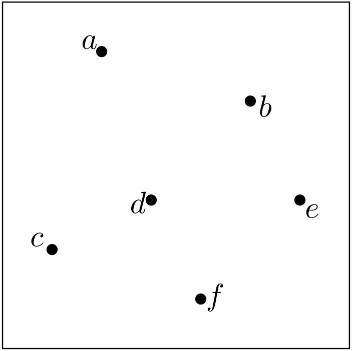
Degree-Driven Design of Geometric Algorithms for Point Location, Proximity, and Volume Calculation

David L. Millman

University of North Carolina at Chapel Hill

October 31, 2012

Numerical Computational Geometry [Y09]



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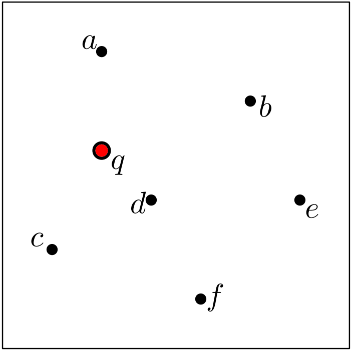

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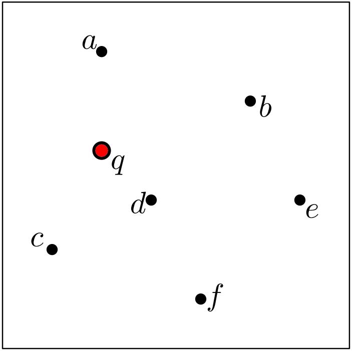
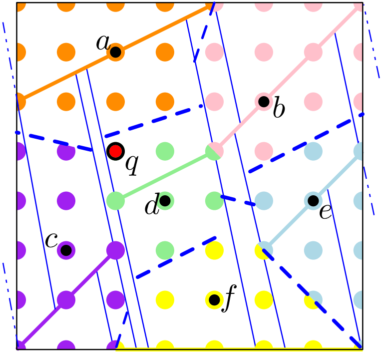
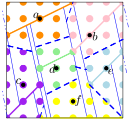
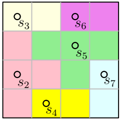




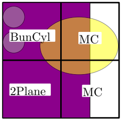
Point location data structure



- Compute point location data structure with double & triple precision

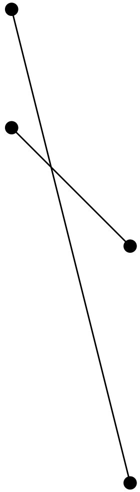


- Compute nearest neighbor transform with double precision



- Compute volumes of CSG models with five-fold precision

A Motivational Problem



DoSegsIntersect:
Given two segments,
defined by their 2D endpoints,
with no three endpoints collinear,
do the segments intersect?

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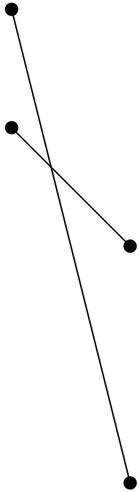

DoSegsIntersect:
Given two segments,
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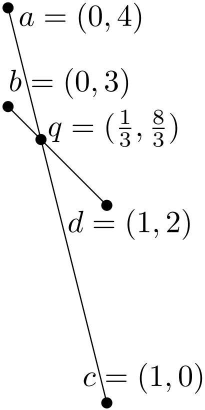
do the segments intersect?

How much arithmetic precision is needed to determine this?

Input Representation

Input: Geometric configuration specified by single precision numerical coordinates and relationships between coordinates.



E.g. DoSegsIntersect problem:

Numerical coordinates:

$(0, 4, 0, 3, 1, 0, 1, 2)$

Relationships between coordinates:

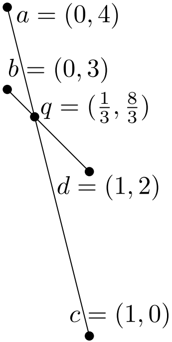
$a = (a_x, a_y) = (0, 4)$
$b = (b_x, b_y) = (0, 3)$
$c = (c_x, c_y) = (1, 0)$
$d = (d_x, d_y) = (1, 2)$
$\overline{ac} = (a, c)$
$\overline{bd} = (b, d)$

Solving DoSegsIntersect with Construction

`InterByConstruction`(a, c, b, d):
Determine if $\overline{ac}$ and $\overline{bd}$ intersect;
if so return INTERSECT, if not return NOINTERSECT

Require: no three points are collinear

if $\overleftrightarrow{ac} \parallel \overleftrightarrow{bd}$ **then**
    return NOINTERSECT
end if
Point $q = \overleftrightarrow{ac} \cap \overleftrightarrow{bd}$
Real $t_1 = (q_x - a_x)/(c_x - a_x)$
Real $t_2 = (q_x - b_x)/(d_x - b_x)$
if $t_1 \in (0, 1)$ and $t_2 \in (0, 1)$ **then**
    return INTERSECT
else
    return NOINTERSECT
end if



Solving DoSegsIntersect with Construction

`InterByConstruction`(a, c, b, d):
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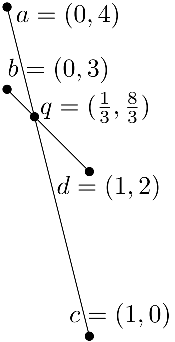
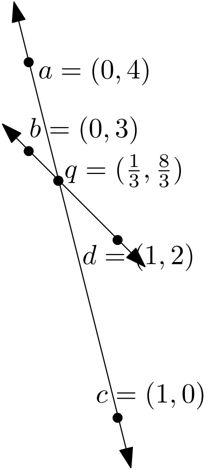
Require: no three points are collinear

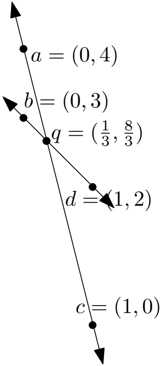
```
if ac⃡ ∥ bd⃡ then
    return NOINTERSECT
end if
Point q = ac⃡ ∩ bd⃡
Real t1 = (qx − ax)/(cx − ax)
Real t2 = (qx − bx)/(dx − bx)
if t1 ∈ (0, 1) and t2 ∈ (0, 1) then
    return INTERSECT
else
    return NOINTERSECT
end if
```



Line 4: Point $q = \overleftrightarrow{ac} \cap \overleftrightarrow{bd}$



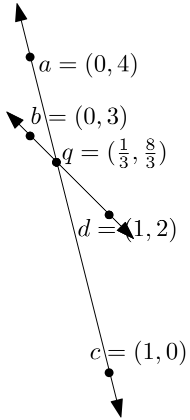
The `Intersect`(a, c, b, d) construction:



Input: single precision coordinates of a, c, b and d defining non-parallel lines $\overleftrightarrow{ac}$ and $\overleftrightarrow{bd}$.
Construct: the intersection q of $\overleftrightarrow{ac}$ and $\overleftrightarrow{bd}$.

$$q_x = \frac{\begin{vmatrix} a_x c_y - c_x a_y & a_x - c_x \\ b_x d_y - d_x b_y & b_x - d_x \end{vmatrix}}{\begin{vmatrix} a_x - c_x & a_y - c_y \\ b_x - d_x & b_y - d_y \end{vmatrix}}, q_y = \frac{\begin{vmatrix} a_x c_y - c_x a_y & a_y - c_y \\ b_x d_y - d_x b_y & b_y - d_y \end{vmatrix}}{\begin{vmatrix} a_x - c_x & a_y - c_y \\ b_x - d_x & b_y - d_y \end{vmatrix}}$$

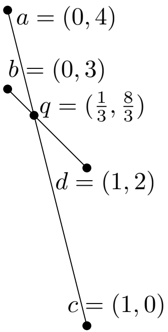
The `Intersect`(a, c, b, d) *construction*:



Input: single precision coordinates of a, c, b and d defining non-parallel lines $\overleftrightarrow{ac}$ and $\overleftrightarrow{bd}$.
Construct: the intersection q of $\overleftrightarrow{ac}$ and $\overleftrightarrow{bd}$.

$$\begin{aligned} q_x &= 0.\overline{3} \\ q_y &= 2.\overline{6} \end{aligned}$$

The `Intersect`(a, c, b, d) construction:

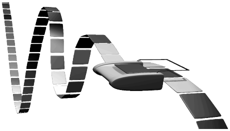


Input: single precision coordinates of a, c, b and d defining non-parallel lines $\overleftrightarrow{ac}$ and $\overleftrightarrow{bd}$.
Construct: the intersection q of $\overleftrightarrow{ac}$ and $\overleftrightarrow{bd}$.
In Python with *numpy.float32* type[a]:

$$
\begin{aligned}
\mathtt{fl}(q_x) &\approx 0.33333334 \\
\mathtt{fl}(q_y) &\approx 2.66666675 \\
\mathtt{fl}(q) &\notin \mathtt{fl}(\overline{ac}) \;\&\; \mathtt{fl}(q) \notin \overline{ac} \\
\mathtt{fl}(q) &\notin \mathtt{fl}(\overline{bd}) \;\&\; \mathtt{fl}(q) \notin \overline{bd}
\end{aligned}
$$

[a] Values are the shortest decimal fraction that rounds correctly back to the true binary value.

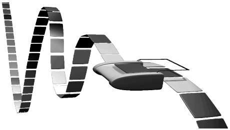
Real-RAM



Real-RAM has 3 unbounded quantities.
The number of:

1. steps an algorithm may take
2. memory cells an algorithm may use
3. bits for representing numbers in cells

Image from: `http://en.wikipedia.org/wiki/File:Maquina.png`



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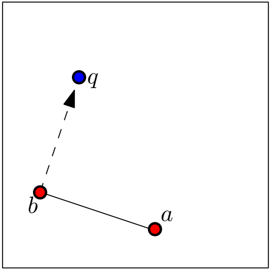
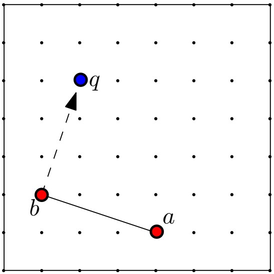
Precision used by the `isRightTurn`:



Input: single precision coordinates of *a*, *b* and *q*.
Return: whether the straight line path from *a* to *b* to *q* forms a right turn.

Analyzing Precision [LPT99]

Precision used by the `isRightTurn`:



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$a, b, q \in \mathbb{U}$
$a = (a_x, a_y)$
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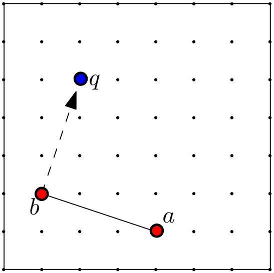

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A *predicate* is a test of the sign of a multivariate polynomial with variables from the input coordinates.

$$\texttt{Orientation}(a, b, q) \quad = \quad \mathrm{sign}(b_x q_y - b_x a_y - a_x q_y - q_x b_y + q_x a_y + a_x b_y)$$

$\texttt{Orientation} < 0$ Right turn
$\texttt{Orientation} > 0$ Left turn
$\texttt{Orientation} = 0$ Collinear

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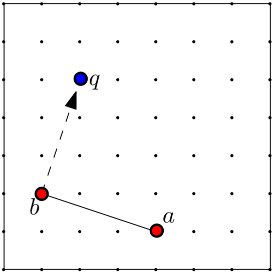

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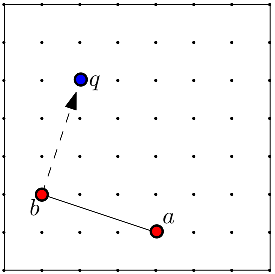

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How the degree relates to precision:

Consider multivariate poly $Q(x_1, \ldots, x_n)$ of deg k and s monomials (for simplicity, assume that coefficient of each monomial is 1).

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Note that ℓk bits is enough to evaluate the sign.

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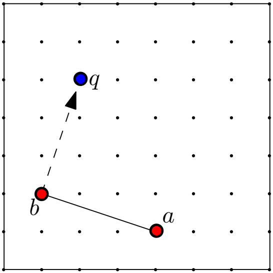
Precision used by the `isRightTurn`:



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`Orientation` is degree 2

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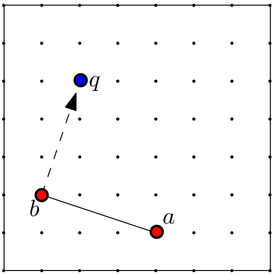

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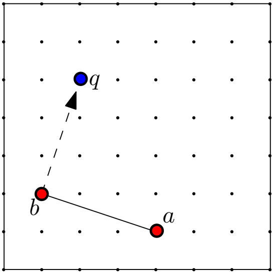
`Orientation` is degree 2

`isRightTurn`(a, b, q):

1: **if** `Orientation`$(a, b, q) < 0$ **then**
2: **return** TRUE
3: **else**
4: **return** FALSE
5: **end if**

Analyzing Precision [LPT99]

Precision used by the `isRightTurn`:



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`isRightTurn` is degree 2

`isRightTurn`(a, b, q):

1: **if** `Orientation`$(a, b, q) < 0$ **then**
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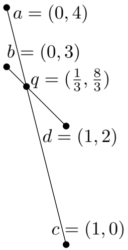
Solving DoSegsIntersect without Construction

`InterByOrientation`(a, c, b, d):
Determine if $\overline{ac}$ and $\overline{bd}$ intersect;
if so return INTERSECT, if not return NOINTERSECT

Require: no three points are collinear

if `Orientation`$(a, c, b) \neq$ `Orientation`(a, c, d) and `Orientation`$(b, d, a) \neq$ `Orientation`(b, d, c) **then**
  return INTERSECT
else
  return NOINTERSECT
end if



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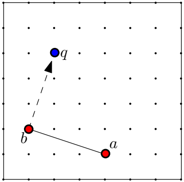
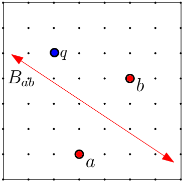
1: **if** `Orientation`$(a, c, b) \neq$ `Orientation`(a, c, d) and `Orientation`$(b, d, a) \neq$ `Orientation`(b, d, c) **then**
2: **return** INTERSECT
3: **else**

In summary:
`Orientation` predicate is degree 2
`InterByOrientation` algorithm is degree 2
`InterByConstruction` algorithm is degree 3

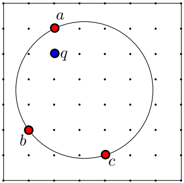
Some other well known predicates:



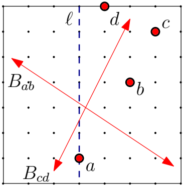
`Orientation`(a, b, q)
degree 2



`SideOfBisector`(B_{ab}, q)
degree 2



`InCircle`(a, b, c, q)
degree 4



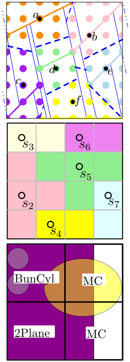
`OrderOnLine`(B_{ab}, B_{cd}, ℓ)
degree 3

Precision/Robust Techniques

Techniques for implementing geometric algorithms
using finite precision computer arithmetic:

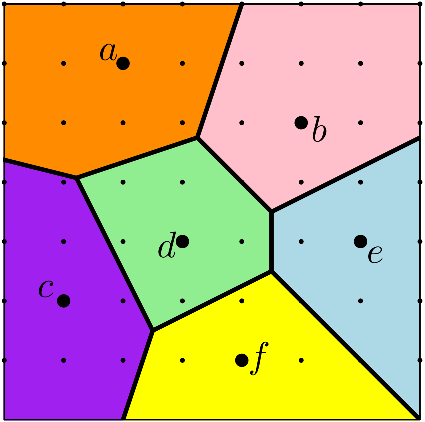
- Rely on machine precision ($+\epsilon$) [NAT90,LTH86,KMP*08]
- Topological Consistency [S99, S01, SI90, SI92, SII*00]
- Exact Geometric Computation [Y97]
  - Software based arithmetic [CORE, LEDA, GMP, MPFR]
  - Predicate eval schemes [ABO*97, FW93, BBP01, S97]
  - Degree-driven algorithm design [LPT99]
    and [BP00,BS00,C00,MS01,MS09,MS10,MV11,MLC*12]

Overview



- Compute point location data structure with double & triple precision
- Compute nearest neighbor transform with double precision
- Compute volumes of CSG models with five-fold precision

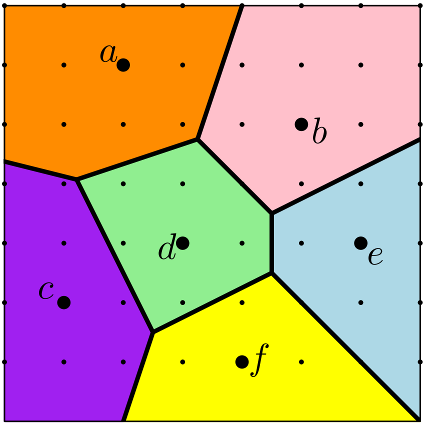
Point Location Data Structure



Given
A grid of size U and
sites $S = \{s_1, \ldots, s_n\} \subset \mathbb{U}$

Compute
A data structure capable of returning the closest $s_i \in S$ to a query point $q \in \mathbb{U}$ in $O(\log n)$ time

Precision of Voronoi Diagram/Trapezoid Graph



Voronoi diagram

- region

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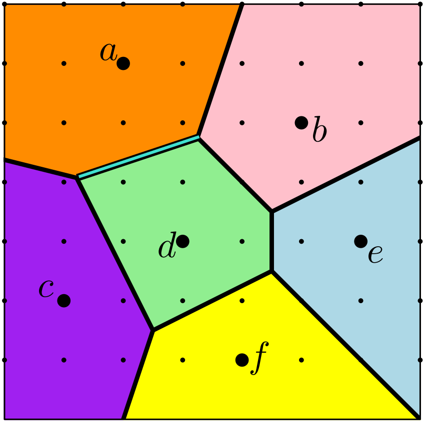
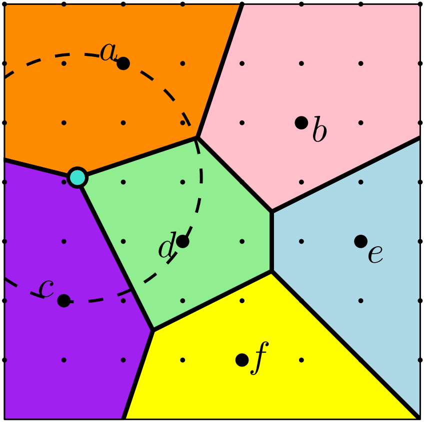


Voronoi diagram

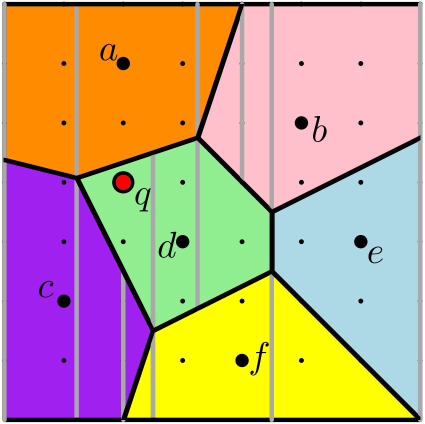
- region
- edge

Precision of Voronoi Diagram/Trapezoid Graph



Voronoi diagram

- region
- edge
- vertex

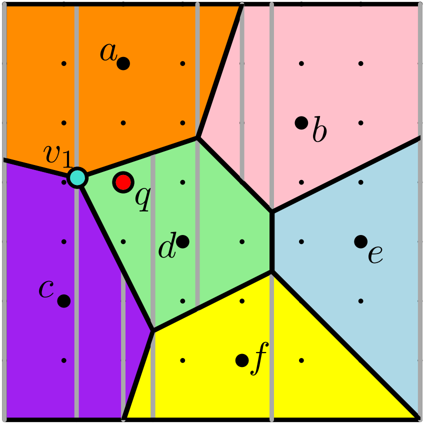


Voronoi diagram

- region
- edge
- vertex

Trapezoid graph for proximity queries

Precision of Voronoi Diagram/Trapezoid Graph

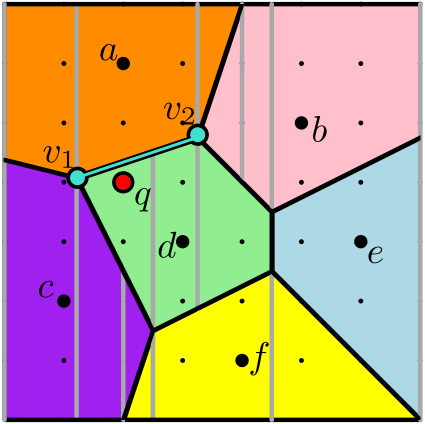


Voronoi diagram

- region
- edge
- vertex

Trapezoid graph for proximity queries

- x-`node`() – degree 3



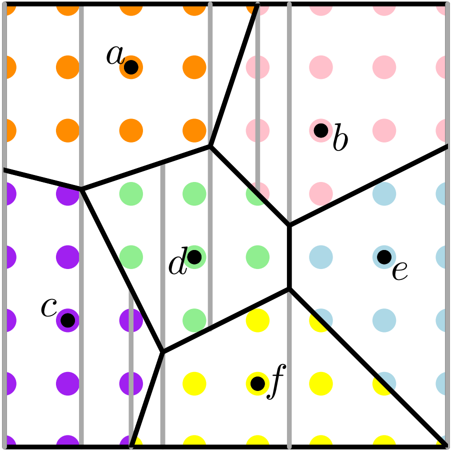
Voronoi diagram

- region
- edge
- vertex

Trapezoid graph for proximity queries

- *x*-`node()` – degree 3
- *y*-`node()` – degree 6

Precision of Voronoi Diagram/Trapezoid Graph



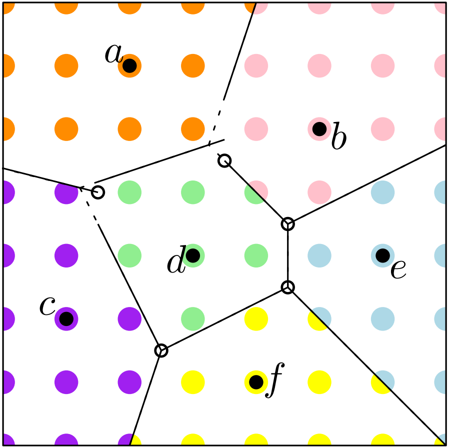
Voronoi diagram

- region
- edge
- vertex

Trapezoid graph for proximity queries

- *x*-`node()` – degree 3
- *y*-`node()` – degree 6

Precision of Voronoi Diagram/Trapezoid Graph



Voronoi diagram

- region
- edge
- vertex

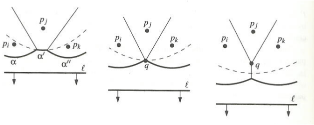
Trapezoid graph for proximity queries [LPT99]

- *x*-`node()` – degree 1
- *y*-`node()` – degree 2

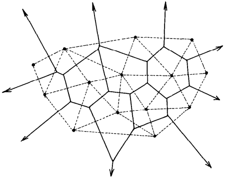
The *Implicit Voronoi diagram* is a degree 2 trapezoid graph.

Precision of Constructing the Voronoi Diagram

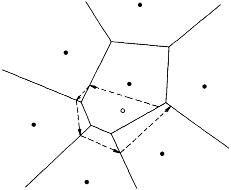
Three well-known Voronoi diagram constructions.



Sweepline[F87]
– degree 6



Divide and Conquer[GS86]
– degree 4



Tracing[SI92]
– degree 4

Three well-known Voronoi diagram constructions.

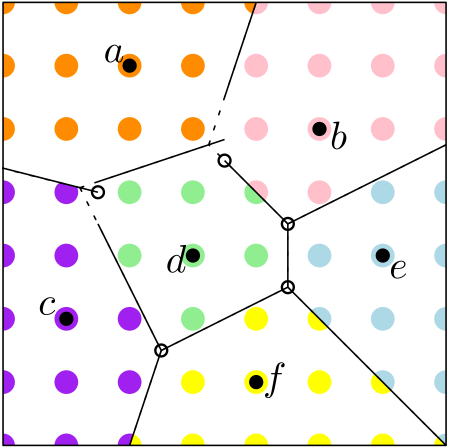
eepline[F87]
degree 6

de and Conquer[GS86]
degree 4

ing[SI92]
degree 4

How do we build
a degree 2 trapezoid graph
for proximity queries
when we can't even construct
a Voronoi vertex?

Implicit Voronoi Diagram [LPT99]



Implicit Voronoi diagram
is disconnected.

Reduced Precision Voronoi [MS09]

Given n sites in $\mathbb{U}$

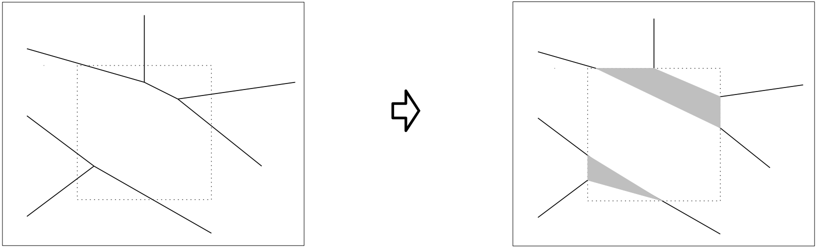
RP-Voronoi randomized incremental construction

- Time: $O(n \log(Un))$ expected
- Space: $O(n)$ expected
- Precision: degree 3

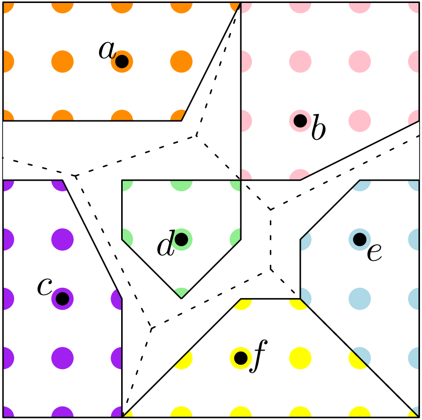
LPT's Implicit Voronoi constructed from RP-Voronoi

- Time: $O(n)$
- Space: $O(n)$
- Precision: degree 3

Reduced Precision Voronoi [MS09]

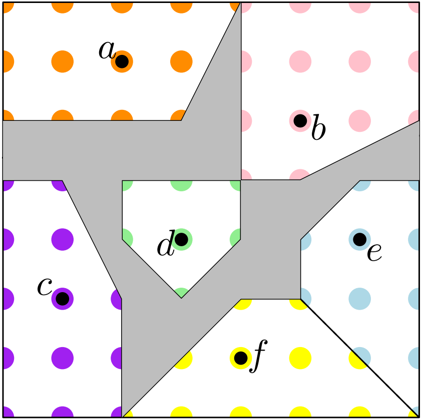


Contract trees of Voronoi vertices that occur in the same grid cell into an *rp-vertex*.



- *Voronoi polygon* is the convex hull of the grid points in a Voronoi cell.

Voronoi Polygon Set



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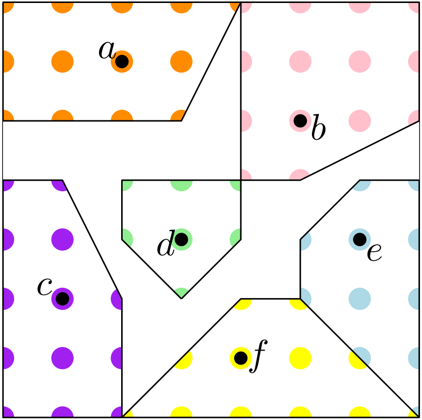
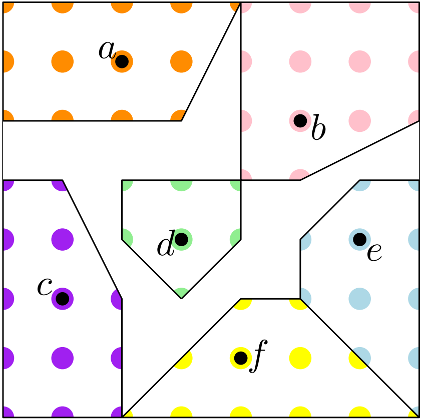
- Gaps

Voronoi Polygon Set



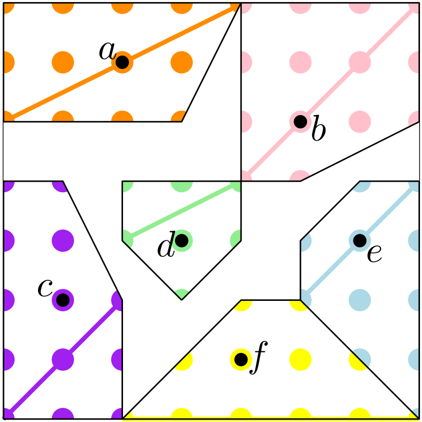
- *Voronoi polygon* is the convex hull of the grid points in a Voronoi cell.
- Gaps
- *Voronoi polygon set* is the collection of the *n* Voronoi polygons.

Voronoi Polygon Set



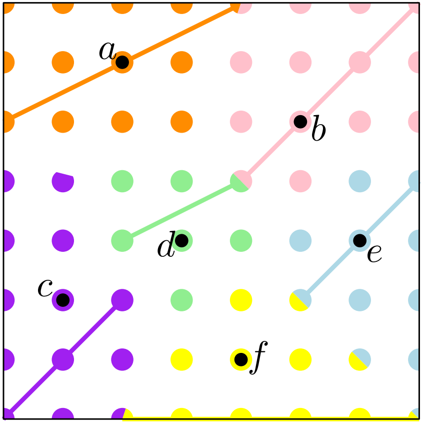
- *Voronoi polygon* is the convex hull of the grid points in a Voronoi cell.
- Gaps
- *Voronoi polygon set* is the collection of the n Voronoi polygons.
- Total size of the Voronoi polygon set is $\Theta(n \log U)$.

Proxy Segments



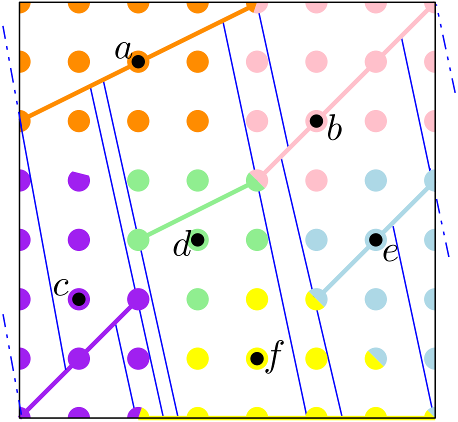
- *Proxy segment* -
  represent Voronoi polygons

Proxy Segments



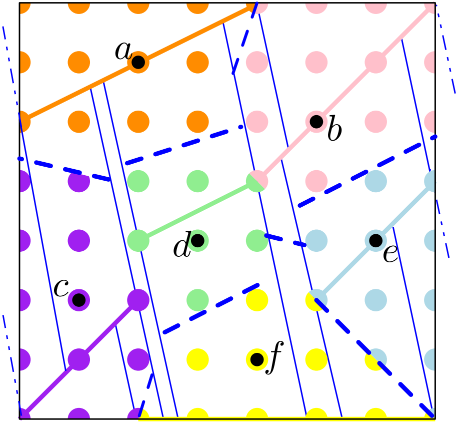
- *Proxy segment* - represent Voronoi polygons

Proxy Segments



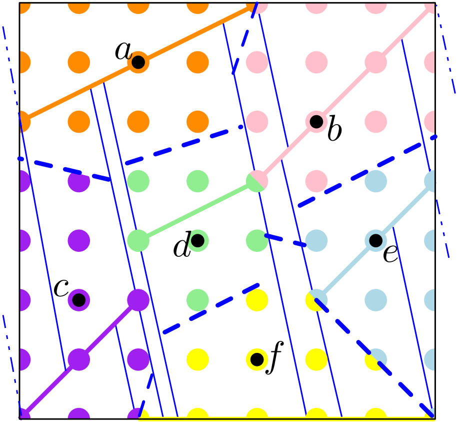
- *Proxy segment* - represent Voronoi polygons
- *Proxy trapezoidation* - trapezoidation of the proxies

Proxy Segments



- *Proxy segment* -
  represent Voronoi polygons
- *Proxy trapezoidation* -
  trapezoidation of the proxies
- *Voronoi trapezoidation* -
  split the trapezoids
  of the proxy trapezoidation
  with bisectors

Proxy Segments



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  with bisectors

Proxy trapezoidation
is a degree 2 trapezoid graph
supporting $O(\log n)$ time and
degree 2 queries.

Point Location[MS09,MS10]

Given n sites in $\mathbb{U}$

RP-Voronoi randomized incremental construction

- Time: $O(n \log(Un))$ expected
- Space: $O(n)$ expected
- Precision: degree 3

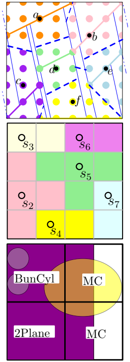
LPT's Implicit Voronoi constructed from RP-Voronoi

- Time: $O(n)$
- Space: $O(n)$
- Precision: degree 3

Queries on Proxy Trapezoidation

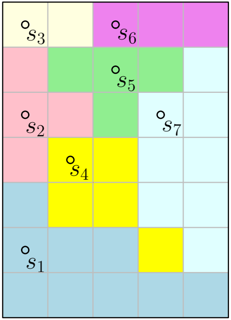
- Time: $O(\log n)$
- Precision: degree 2

Overview



- Compute point location data structure with double & triple precision
- Compute nearest neighbor transform with double precision
- Compute volumes of CSG models with five-fold precision

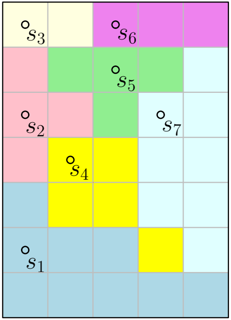
Nearest Neighbor Transform



Given
A grid of size U and
Sites $S = \{s_1, \ldots, s_n\} \subset \mathbb{U}$

Label
Each grid point of $\mathbb{U}$ with the closest site of S

Nearest Neighbor Transform



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Label
Each grid point of $\mathbb{U}$ with the closest site of S

Algorithm	Precision	Time
Brute Force	degree 2	$O(nU^2)$
Nearest Neighbor Trans. [B90]	degree 5	$O(U^2)$
Discrete Voronoi diagram [C06, MQR03]	degree 3	$O(U^2)$
GPU Hardware [H99]	-	$\Theta(nU^2)$

Problem Transformations: Part 1

Problem (NNTrans-min)

For each pixel $q \in U^2$, find the site with lowest index $s_i \in S$ minimizing $\|q - s_i\|$.

Problem Transformations: Part 1

Problem (NNTrans-min)

For each pixel $q \in U^2$, find the site with lowest index $s_i \in S$ minimizing $\|q - s_i\|$.

$$
\begin{gathered}
\|q - s_i\|^2 < \|q - s_j\|^2 \\
q \cdot q - 2q \cdot s_i + s_i \cdot s_i < q \cdot q - 2q \cdot s_j + s_j \cdot s_j \\
2x_i x_q + 2y_i y_q - x_i^2 - y_i^2 > 2x_j x_q + 2y_j y_q - x_j^2 - y_j^2 .
\end{gathered}
$$

Problem Transformations: Part 1

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$$q \cdot q - 2q \cdot s_i + s_i \cdot s_i < q \cdot q - 2q \cdot s_j + s_j \cdot s_j$$
$$2x_i x_q + 2y_i y_q - x_i^2 - y_i^2 > 2x_j x_q + 2y_j y_q - x_j^2 - y_j^2.$$

Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ maximizing $2x_i x_q + 2y_i y_q - x_i^2 - y_i^2$.

Problem Transformations: Part 2

Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ maximizing $2x_i x_q + 2y_i y_q - x_i^2 - y_i^2$.

Problem Transformations: Part 2

Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ maximizing $2x_ix_q + 2y_iy_q - x_i^2 - y_i^2$.

For a fixed, $y_q = Y$

$$2x_ix_q + 2y_iy_q - x_i^2 - y_i^2 > 2x_jx_q + 2y_jy_q - x_j^2 - y_j^2$$
$$2x_ix_q + (2y_iY - x_i^2 - y_i^2) > 2x_jx_q + (2y_jY - x_j^2 - y_j^2)$$
$$①x_q + ② > ①x_q + ②$$

Problem Transformations: Part 2

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$$①x_q + ② > ①x_q + ②$$

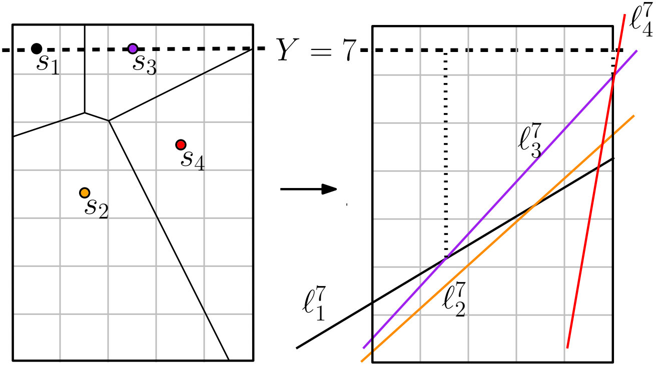
Problem (DUE-Y)

For a fixed $1 \leq Y \leq U$, and for each $1 \leq X \leq U$, find the smallest index of a line of L_Y with maximum y coordinate at $x = X$.

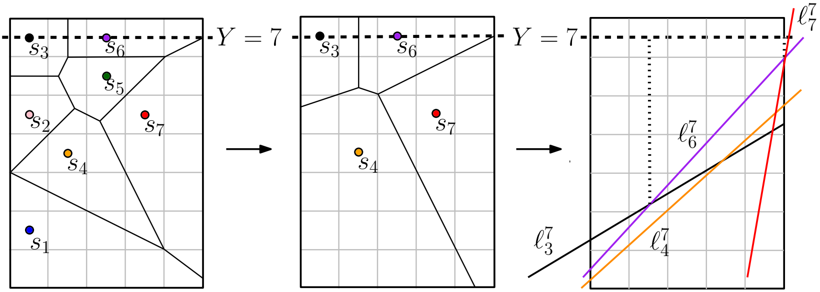
Problem Transformations: Part 2

Problem (DUE-Y)

For a fixed $1 \leq Y \leq U$, and for each $1 \leq X \leq U$, find the smallest index of a line of L_Y with maximum y coordinate at $x = X$.



Sketch of NNTransform Algorithm



Three Algorithms for Computing the DUE [MLCS12]

Given m lines of the form $y = ①x + ②$

Discrete Upper Envelope construction

- `DUE-DEG3`: $O(m + U)$ time and degree 3
- `DUE-ULgU`: $O(m + U \log U)$ time and degree 2
- `DUE-U`:$O(m + U)$ expected time and degree 2

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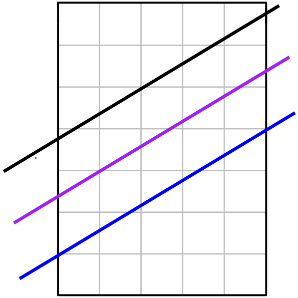
- `DUE-U`: $O(m + U)$ expected time and degree 2

For each algorithm:

1. Reduce to at most $O(U)$ lines.
2. Compute DUE of lines.



Three Algs for Computing the NNTransform [MLCS12]

Given n sites from $\mathbb{U}$

Nearest Neighbor Transform construction

- `Deg3`: $O(U^2)$ time and degree 3
- `UsqLgU`: $O(U^2 \log U)$ time and degree 2
- `Usq`: $O(U^2)$ expected time and degree 2

Three Algs for Computing the NNTransform [MLCS12]

Given *n* sites from $\mathbb{U}$

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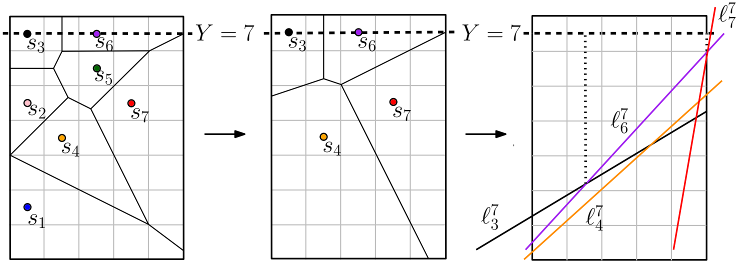
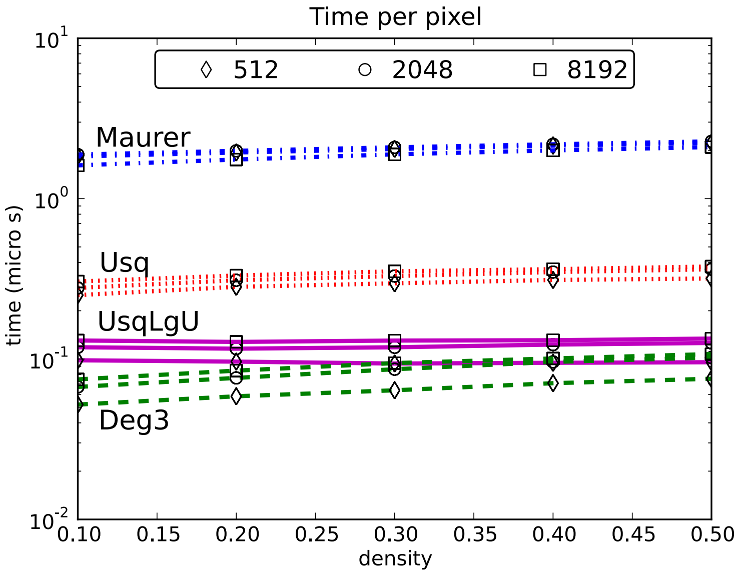
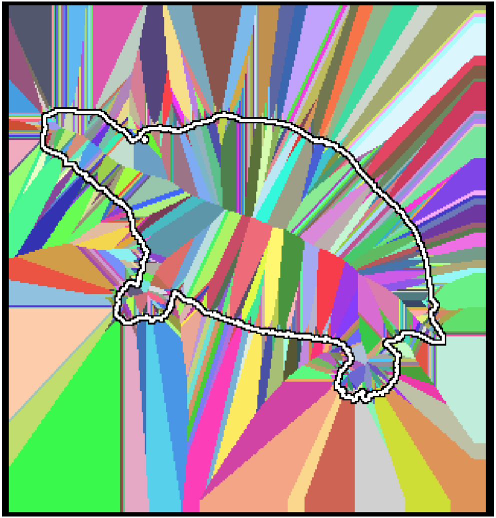


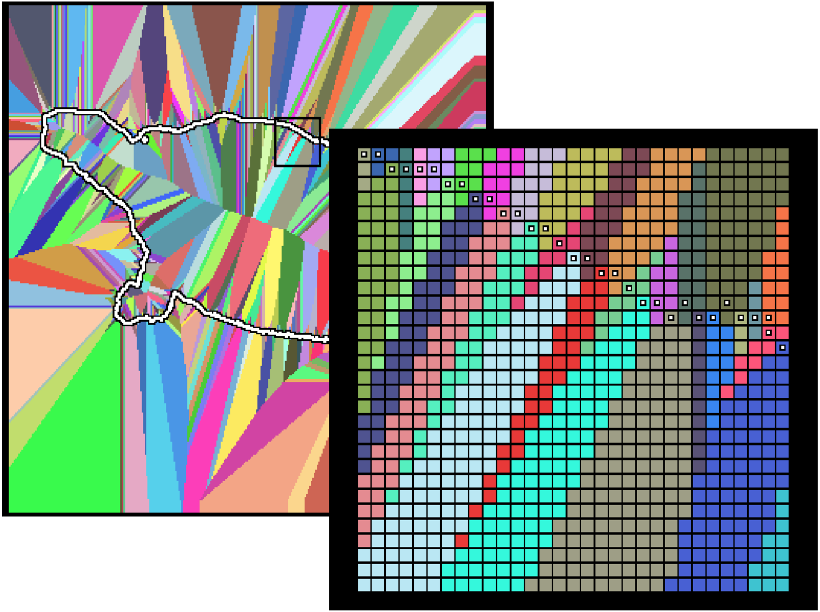

Time per pixel
◊ 512
○ 2048
□ 8192
Maurer
Usq
UsqLgU
Deg3
time (micro s)
10^{1}
10^{0}
10^{-1}
10^{-2}
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
density

Experiments: Part 2



Boundaries extracted from 120 images of the MPEG 7 CE Shape-1 Part B data set.

Experiments: Part 2



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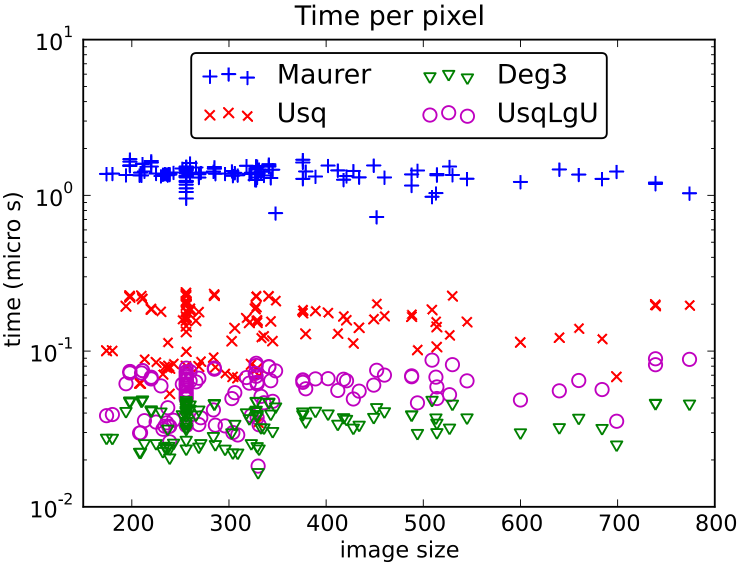

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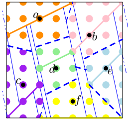
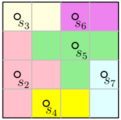
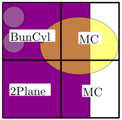


- Compute point location data structure with double & triple precision



- Compute nearest neighbor transform with double precision



- Compute volumes of CSG models with five-fold precision

Motivation and Background

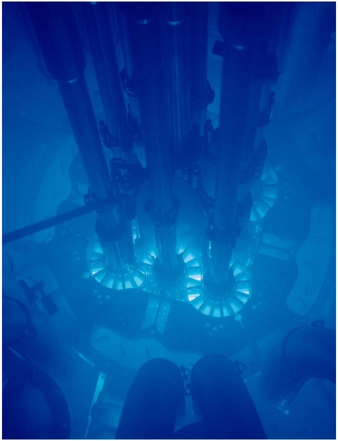


Image from Idaho National Lab, Flickr

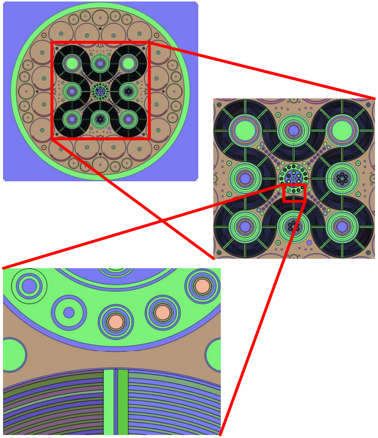
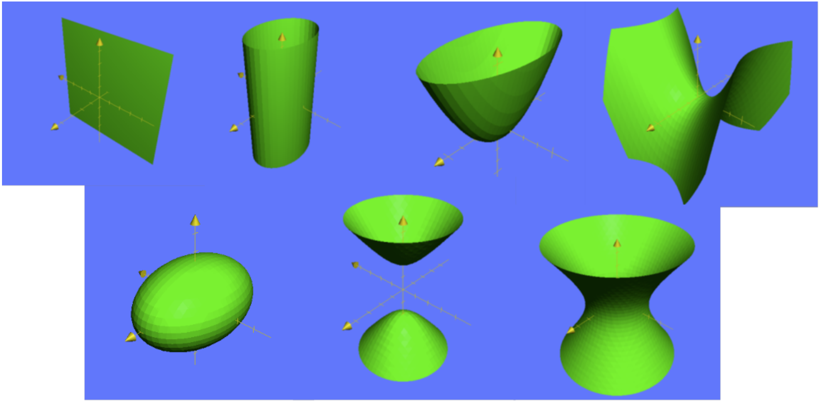


Image from: T.M. Sutton, et al.,
The MC21 Monte Carlo Transport Code,
Proceedings of M&C + SNA 2007

Primitives: Signed Quadratic Surfaces

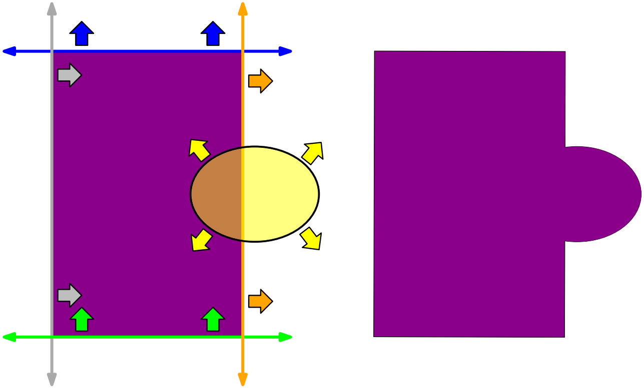


$$
\begin{aligned}
f(x, y, z) < A_1 x^2 + A_2 y^2 + A_3 z^2 \\
+ A_4 xy + A_5 xz + A_6 yz \\
+ A_7 x + A_8 y + A_9 z + A_{10}
\end{aligned}
$$

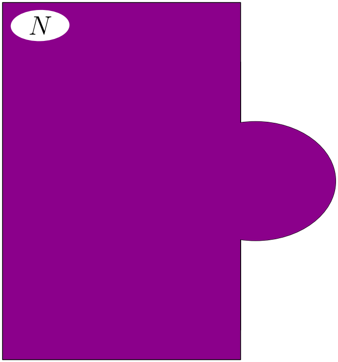
Model Representation
Basic Component: Boolean Formula

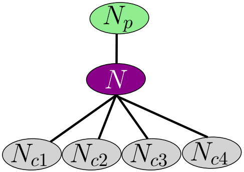
A *basic component* defined by
intersections and unions of signed surfaces.

$$\left(-s_{blue} \cap s_{grey} \cap s_{green} \cap -s_{orange}\right) \cup -s_{yellow}$$

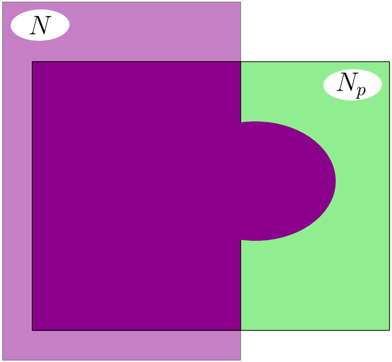


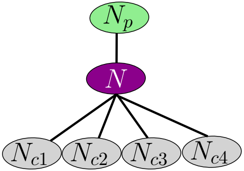
Basic comp: $B(N)$, $\cup$ and $\cap$ of signed surfaces





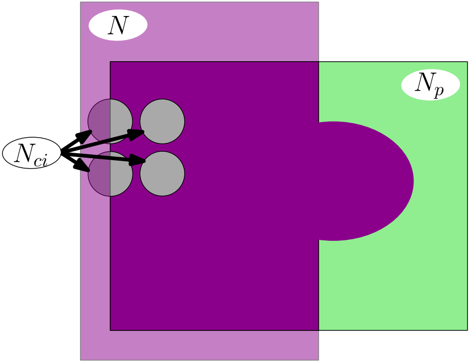
Basic comp: $B(N)$, $\cup$ and $\cap$ of signed surfaces
Restricted comp: $R(N) = B(N) \cap R(N_p)$

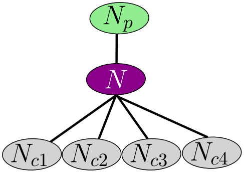




Model Representation
Component Hierarchy: Boolean Formulae

Basic comp: $B(N)$, $\cup$ and $\cap$ of signed surfaces
Restricted comp: $R(N) = B(N) \cap R(N_p)$
Hierarchical comp: $H(N) = R(N) \setminus (\bigcup_i R(N_{ci}))$





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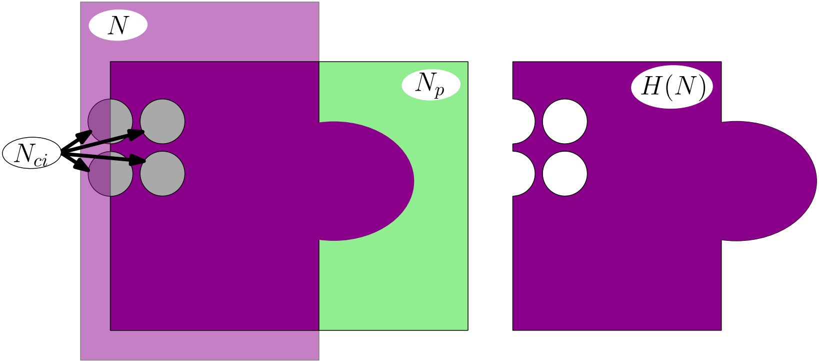
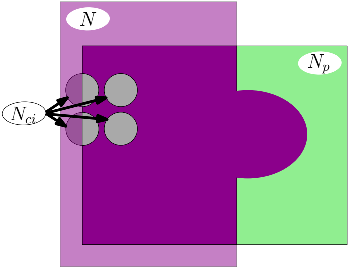
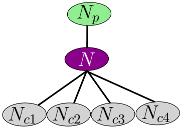




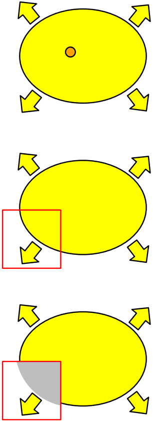


Given
A component hierarchy and an accuracy

Compute
The volume of each hierarchical component to accuracy

Operations on Surfaces

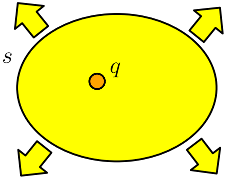
Operations on signed surfaces s with a query point q or an axis-aligned box b:



- `Inside`(s, q) – return if q is inside s.
- `Classify`(s, b) – return if the points in b are inside, outside or both with respect to s.
- `Integrate`(s, b) – return the volume of $s \cap b$.

  .

`Inside`(s, q) – return if query point q is inside signed surface s.

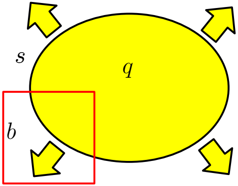


$$q = (q_1, q_2, q_3)$$
$$s = (s_1, s_2, \ldots, s_{10})$$
$$p_i, s_i \in \{-U, \ldots, U\}$$

$$\begin{aligned}\texttt{PointInside}(s, q) &= s_1 q_1^2 + s_2 q_2^2 + s_3 q_3^2 \\ &+ s_4 q_1 q_2 + s_5 q_1 q_3 + s_6 q_2 q_3 \\ &+ s_7 q_1 + s_8 q_2 + s_9 q_3 + s_{10} \\ &= \mathrm{sign}(3)\end{aligned}$$

`Classify` Test

`Classify`(s, q) – return if the points in axis-aligned box b are inside, outside or both with respect to signed surface s.



$$b = (b_1, b_2, \ldots, b_6)$$
$$s = (s_1, s_2, \ldots, s_{10})$$
$$b_i, s_i \in \{-U, \ldots, U\}$$

`Classify`(s, b), check if:

1. any `Vertices` of b are on different sides of s. – Degree 3
2. any `Edge` of b intersects s. – Degree 4
3. any `Face` b intersects s. – Degree 5

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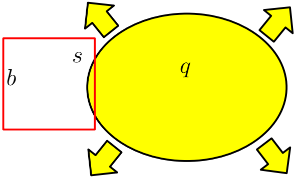

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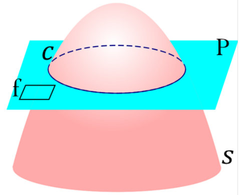
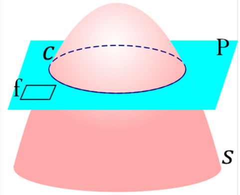
Test if a face f intersects s.



Let c be the intersection curve of the plane P containing f and s.

$$c(x, y) = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} ① & ① & ② \\ ① & ① & ② \\ ② & ② & ③ \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Test if a face f intersects s.



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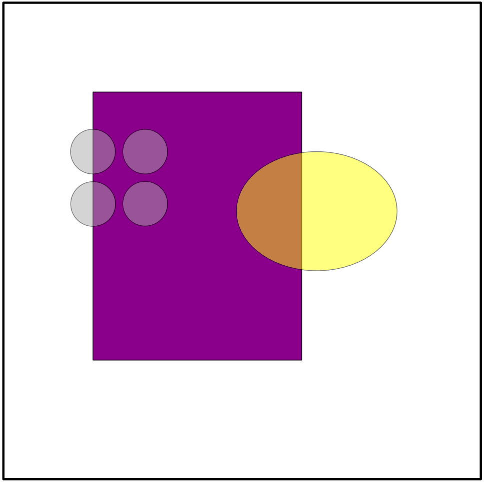
$$c(x,y) = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} ① & ① & ② \\ ① & ① & ② \\ ② & ② & ③ \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

To determine if s intersects f, test properties of the matrix.

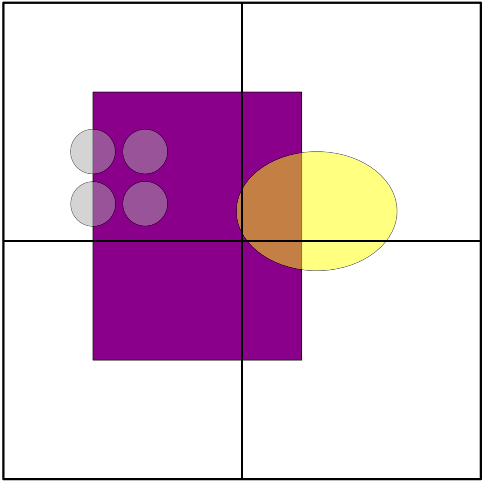
Test if c is an ellipse: $\operatorname{sign}\left(\begin{vmatrix} ① & ① \\ ① & ① \end{vmatrix}\right) = \operatorname{sign}(②)$

Test if c is real or img: $\operatorname{sign}\left(\begin{vmatrix} ① & ① & ② \\ ① & ① & ② \\ ② & ② & ③ \end{vmatrix}\right) = \operatorname{sign}(⑤)$

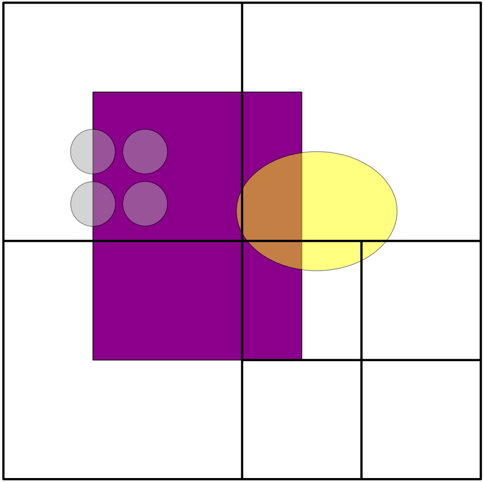
Algorithm Animation



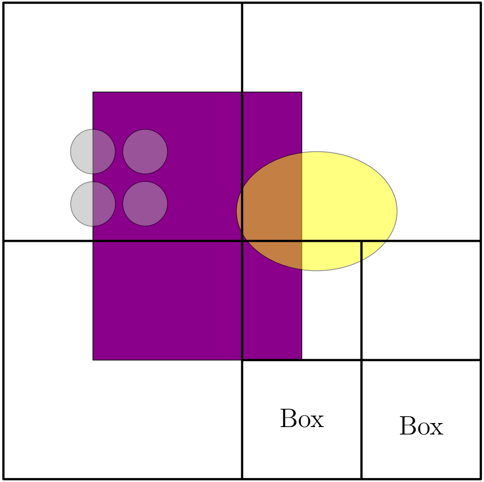
Algorithm Animation



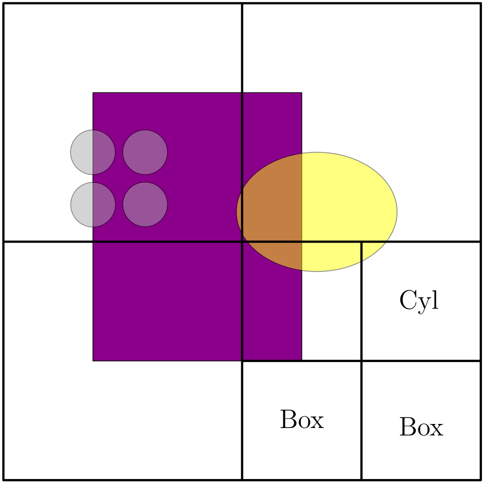
Algorithm Animation



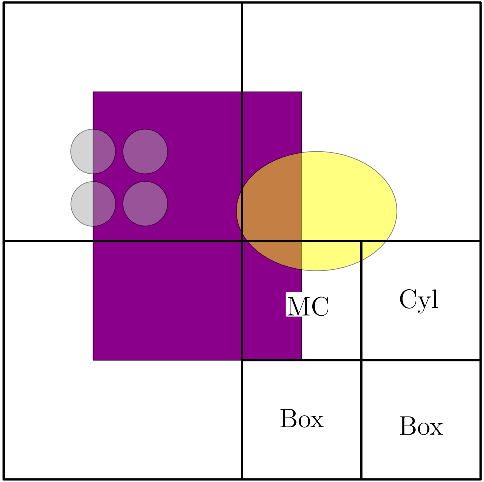
Algorithm Animation



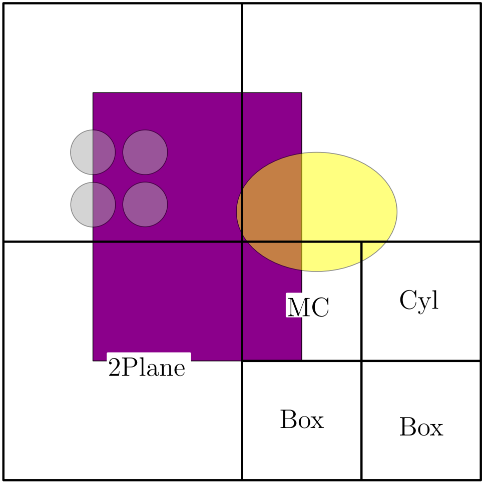
Algorithm Animation



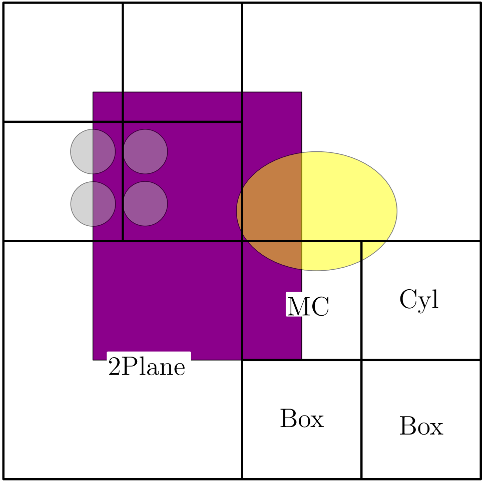
Algorithm Animation



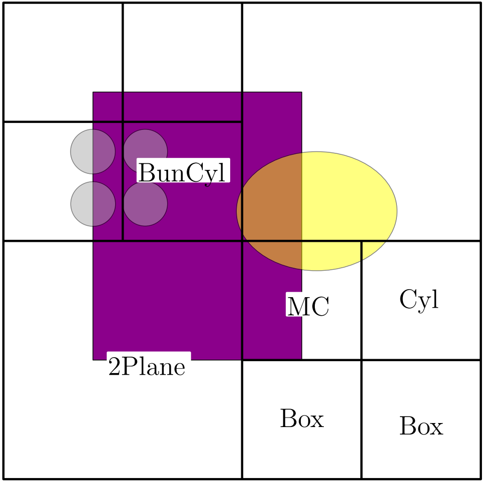
Algorithm Animation



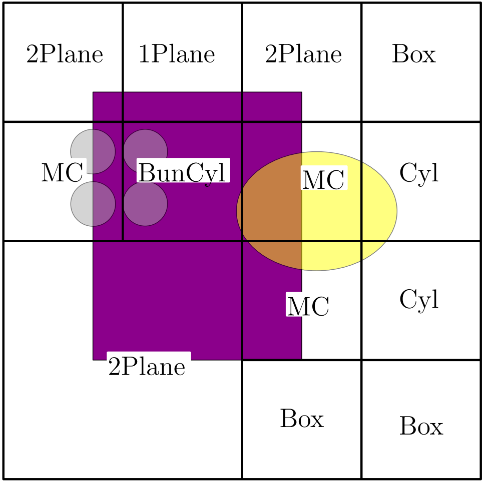
Algorithm Animation



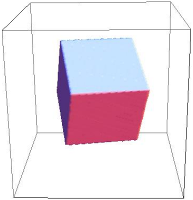
Algorithm Animation



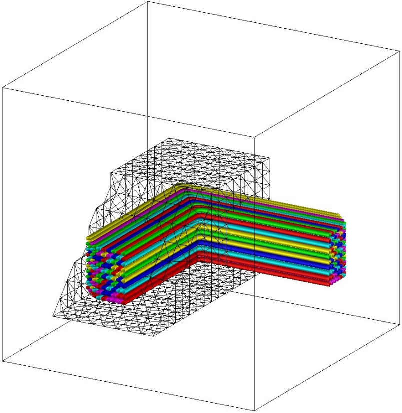
Algorithm Animation

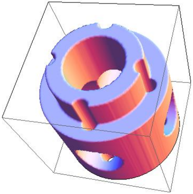


Experiments: Models



Cube

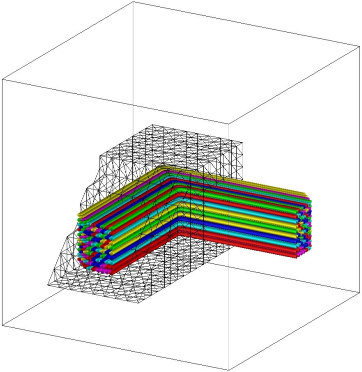




DrillCyl

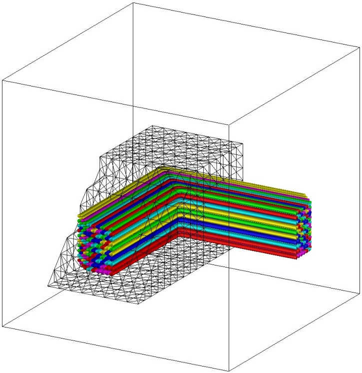
L-Pipe12,
L-Pipe100, and L-Pipe10k

Experiments: Accuracy and Time for L-Pipe100



Algorithm	Requested Accuracy	Error	Time (sec)
MC	1e-4	$<$1e-4	790.28
New	1e-4	$<$1e-6	1.41

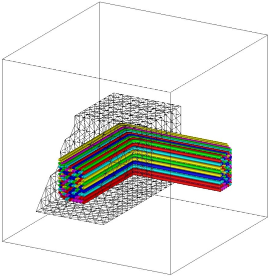
Experiments: Accuracy and Time for L-Pipe100



Algorithm	Requested Accuracy	Error	Time (sec)
MC	1e-4	$<$1e-4	790.28
New	1e-4	$<$1e-6	1.41

Experiments: Larger Model for L-Pipe10k

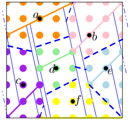
L-Pipe10k is similar to L-Pipe100 but defined by over 40k surfaces.

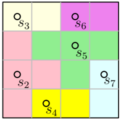


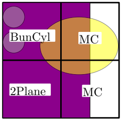
Algorithm	Requested Accuracy	Error	Time
MC	1e-4	-	$> 12.00\mathrm{h}^*$
New	1e-4	$<$1e-6	$9.43\mathrm{s}$

*Halted after 12 hours. Extrapolating from other experiments, 76 hours.

Overview







- Compute point location data structure with double & triple precision
- Compute nearest neighbor transform with double precision
- Compute volumes of CSG models with five-fold precision

Contact

David L. Millman
dave@cs.unc.edu
`http://cs.unc.edu/~dave`