Properties of Bezier Curves

- Invariance under affine parameter transformation

\[ \sum_{i=0}^{n} P_i B_{i,n}(u) = \sum_{i=0}^{n} P_i B_{i,n}( (u-a)/(b-a) ) \]

- Invariance under barycentric combinations (weighted average):

\[ \sum_{i=0}^{n} (\alpha Q_i + \beta R_i)B_{i,n}(u) = \alpha \sum_{i=0}^{n} Q_i B_{i,n}(u) + \beta \sum_{i=0}^{n} R_i B_{i,n}(u), \]

\[ \alpha + \beta = 1 \]

- Pseudo-local control: \( B_{i,n}(u) \) has a max at \( u = i/n \). If we move the control point \( P_i \), then the curve is most affected in the region around the parameter value \( i/n \).
Derivatives of Bezier Curve

- Derivative of a Bezier curve:

\[
\frac{d}{du} P(u) = n \sum_{0}^{n-1} \Delta P_i B_{i,n-1}(u) = P'(u),
\]

where \( \Delta P_i = P_{i+1} - P_i \).

\( P'(u) \) is also called the *hodograph* curve.

- Higher order derivatives can also be defined in terms of lower order Bezier curves.

- Based on the derivatives, we can place constraints on the control points for \( C^1 \) or \( G^1 \) continuity.
Degree Elevation

- Geometric representation of a degree n curve in terms of n+1 degree curve
  - Compute the control points ($P_i$) of the elevated curve

$$\sum_{i=0}^{n+1} P_i B_{i,n+1}(u) = \sum_{i=0}^{n} P_i B_{i,n}(u)$$

where $P_i = \left(\frac{i}{n+1}\right) P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i$, where $i=0, \ldots, n+1$

- What happens if degree elevation is applied repeatedly?
Truncating a Bezier Curve

- **Truncation** and subsequent reparametrization: Given a Bezier curve, find the new set of control points of a Bezier curve that define a segment of this curve in the parametric interval: \( u \in [u_i, u_j] \)

- **Subdivision**: Given a Bezier curve, \( \mathbf{P}(u) \), subdivide at a parameter value \( u_i \). Compute the control points of two Bezier curves: \( \mathbf{P}_1(s) \) and \( \mathbf{P}_2(t) \), so that \( \mathbf{P}_1(s), s \in [0,1] \) corresponds to \( \mathbf{P}(u), u \in [0,u_i] \), and \( \mathbf{P}_2(t), t \in [0,1] \) corresponds to \( \mathbf{P}(u), u \in [u_i,1] \).

- Subdivision can be used to truncate a curve. The control points of the subdivided curve are computed using de Casteljau’s algorithm.
Subdividing a Bezier Curve

- Subdivision doesn’t change the shape of a Bezier curve
- It can be used for local refinement: subdivide a curve and change the control point(s) of one of the subdivided curve
- The union of convex hulls of the subdivided curve is a subset of the convex hull of the original curve (i.e. the convex hulls are a better approximation of the Bezier curve).
- Asymptotically the control polygons of the subdivided curve converge to the actual curve (at a quadratic rate)
- Subdivision and convex hulls are frequently used for intersection computations