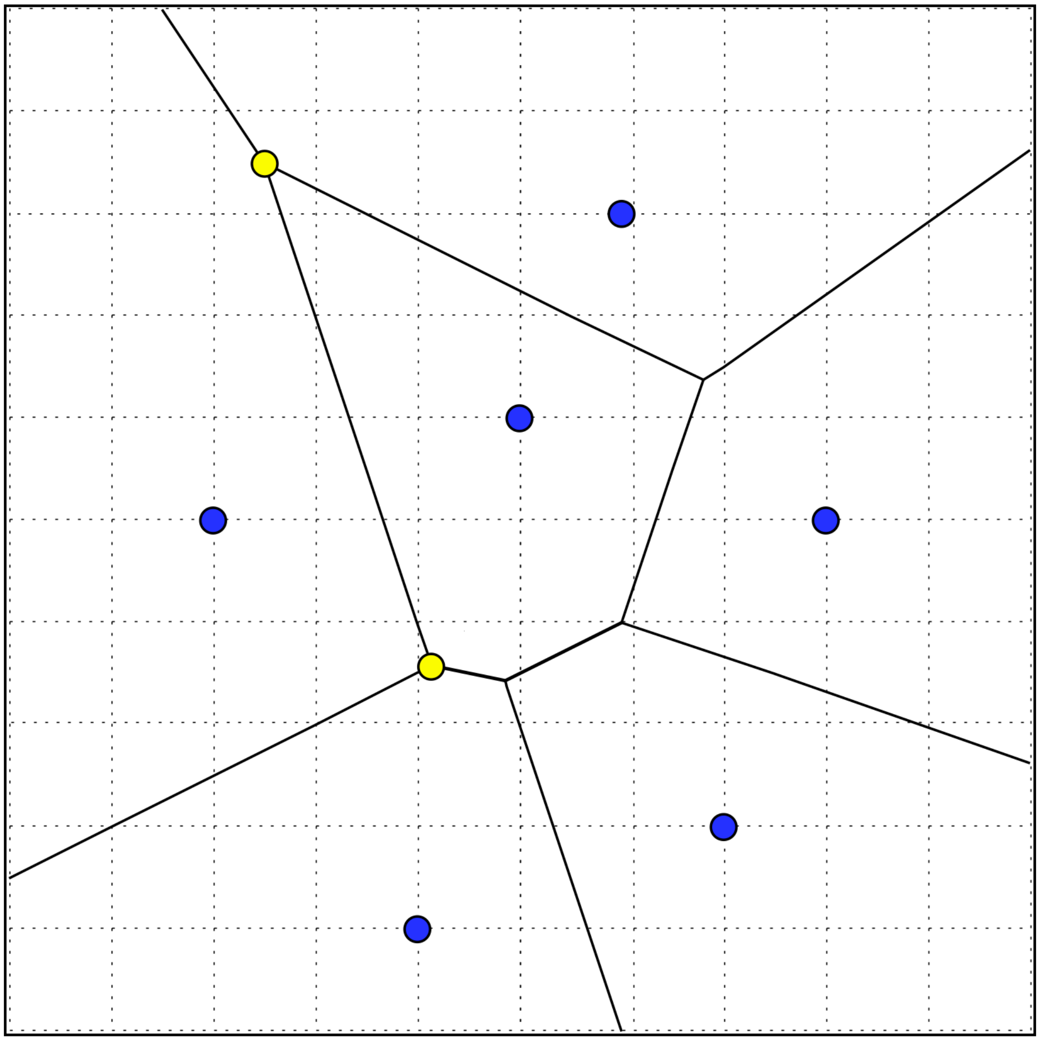
Voronoi edges

Voronoi vertices

Voronoi cell

Voronoi diagram

Voronoi Diagram



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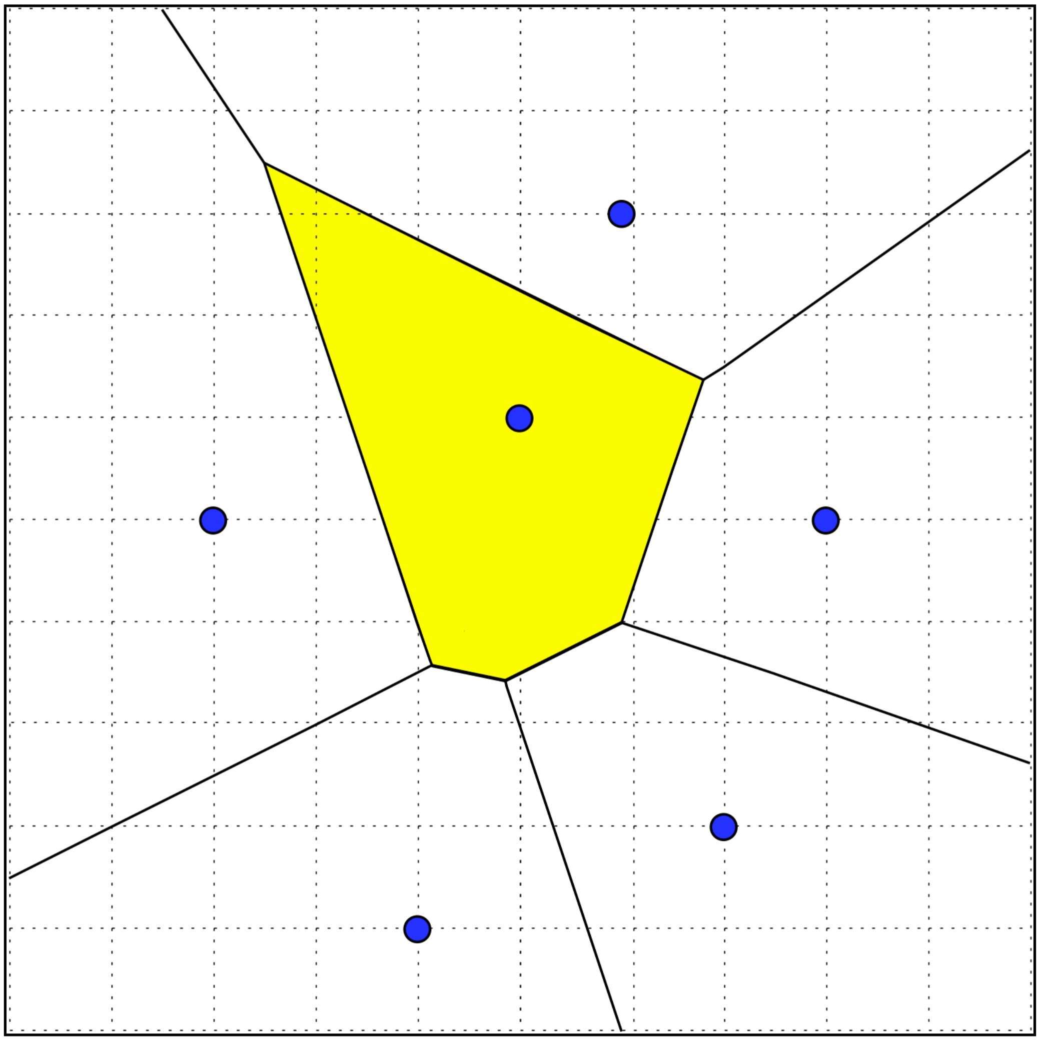
Voronoi Diagram



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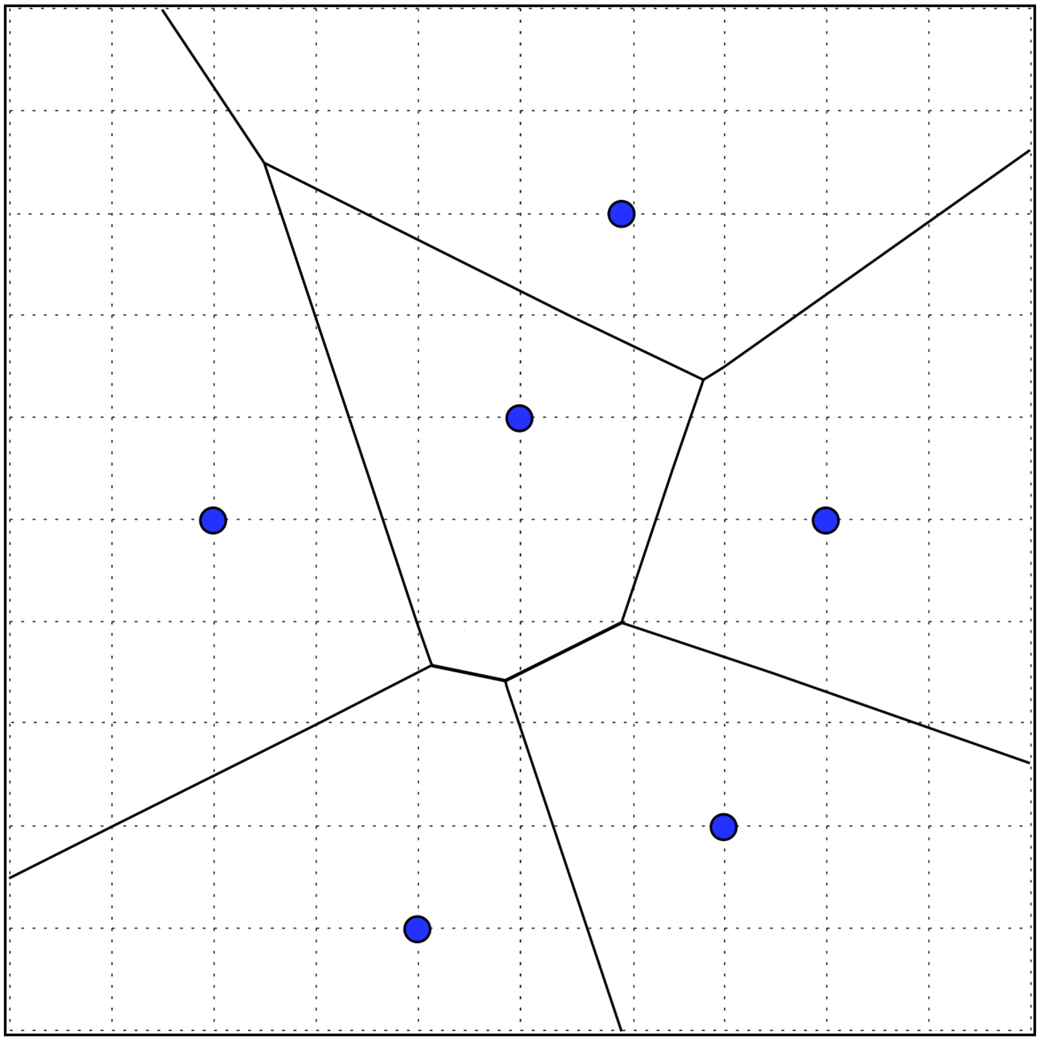
Voronoi Diagram



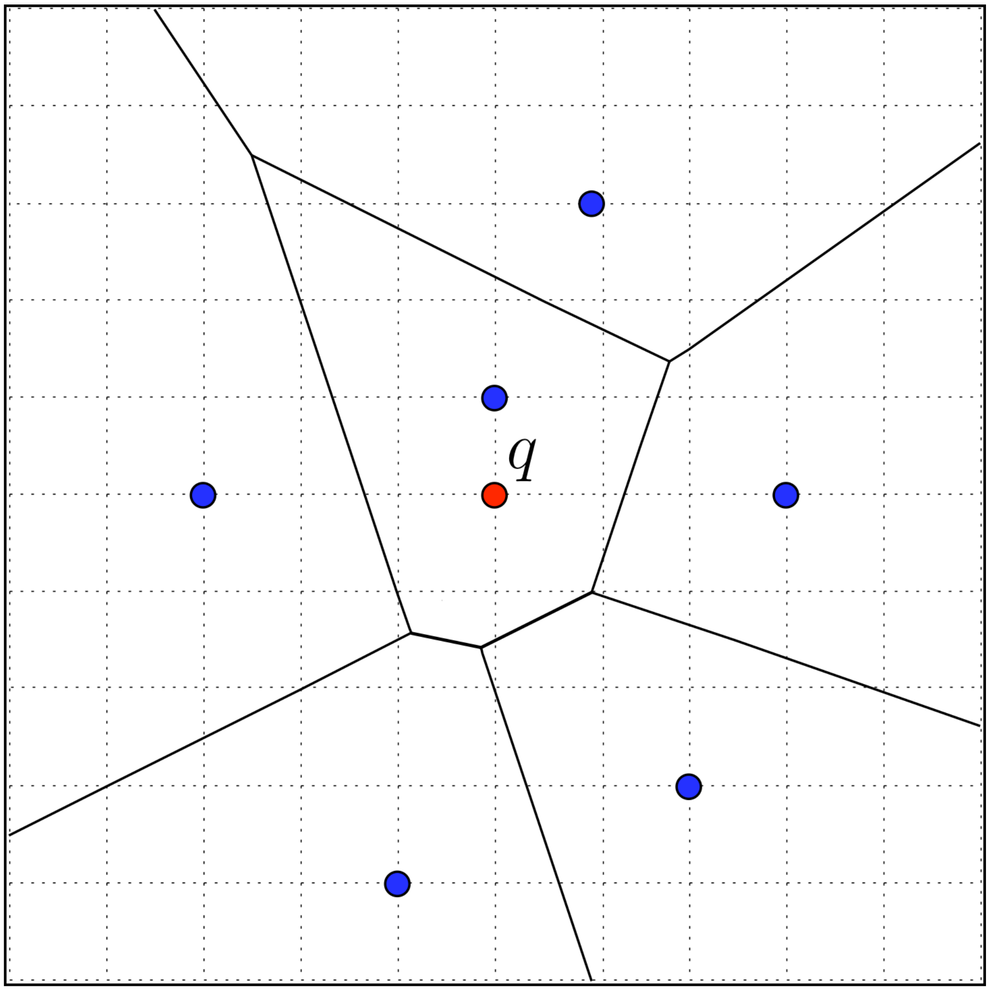
Voronoi edges

Voronoi vertices

Voronoi cell

Voronoi diagram

Point location returns nearest neighbor



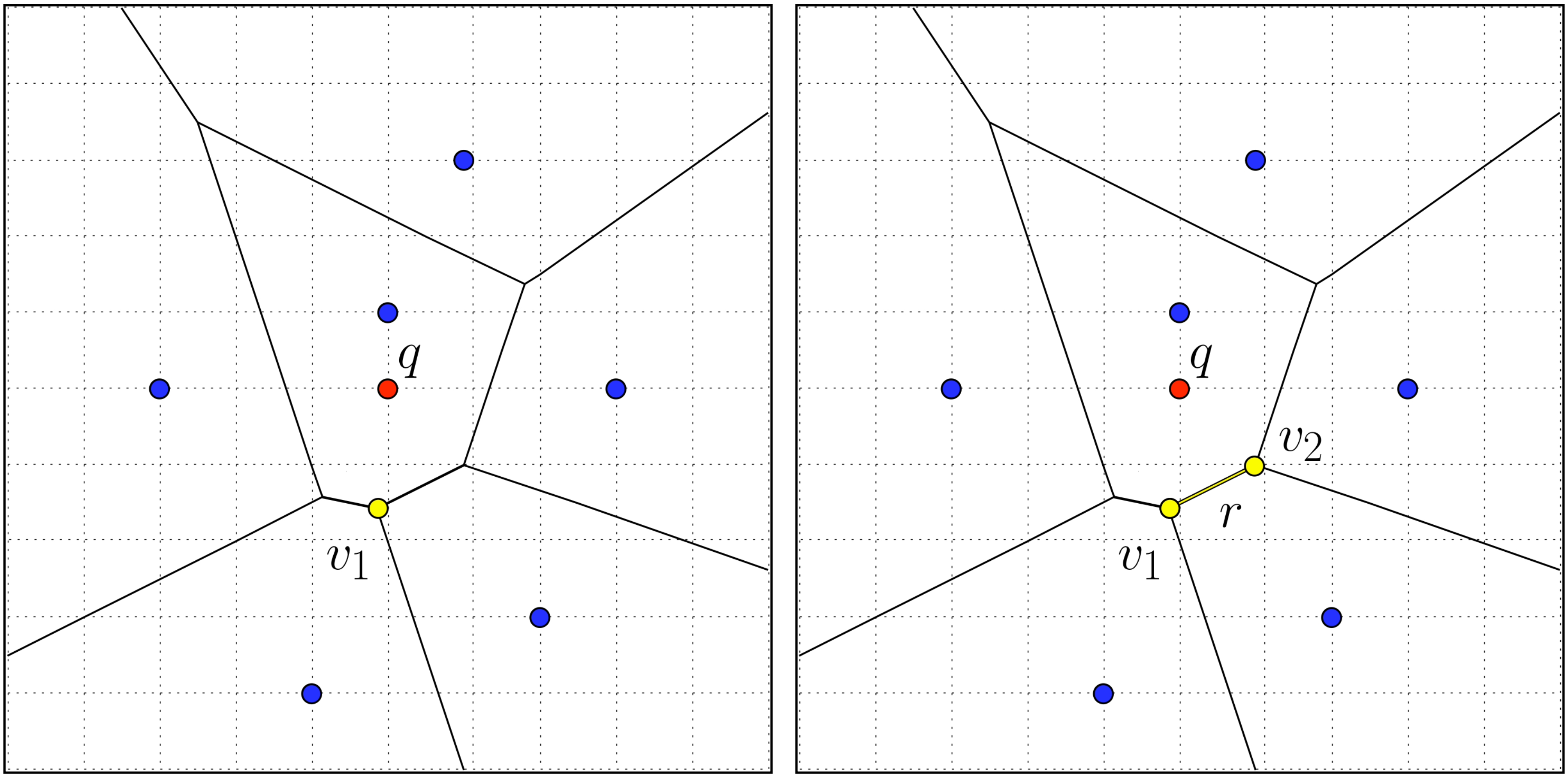
Given:

A set of sites S the Voronoi diagram of S & query point q

Determine:

the site of S closest to q

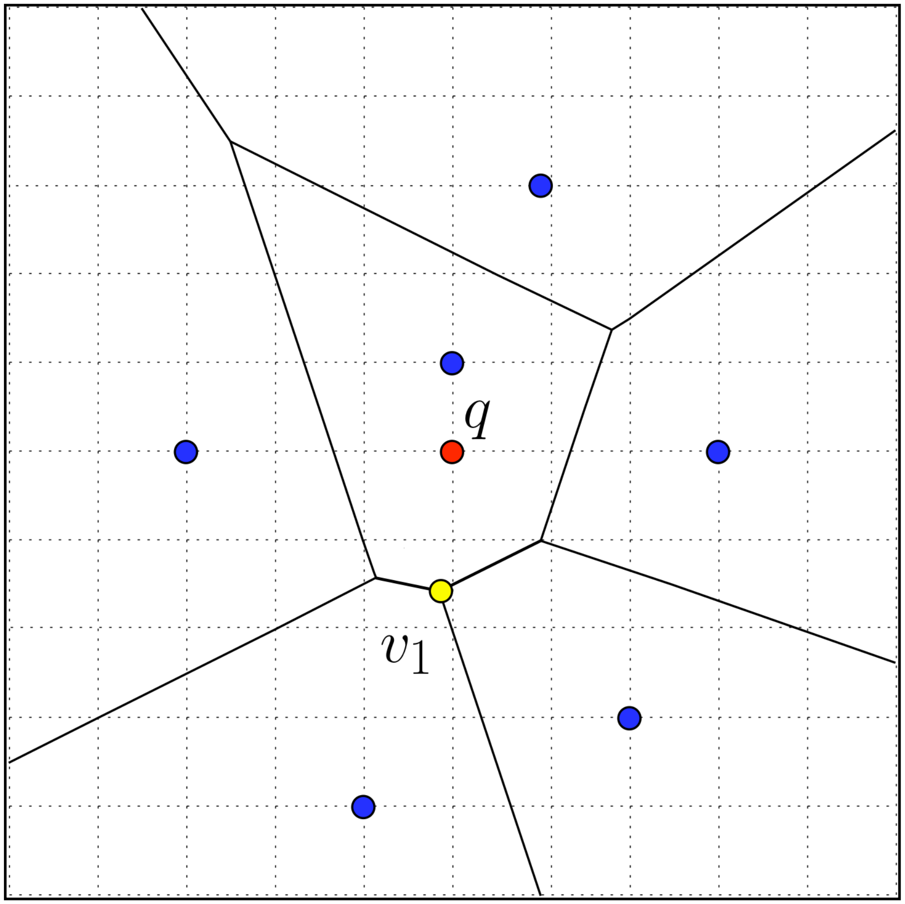
Predicates and their precision

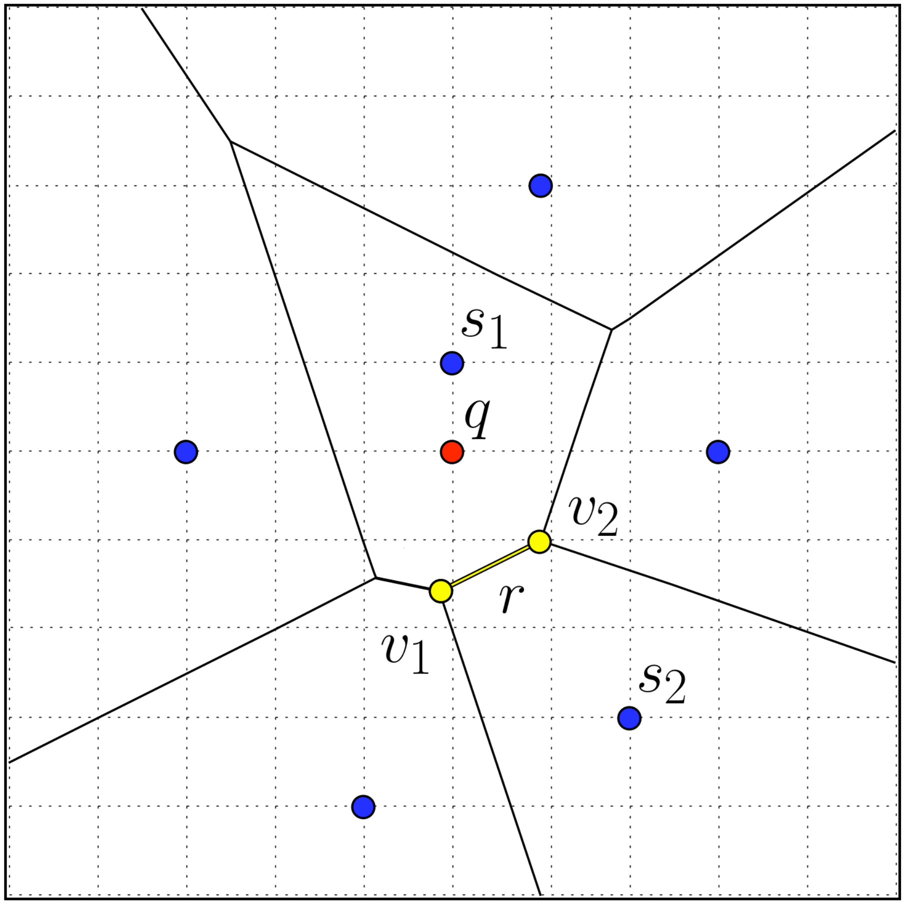


Is $q.x$ left/right of vertex $v_1.x$?
3× precision

Is q above/below segment r?
6× precision [LPT99]

Predicates and their precision



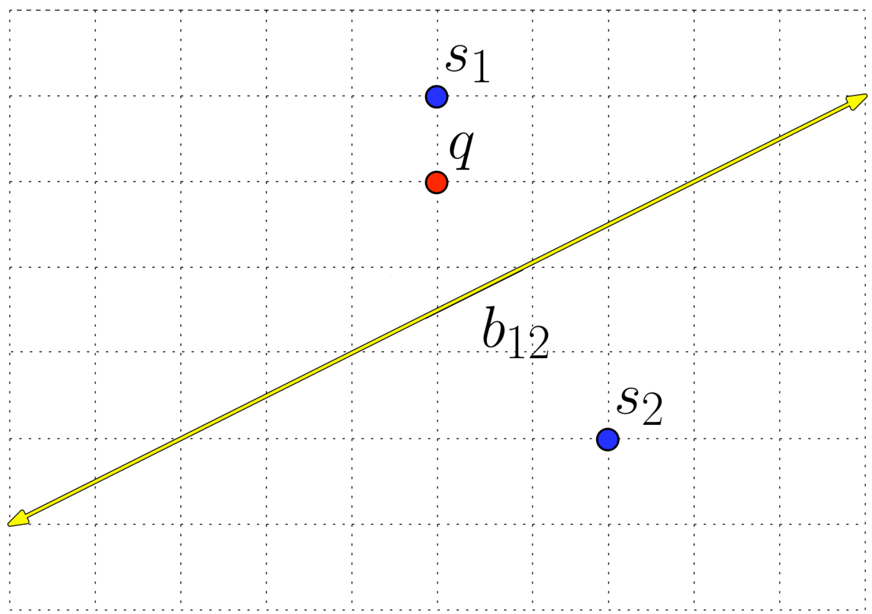


Is *q.x* left/right of vertex $v_1.x$?
3× precision

Is *q* above/below segment *r*?
6× precision [LPT99]

Arithmetic Degree – Side of Bisector

Is q closer to s_1?

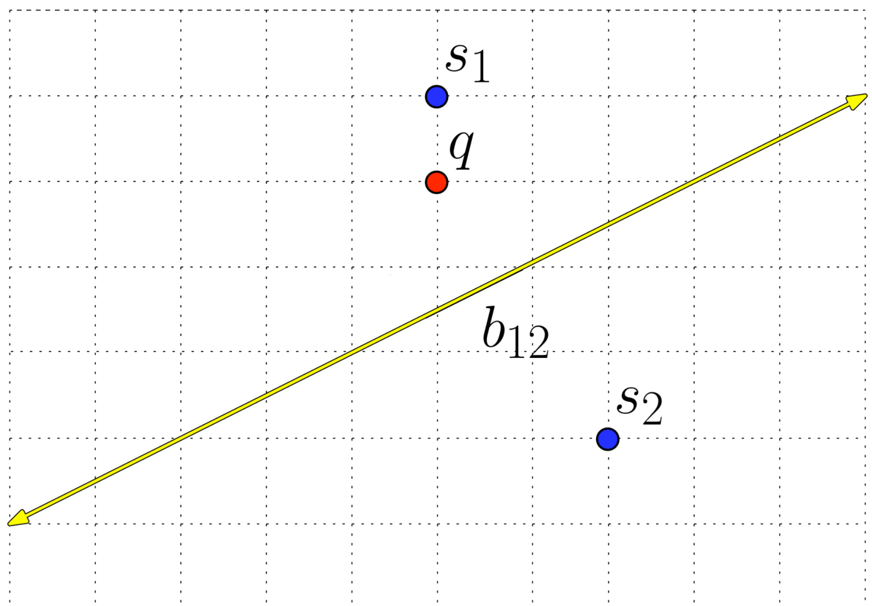


$$\begin{aligned} q &= (x_q, y_q) \\ s_1 &= (x_1, y_1) \\ s_2 &= (x_2, y_2) \end{aligned}$$

$$\|q - s_1\|_2 \lesseqgtr \|q - s_2\|_2$$

Arithmetic Degree – Side of Bisector

Is *q* closer to s_1?



$$q = (x_q, y_q)$$
$$s_1 = (x_1, y_1)$$
$$s_2 = (x_2, y_2)$$

$$\|q - s_1\|_2 \lesseqgtr \|q - s_2\|_2$$

$$(x_q - x_1)^2 + (y_q - y_1)^2 \lesseqgtr (x_q - x_2)^2 + (x_q - y_2)^2$$

Arithmetic Degree – Side of Bisector

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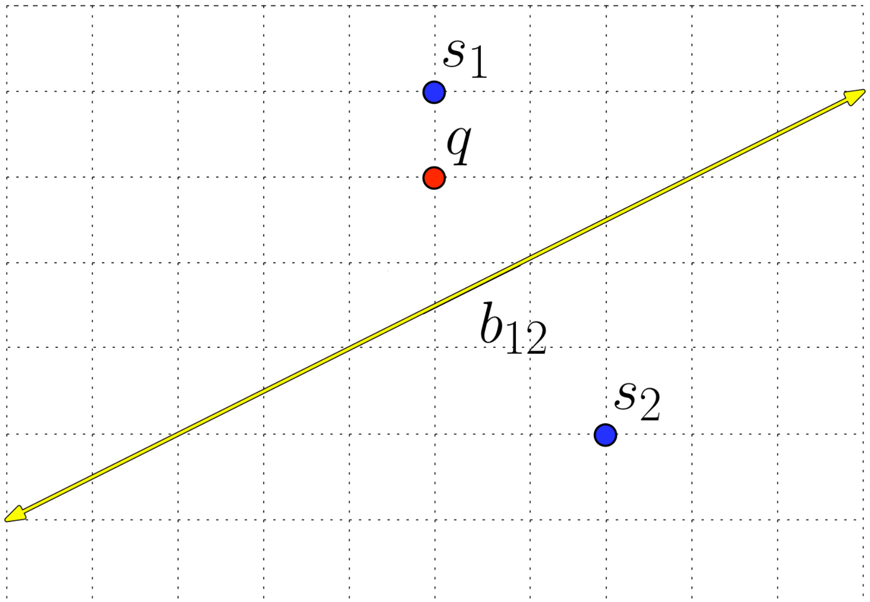

$$
\begin{aligned}
q &= (x_q, y_q) \\
s_1 &= (x_1, y_1) \\
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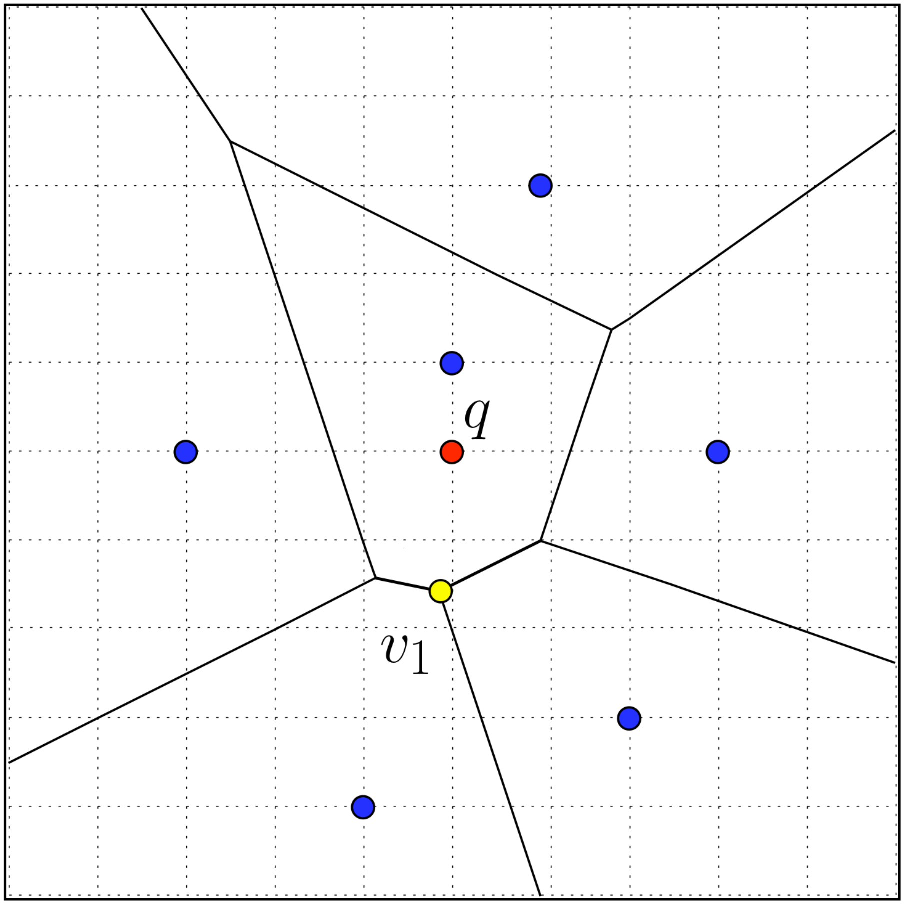
\end{aligned}
$$

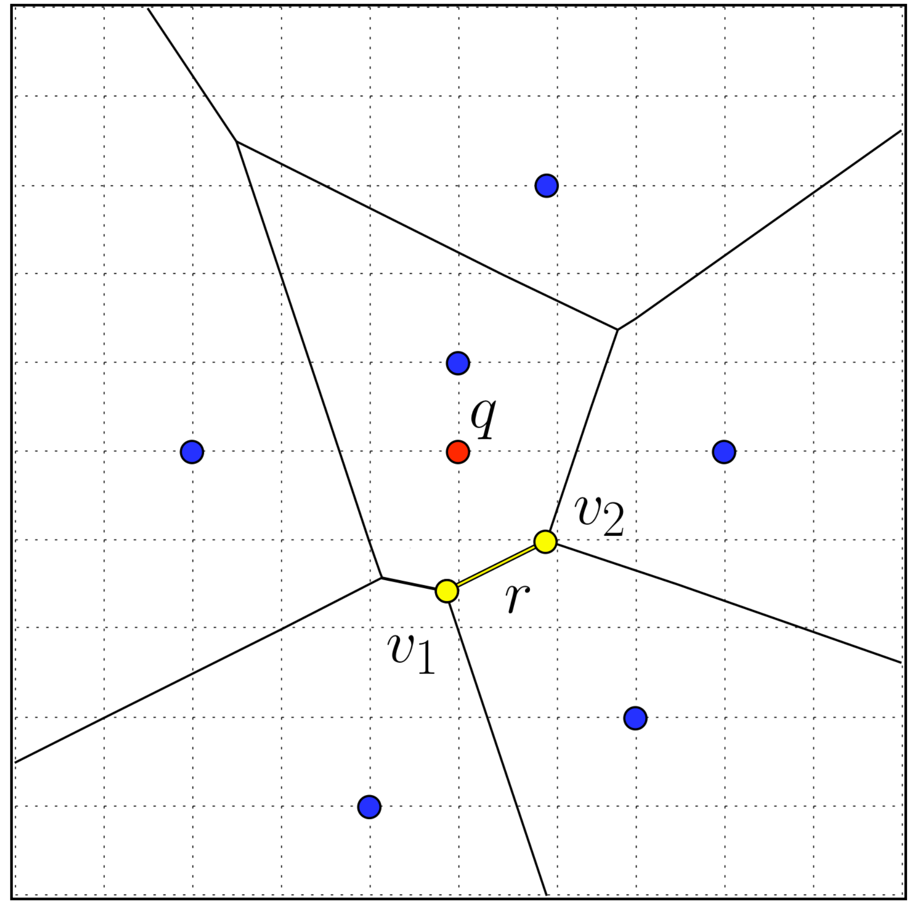
$$\|q - s_1\|_2 \lesseqgtr \|q - s_2\|_2$$

$$(x_q - x_1)^2 + (y_q - y_1)^2 \lesseqgtr (x_q - x_2)^2 + (x_q - y_2)^2$$

Degree 2

Predicates and their precision

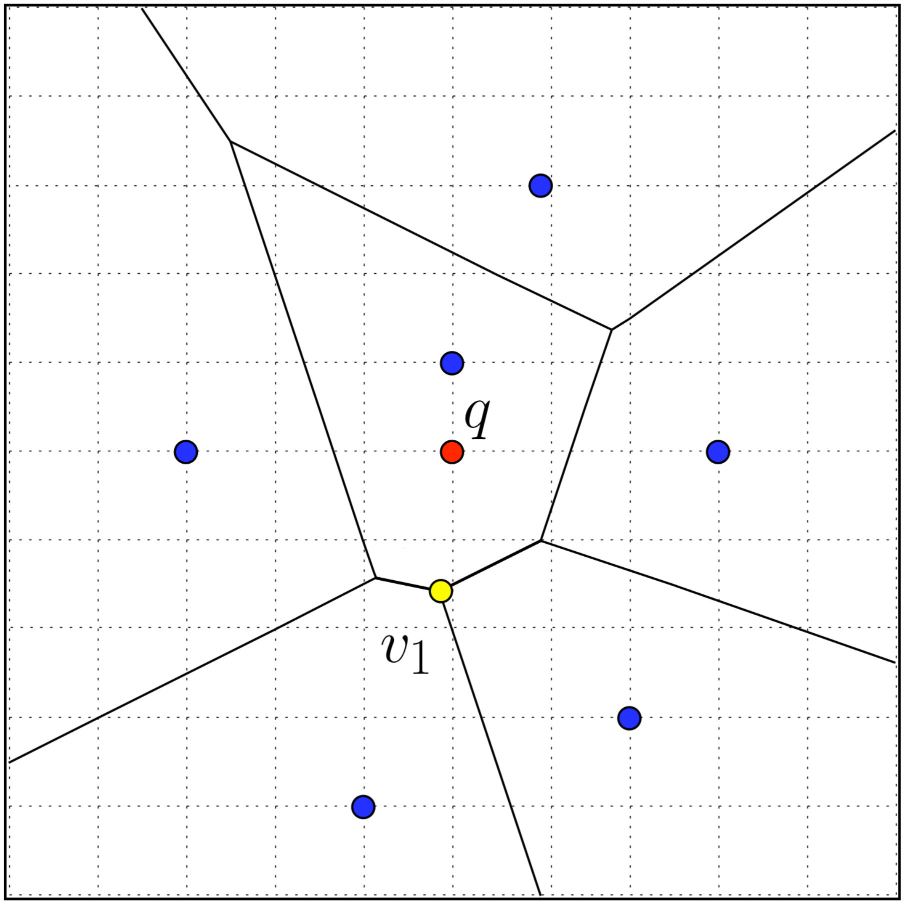


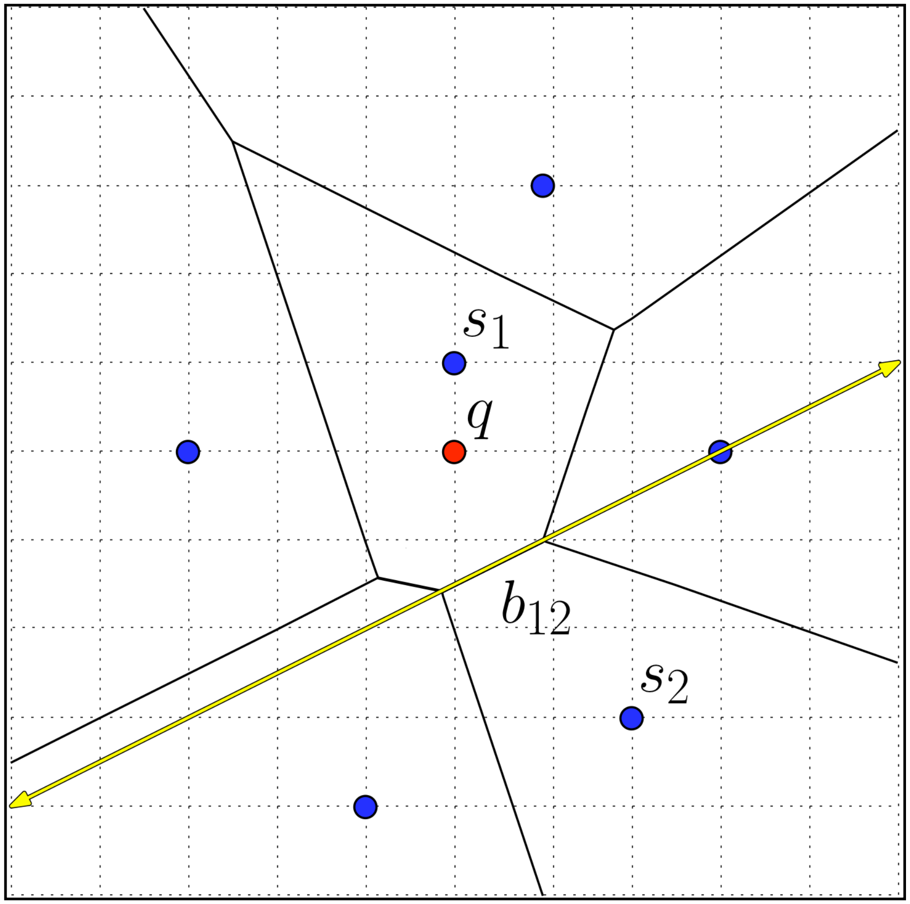


Is *q.x* left/right of vertex $v_1.x$?
degree 3

Is *q* above/below segment *r*?
degree 6

Predicates and their precision

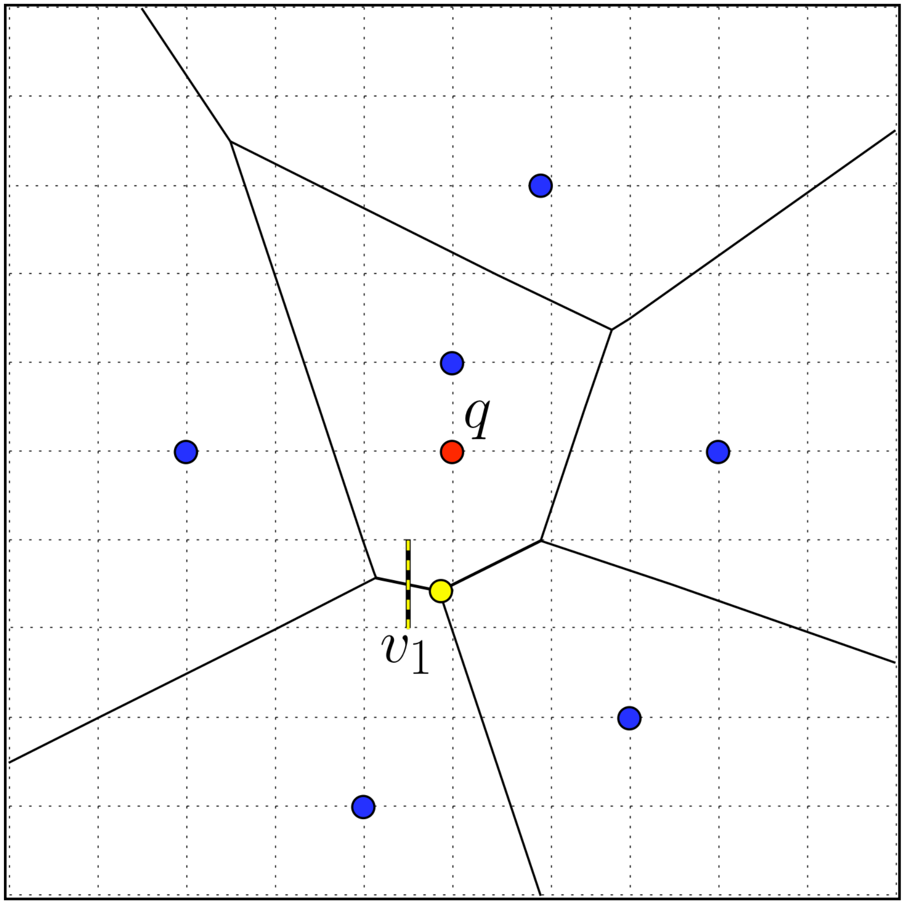


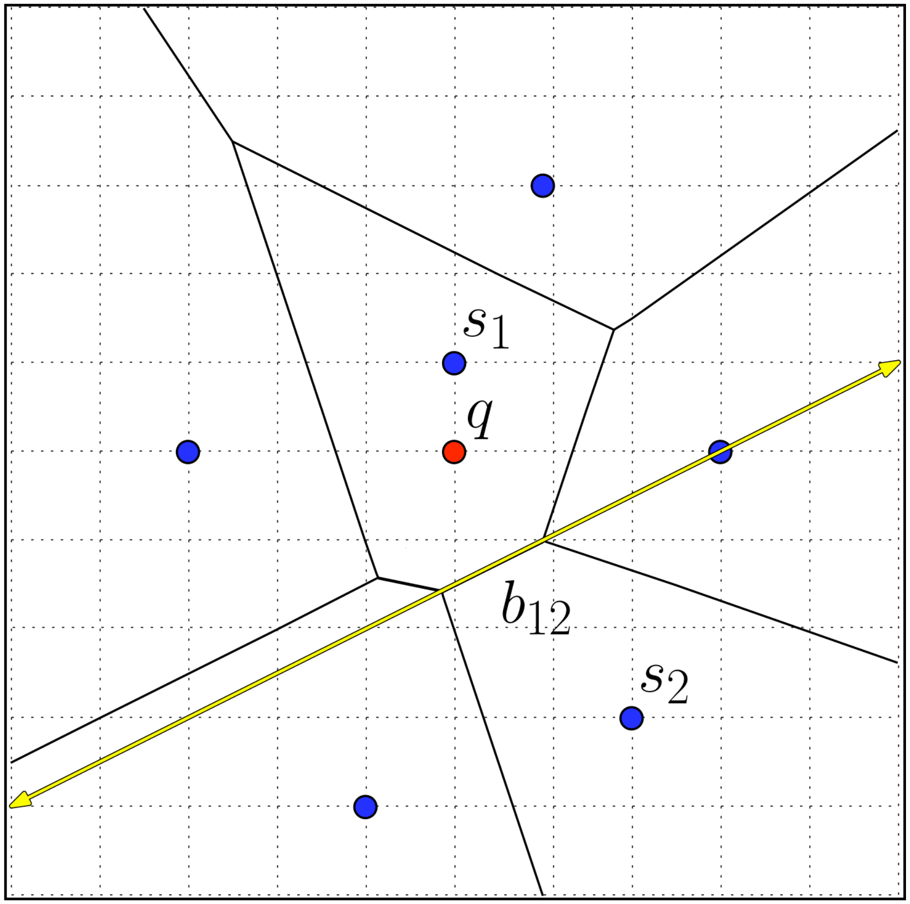


Is *q.x* left/right of vertex $v_1.x$?
degree 3

Is *q* closer to s_1 or s_2?
degree 2

Predicates and their precision

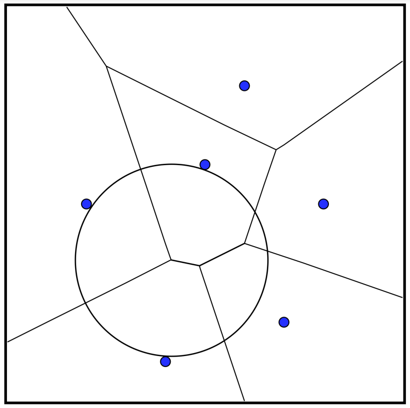




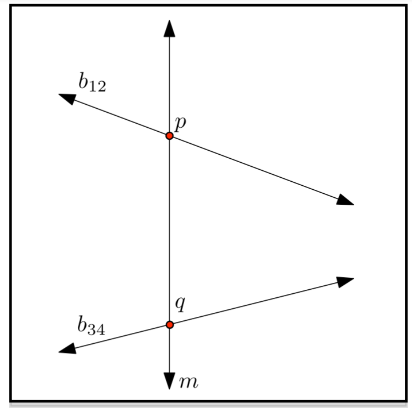
Is $q.x$ left/right of the grid cell containing v_1? [LPT99]
degree 1

Is q closer to s_1 or s_2?
degree 2

Preds, Ops, Constructions and their precision

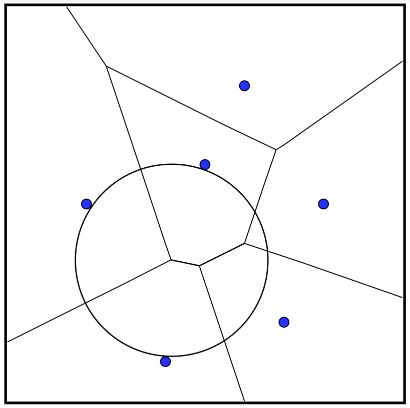


InCircle
degree 4



Vertical order
degree 3

Preds, Ops, Constructions and their precision



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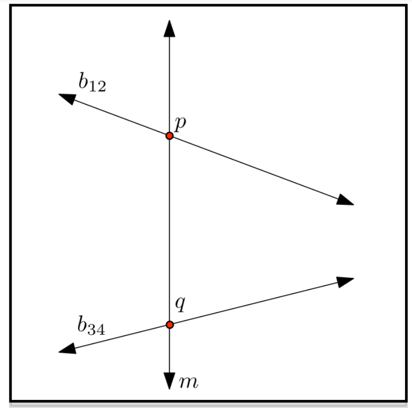

Vertical order
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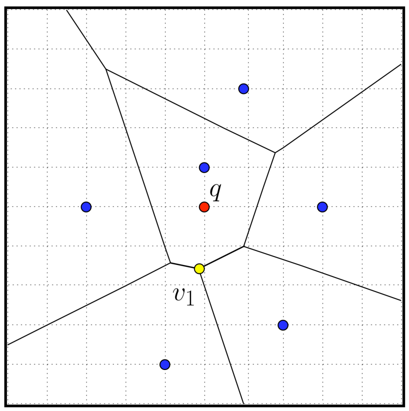
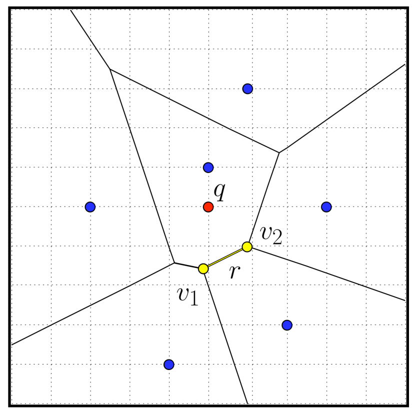
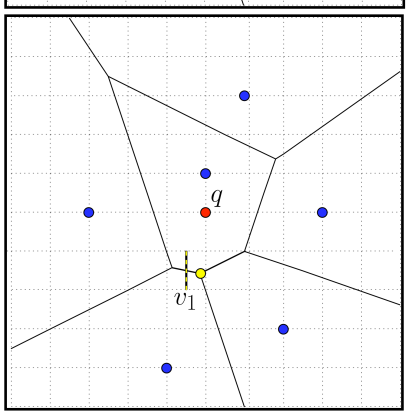


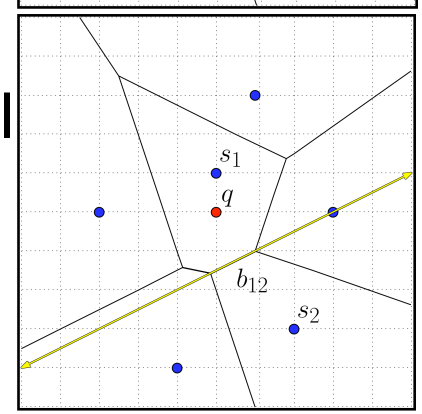
Left/right vertex
degree 3



Above/below seg
degree 6

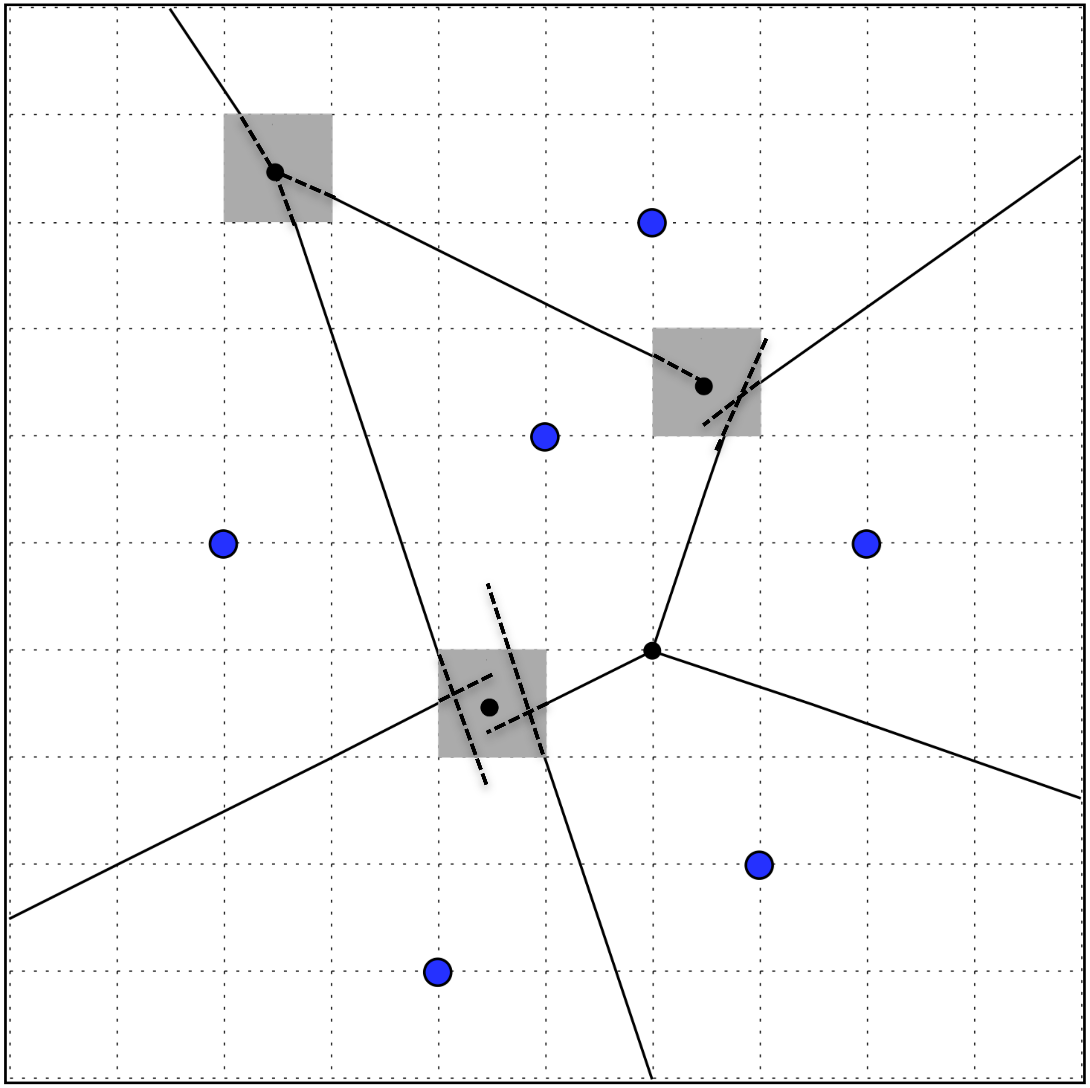


Left/right grid cell
[LPT99]
degree 1

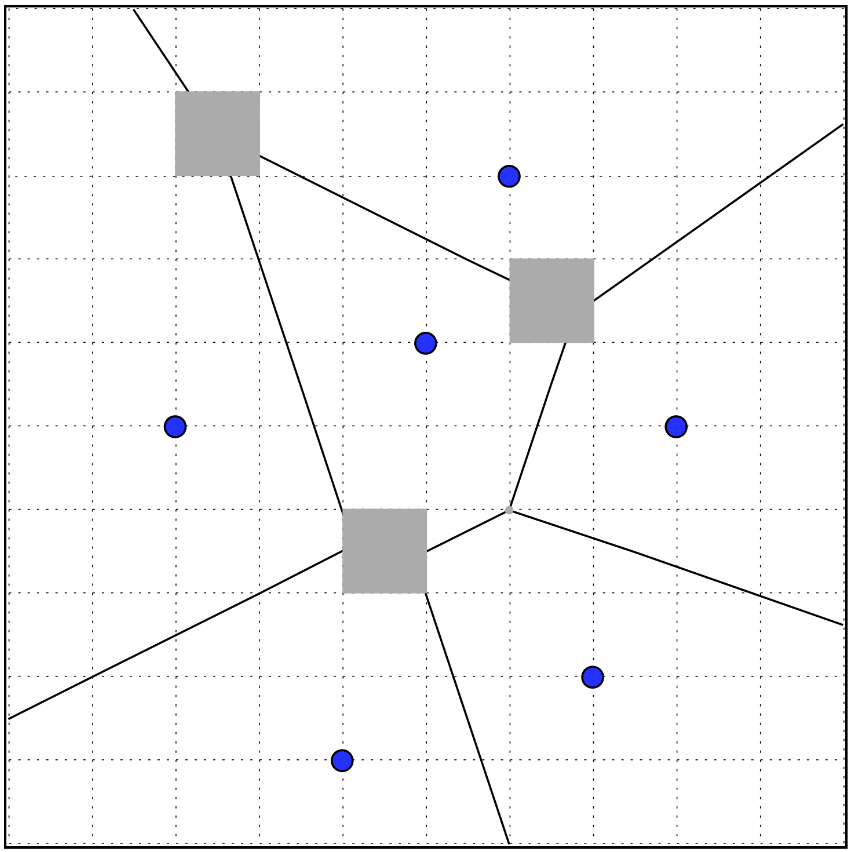


Side of a bisector
degree 2

Implicit Voronoi diagram [LPT99]



Proximity queries w/ min arith precision?



Given:

sites $S = \{s_1, s_2, \ldots, s_n\}$
w/ *b*-bit integer coords

Construct:

Implicit Voronoi with
minimum precision.

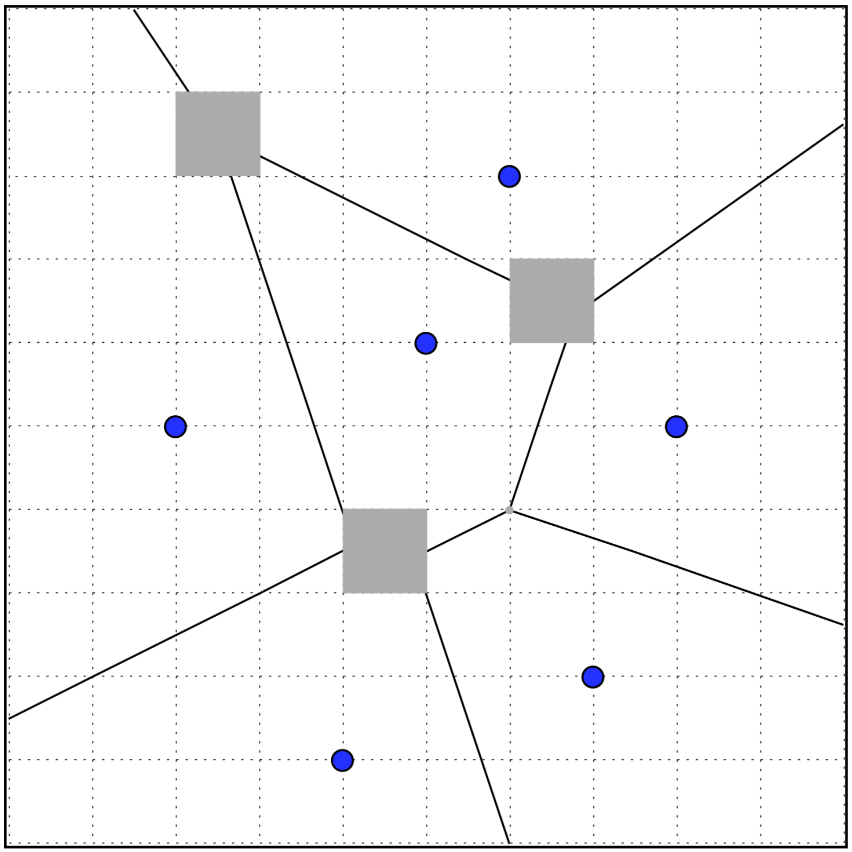
Note: precision < 4*b* bits
precludes computing the
Voronoi Diagram…

Previous work on handling numerics in CG

Implementing geometric algorithms in finite precision computer arithmetic:

- Rely on machine precision (+epsilon)
- Exact Geometric Computation [Y97]
- Arithmetic Filters [FW93][DP99]
- Adaptive Predicates [P92][S97]
- Topological Consistency [SI92]
- Restricted precision algorithm design [LPT99]

Proximity queries w/ min arith precision?



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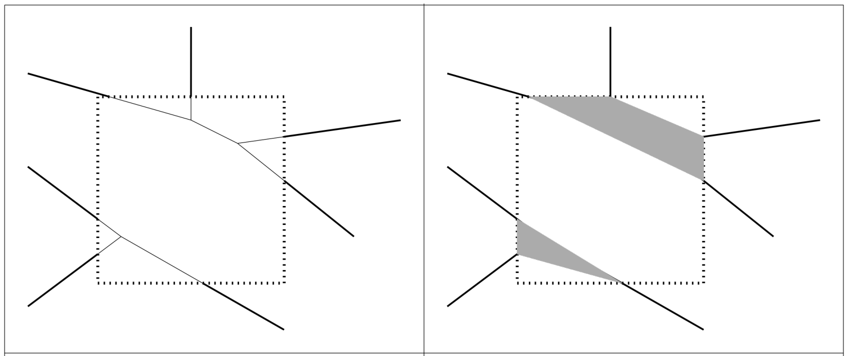
Note: precision < $4b$ bits precludes computing the Voronoi Diagram…

Reduced-precision Voronoi diagram

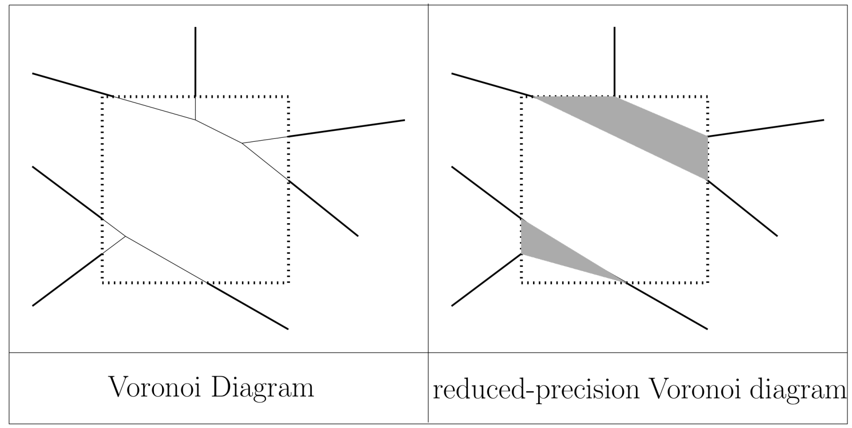
Replace connected subtrees of Vor edges inside a cell with their convex hulls.

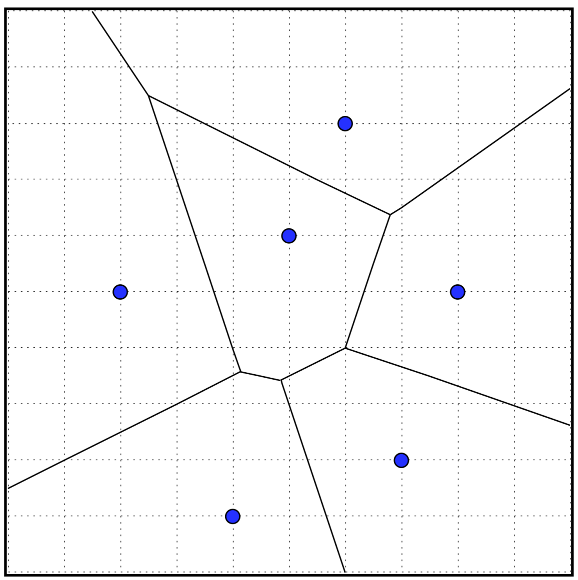


Voronoi Diagram	reduced-precision Voronoi diagram

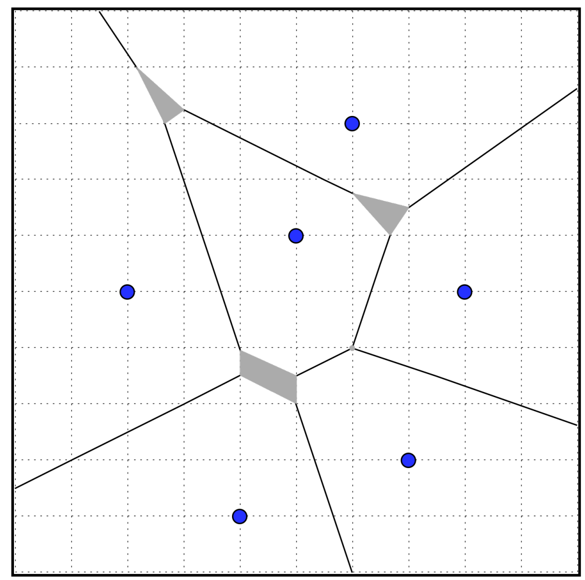
Reduced-precision Voronoi diagram

Replace connected subtrees of Vor edges inside a cell with their convex hulls.





Voronoi diagram



reduced-precision Voronoi

Results (preview)

Randomized Incremental Construction
of the *reduced-precision Voronoi diagram*
of n points on an $m \times m$ grid.

Time: $O(n (\log n + \log m))$ expected

Space: O(n) expected

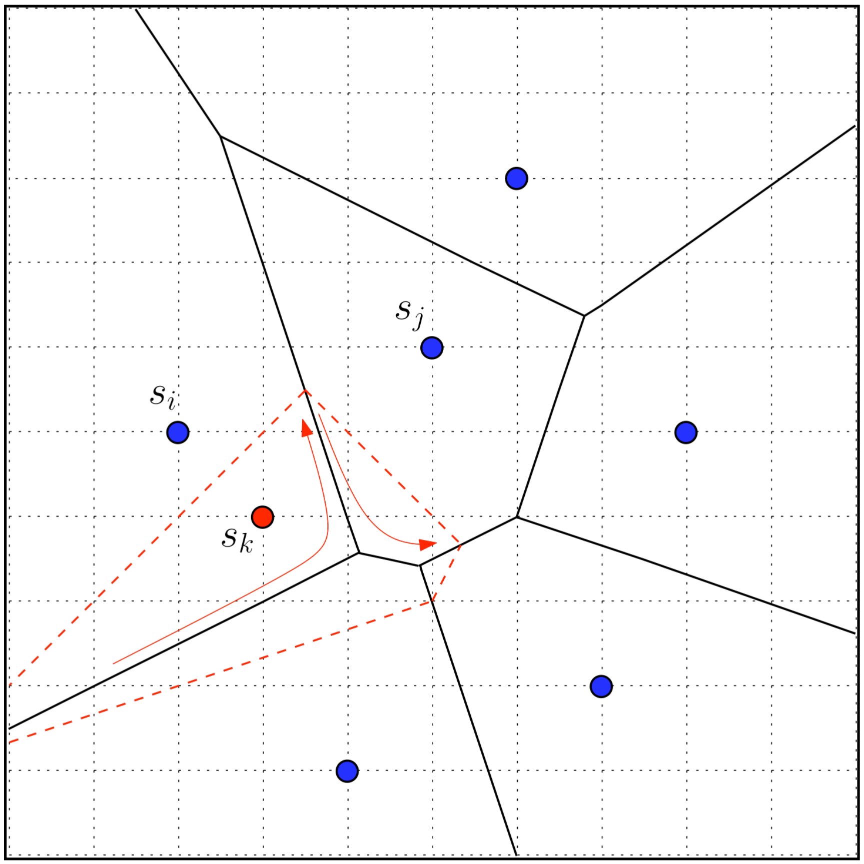
Precision: 3× precision of the input.

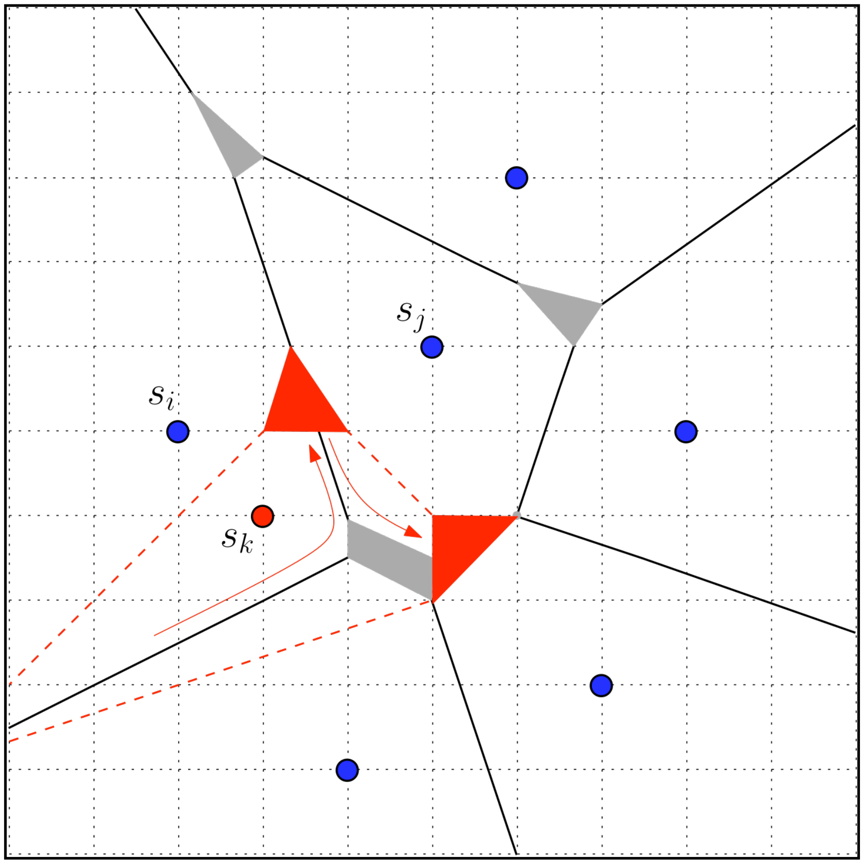
Construction of LPT's implicit Voronoi
w/o computing the full Voronoi diagram.

Rand. Incremental Construction

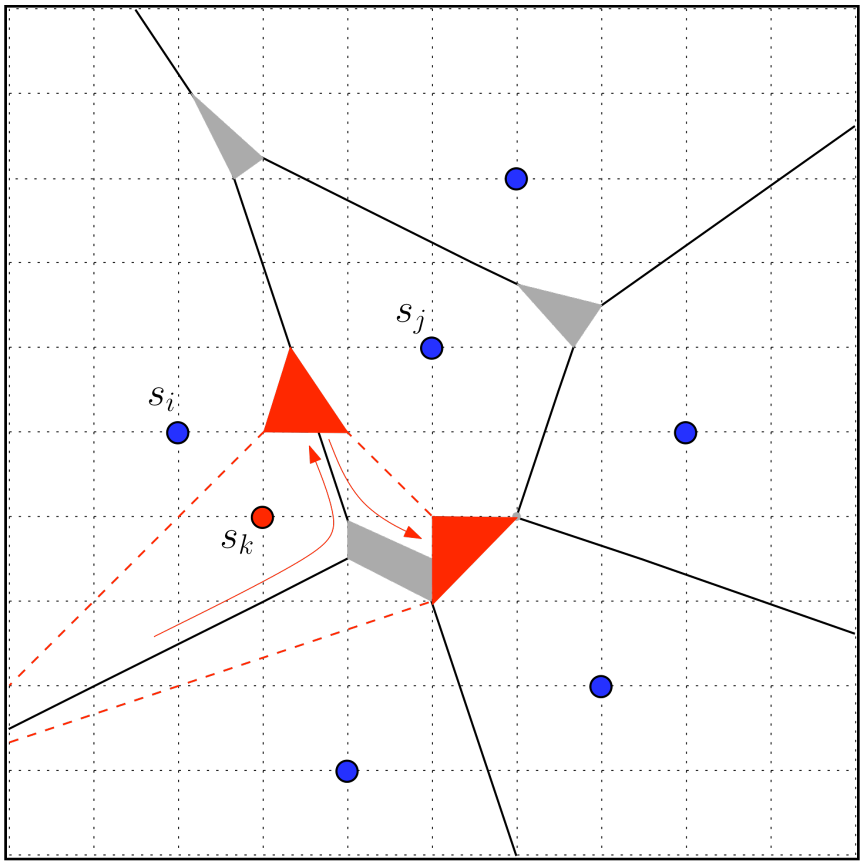
Invariant: Maintain reduced-precision Voronoi as each new site is added.

Update step: Extension of [SI92], walk the deleted tree.





Operations for RIC



What we need:

- Low precision construction for bisector intersection
- Determine the next edge in the tree walk

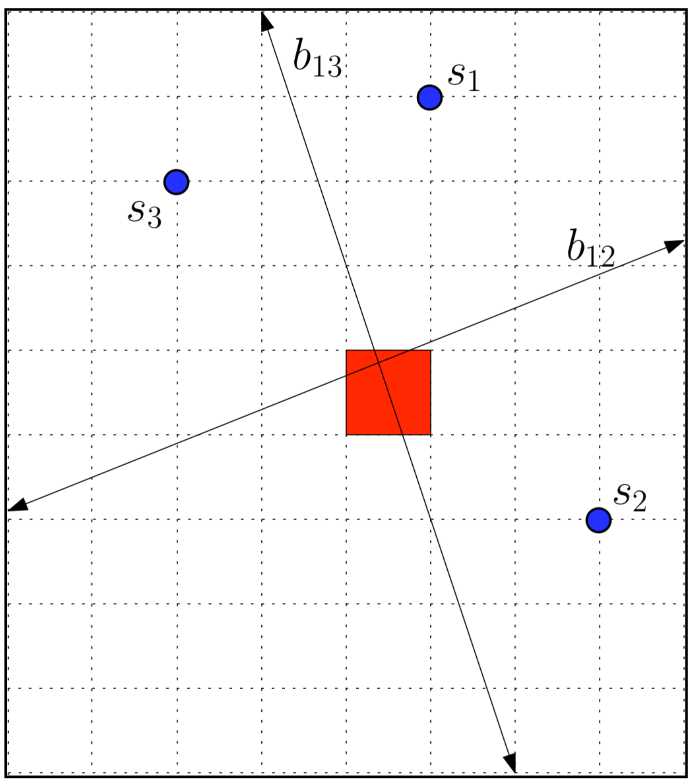
Low precision bisector intersection

Given:

Three sites s_1, s_2 and s_3

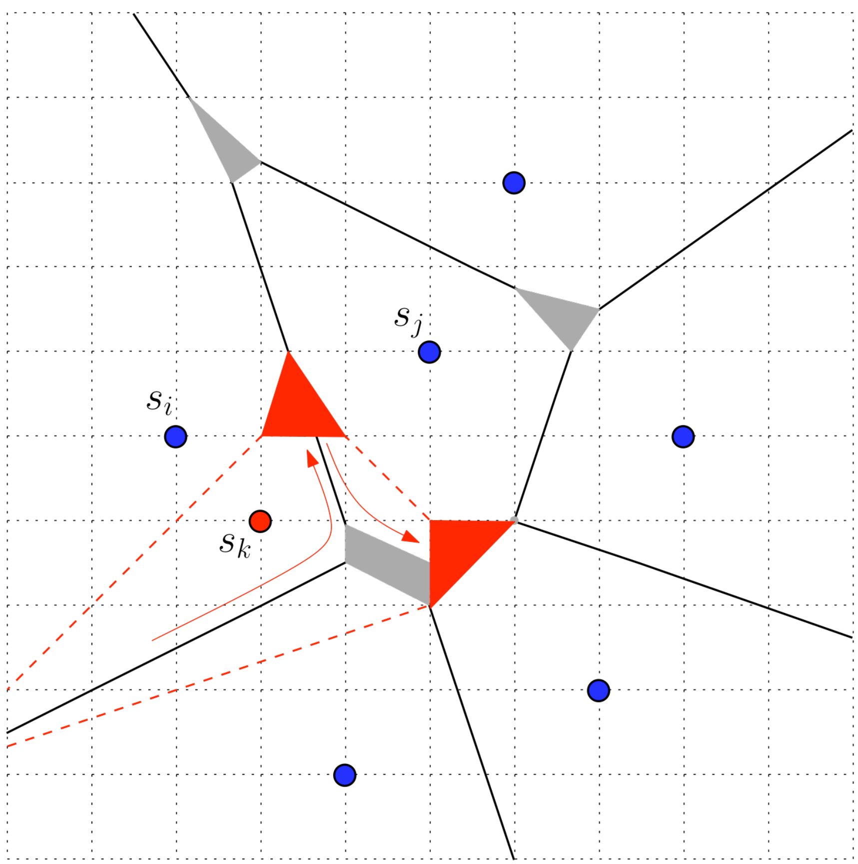
Find:

The grid cell that contains the intersection of bisectors b_{12} and b_{13}



Time: O(log *m*)
Precision: degree 3

Operations for RIC



What we needed:

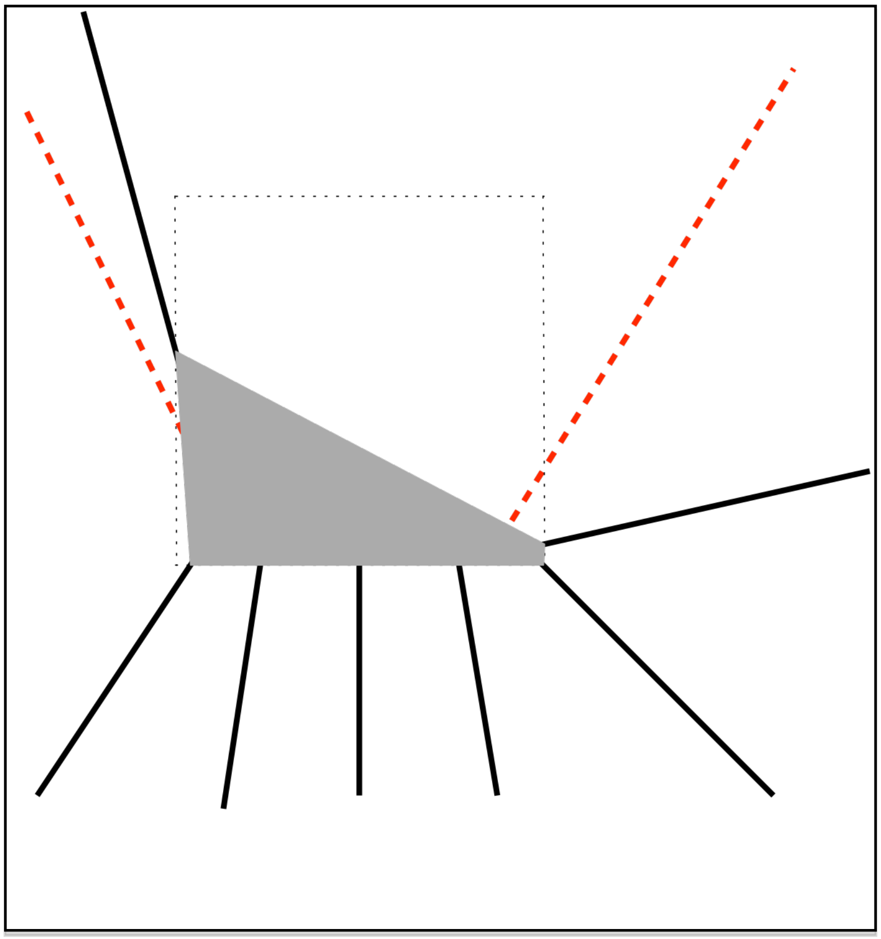
- Low precision construction for bisector intersection
- Determine the next edge in the tree walk

What we have:

- Bisector intersection degree 3, $O(\log m)$

Where to walk next

Where do we walk once we have found an intersection?



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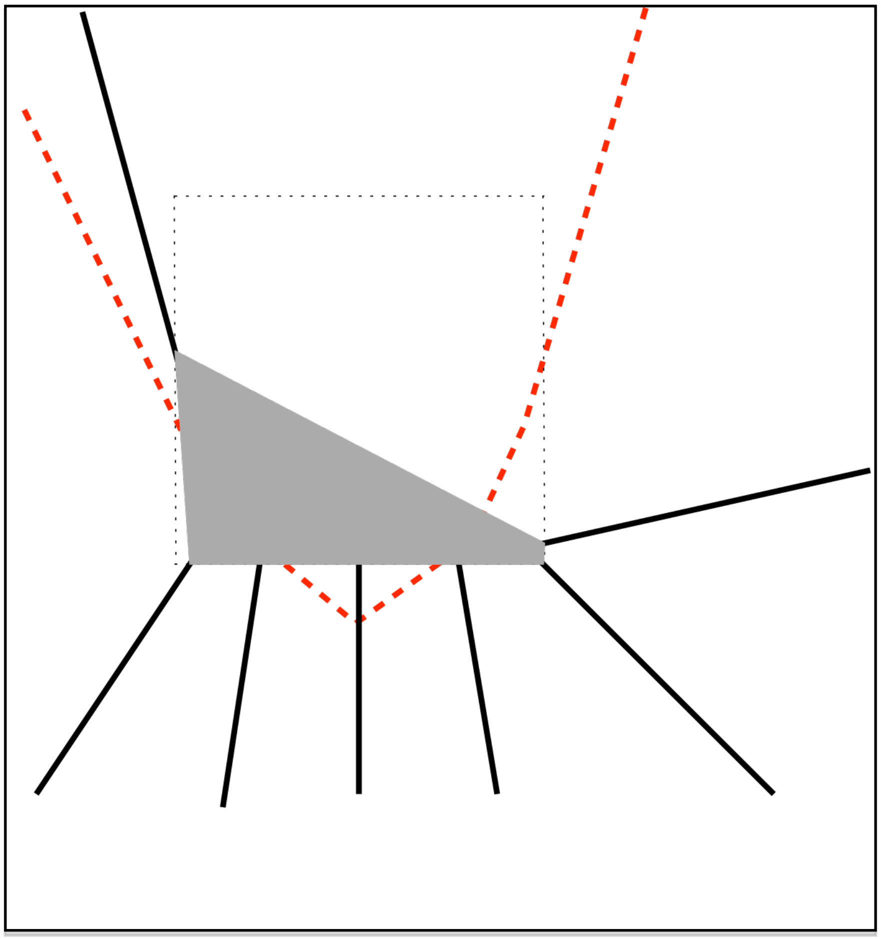

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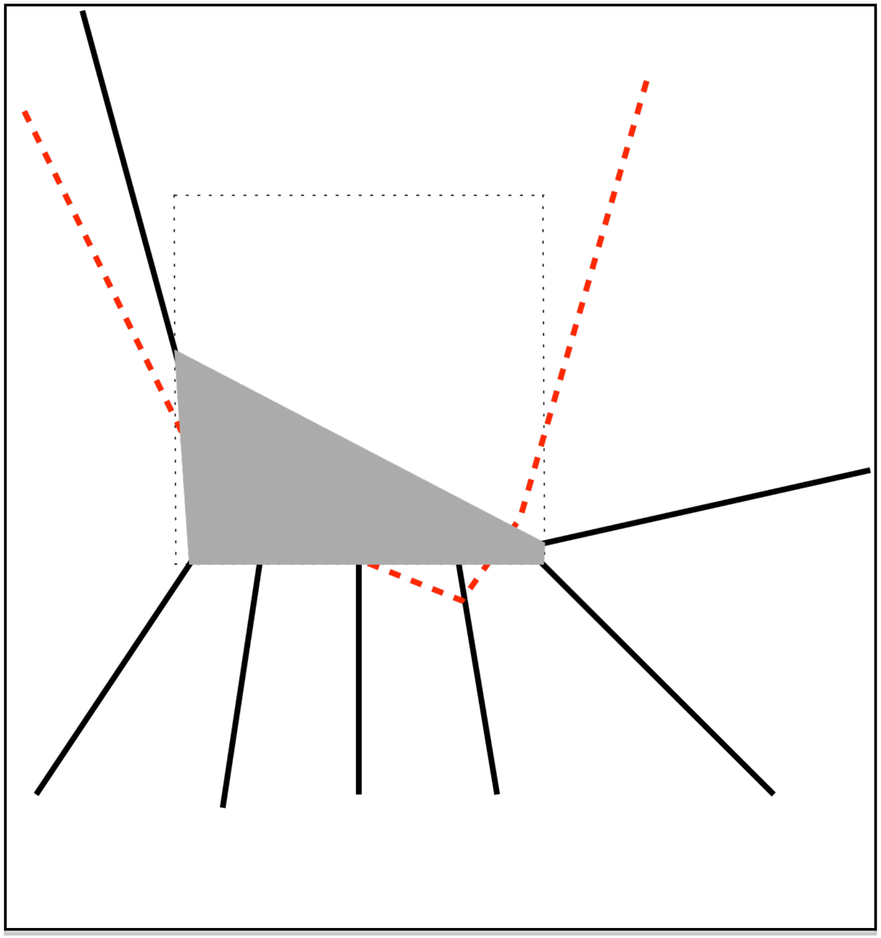

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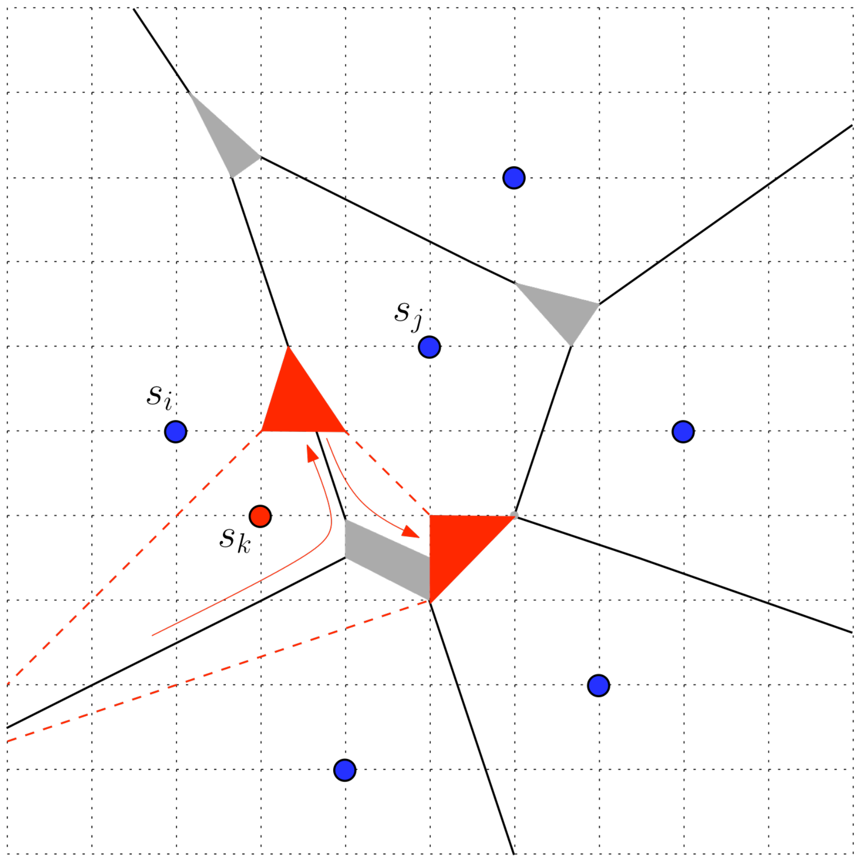


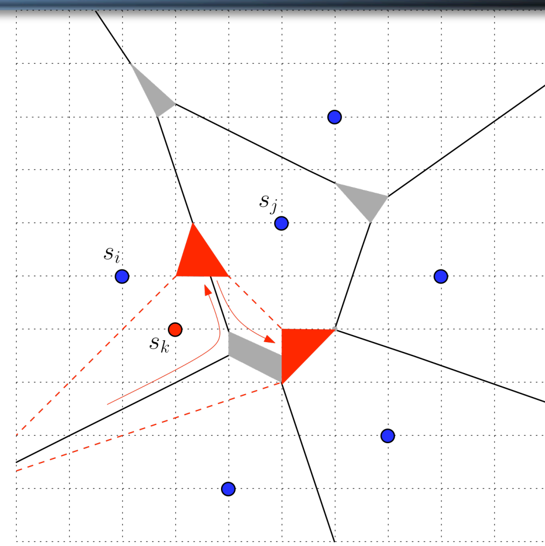
What we needed:

- Low precision construction for bisector intersection
- Determine the next edge in the tree walk

What we have:

- Bisector intersection degree 3, $O(\log m)$
- Next tree edge degree 3, $O(\log n)$

RIC alg for reduced-precision Voronoi



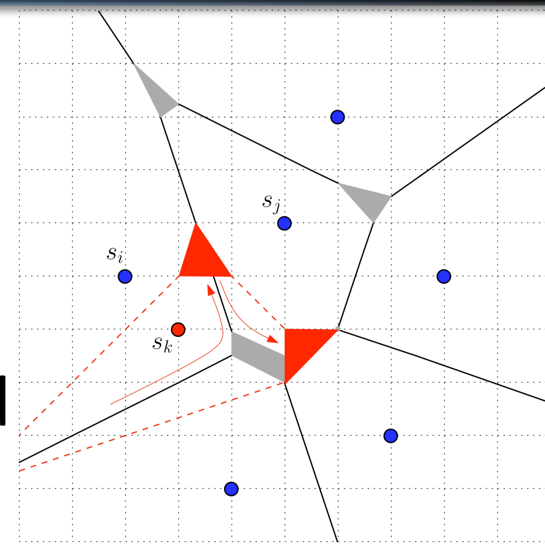
Update step to add a new site s_k

1. Find cell containing s_k
2. Identify bisectors and cells in the new cell of s_k
3. Walk tree inside cell of s_k

   3a. Binary search bisector for crossing

   3b. Compute grid intersection with cell s_k

Analysis



RIC Facts

- Creates θ(*n*) vertices expected
- Point location takes θ(*n* log *n*) expected

Charging scheme

- Binary searchers for (3a) (log *m*) and (3b) (log *n*) are charged to vertex creation.

Therefore,

- It takes O(*n*) space and O(*n* (log *n* + log *m*)) time to build the reduced-precision Voronoi diagram of *n* sites on a grid of size *m* using 3× input precision.

Results

Randomized Incremental Construction
of the *reduced-precision Voronoi diagram*
of n points on an $m \times m$ grid.

Time: $O(n (\log n + \log m))$ expected

Space: O(n) expected

Precision: 3× precision of the input.

Construction of LPT's implicit Voronoi
w/o computing the full Voronoi diagram.

Open problems

Can we…

- create a reduced-precision Voronoi diagram with $2\times$ precision of the input?
- generalize the algorithm to higher dimensions?
- Reduced-precision Voronoi diagrams with other input sites?
- treat precision as a limited resource (like time & space) when solving other algorithmic problems?

THANK YOU!