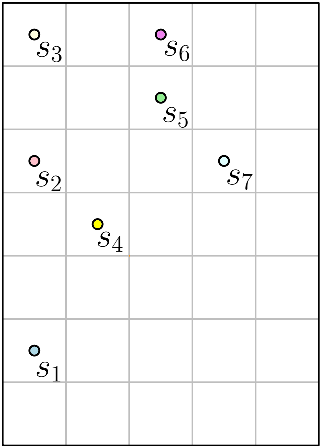
Computing the Nearest Neighbor Transform Exactly with only Double Precision

DAVID L. MILLMAN Steven Love
Timothy M. Chan Jack Snoeyink

20 April 2013

The Problem



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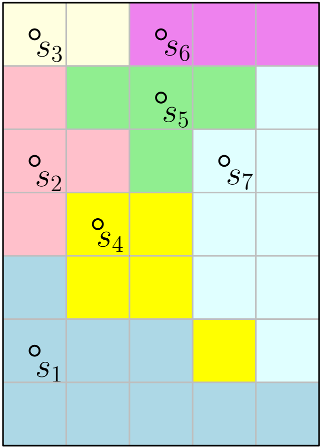

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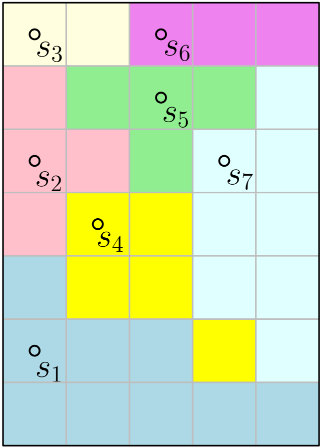

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How much precision is need to determine this?

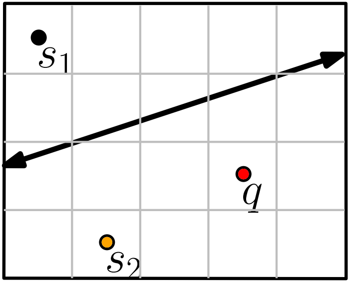
Other precision/robust approaches

Techniques for implementing geometric algorithms with finite precision computer arithmetic:

- Rely on machine precision (+epsilon)
- Topological Consistency [S99, S01, SI90, SI92, SII*00]
- Exact Geometric Computation [Y97]
  - Software based arithmetic [CORE, LEDA, MPFR]
  - Predicate eval. schemes [C92, FW93, ABO*97, S97]
  - Degree-driven algorithm design [LPT99]

Analyzing Precision[LPT99]

Is q closer to s_1?



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\mathbb{U} &= \{1, 2, \ldots, U\} \\
s_1,\ s_2,\ &q \in \mathbb{U}^2 \\
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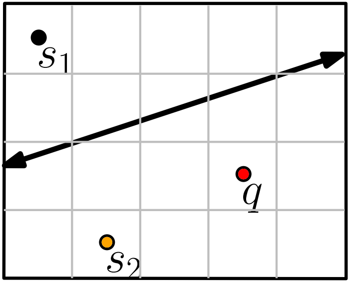

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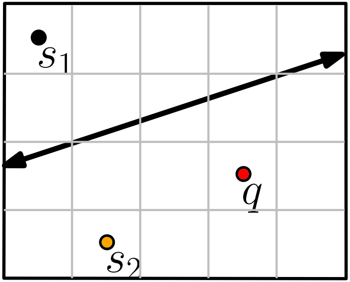

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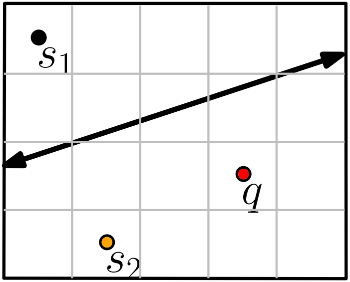

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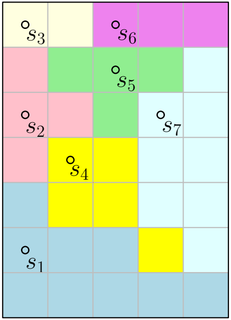
Note that ℓk bits is enough to evaluate the sign.



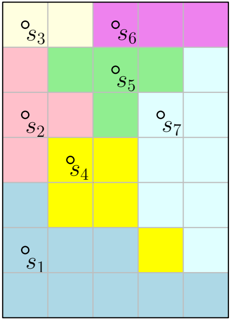
Given

A grid of size U and
Sites $S = \{s_1, \ldots, s_n\} \subset \mathbb{U}^2$

Label

Each grid point of $\mathbb{U}^2$ with the closest site of S

Nearest Neighbor Transform



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	Alg	Time
Brute Force	deg 2	$O(nU^2)$
Nearest Neighbor Trans. [B90]	deg 5	$O(U^2)$
Discrete Voronoi diagram [C06, MQR03]	deg 3	$O(U^2)$
GPU Hardware [H99]	-	$\Theta(nU^2)$

Problem Transformations–Part 1

Problem (NNTrans-min)

For each pixel q, find the site with lowest index $s_i \in S$ minimizing $\|q - s_i\| < \|q - s_j\|$.

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$$\|q - s_i\|^2 < \|q - s_j\|^2$$
$$q \cdot q - 2q \cdot s_i + s_i \cdot s_i < q \cdot q - 2q \cdot s_j + s_j \cdot s_j$$
$$2x_i x_q + 2y_i y_q - x_i^2 - y_i^2 > 2x_j x_q + 2y_j y_q - x_j^2 - y_j^2.$$

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Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ maximizing $2x_i x_q + 2y_i y_q - x_i^2 - y_i^2$.

Problem Transformations–Part 2

Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ *maximizing* $2x_i x_q + 2y_i y_q - x_i^2 - y_i^2$.

Problem Transformations–Part 2

Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ maximizing $2x_i x_q + 2y_i y_q - x_i^2 - y_i^2$.

For a fixed row, $y_q = Y$

$$2x_i x_q + 2y_i y_q - x_i^2 - y_i^2 > 2x_j x_q + 2y_j y_q - x_j^2 - y_j^2$$

$$2x_i x_q + (2y_i Y - x_i^2 - y_i^2) > 2x_j x_q + (2y_j Y - x_j^2 - y_j^2)$$

$$①x_q + ② > ①x_q + ②$$

Problem Transformations–Part 2

Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ maximizing $2x_ix_q + 2y_iy_q - x_i^2 - y_i^2$.

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Problem (DUE-Y)

For a fixed $1 \leq Y \leq U$, and for each $1 \leq X \leq U$, find the line with lowest index $\ell_i^Y \in L_Y$ with maximum y-coordinate.

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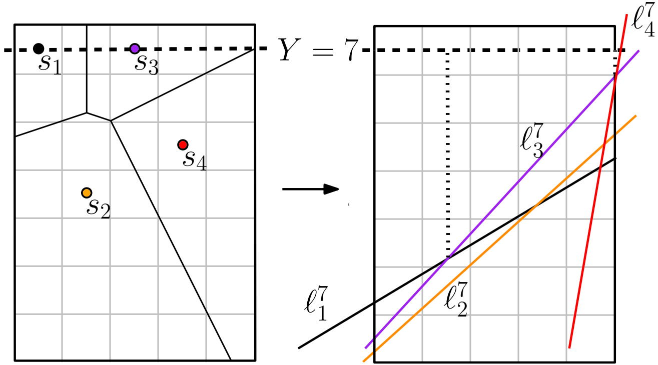
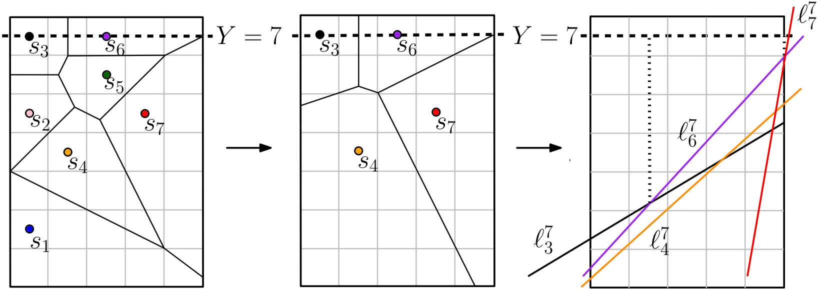


Sketch of NNTransform Algorithm



Three Algorithms for Computing the DUE

Given m lines of the form $y = ①x + ②$

Discrete Upper Envelope construction

- `DUE-DEG3`: $O(m + U)$ time and degree 3
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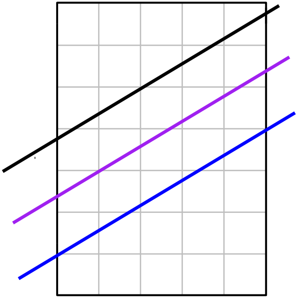
For each algorithm:

1. Reduce to at most $O(U)$ lines.
2. Compute DUE of lines.



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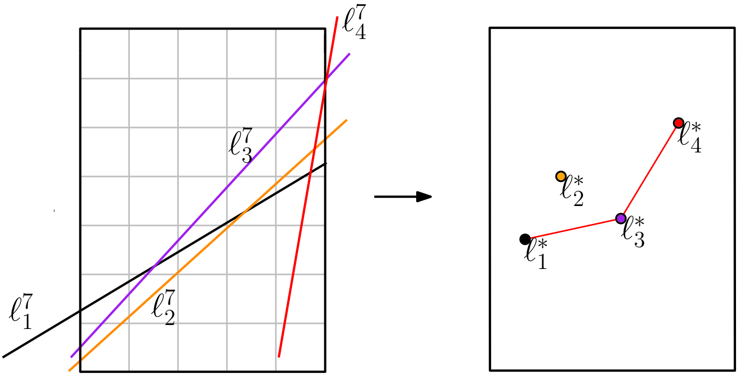
Discrete Upper Envelope Lemma (DUE−DEG3)

Two steps:

1. Compute upper env. via the lower hull of dual points
   line $y = mx + b$ maps to dual point $(m, -b)$.
2. Discretize upper env to DUE.



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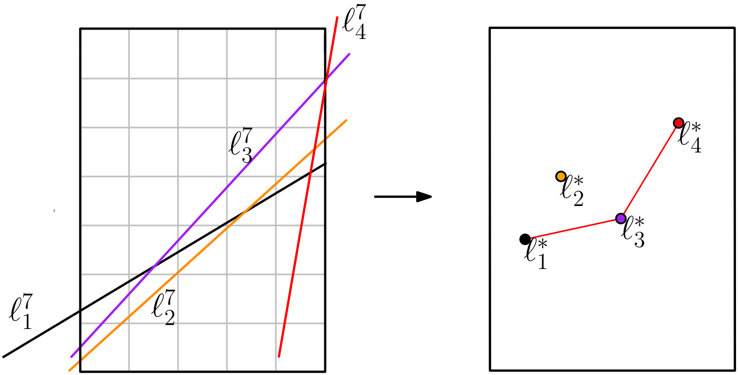
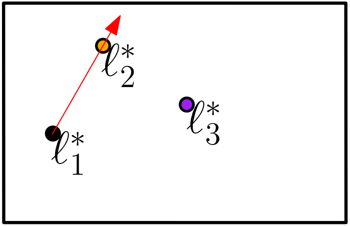


line form is $y = ①x + ②$ $\implies$ dual point form is $(①, ②)$

Precision of DUE-DEG3

Orientation test



ℓ_1^*, ℓ_2^* and ℓ_3^* have form (①, ②)

$$\texttt{orient}(\ell_1^*, \ell_2^*, \ell_3^*) = \operatorname{sign}\left(\begin{vmatrix} ⓪ & ① & ② \\ ⓪ & ① & ② \\ ⓪ & ① & ② \end{vmatrix}\right) = \operatorname{sign}(③)$$

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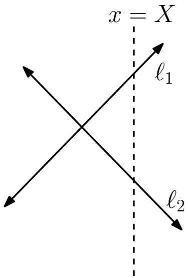
Main predicate `OrderOnALine`

Order on a Line



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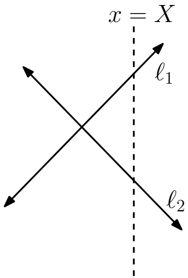

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Lemma `IntersectCol`

Given two lines ℓ_1 and ℓ_2 of the form $y = ①x + ②$, construction `IntersectCol`(ℓ_1, ℓ_2) returns the column containing $\ell_1 \cap \ell_2$ in $O(logU)$ time and degree 2.

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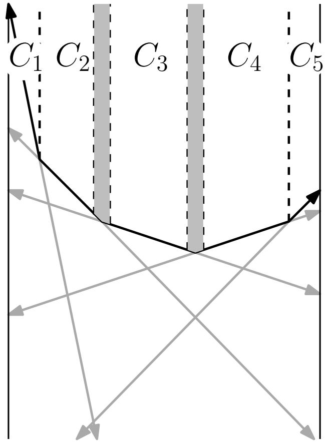

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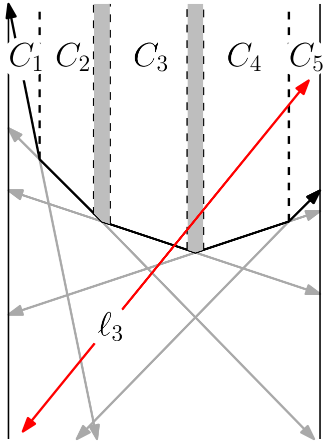

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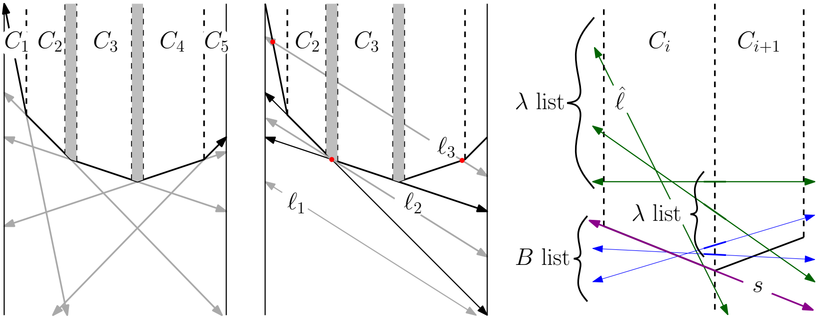

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Given n sites from $\mathbb{U}$

Nearest Neighbor Transform construction

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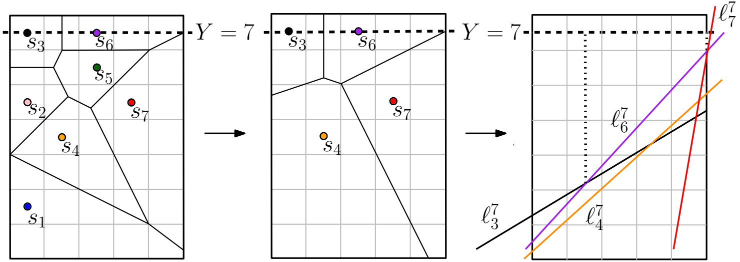
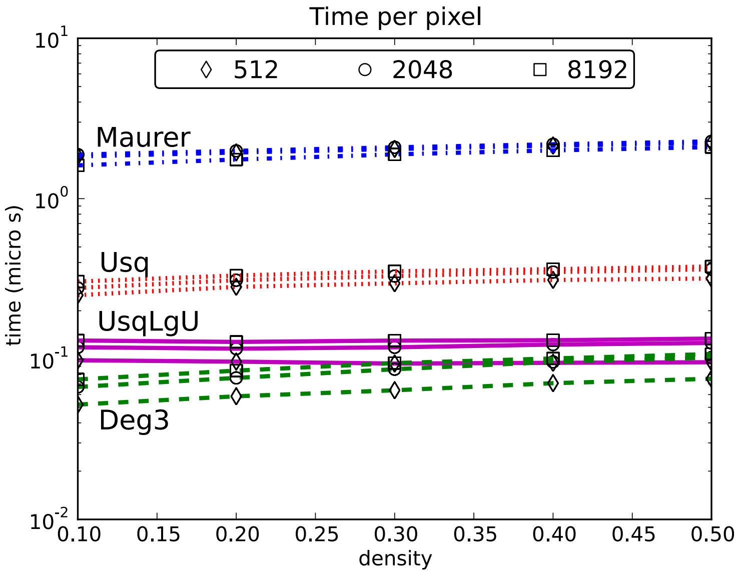
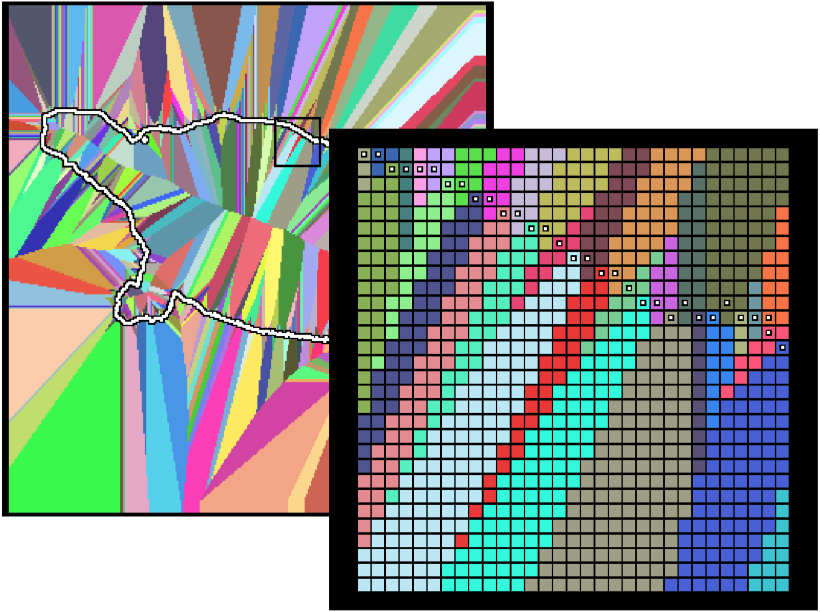


Experiments Part 1



Experiments Part 2



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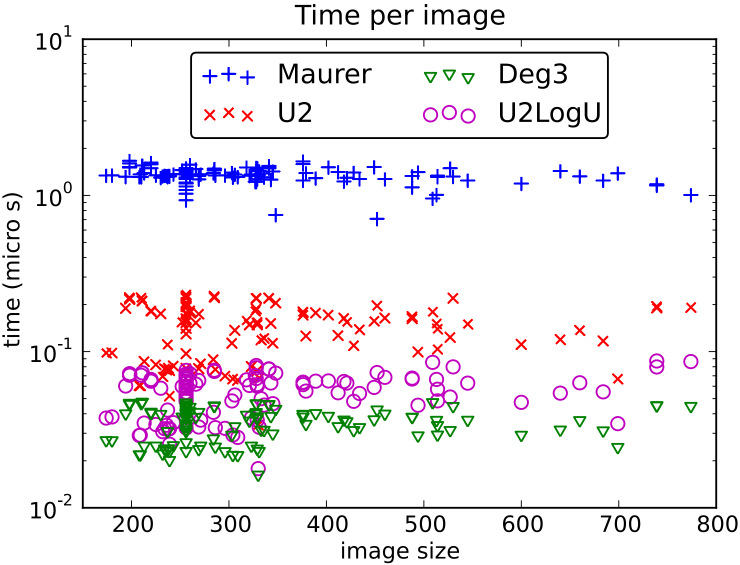

Summary

Described and implemented three algorithms
for computing the DUE of lines of the form $y = ①x + ②$:

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Which gave us three algorithms for computing the NNTransform:

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Conclusions

Can we compute the NNTraform with degree 2 without randomization?

What about L_1 or L_∞?

What other geometric problems can be considered using degree-driven algorithm design?

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Contact

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web: `http://cs.unc.edu/~dave`