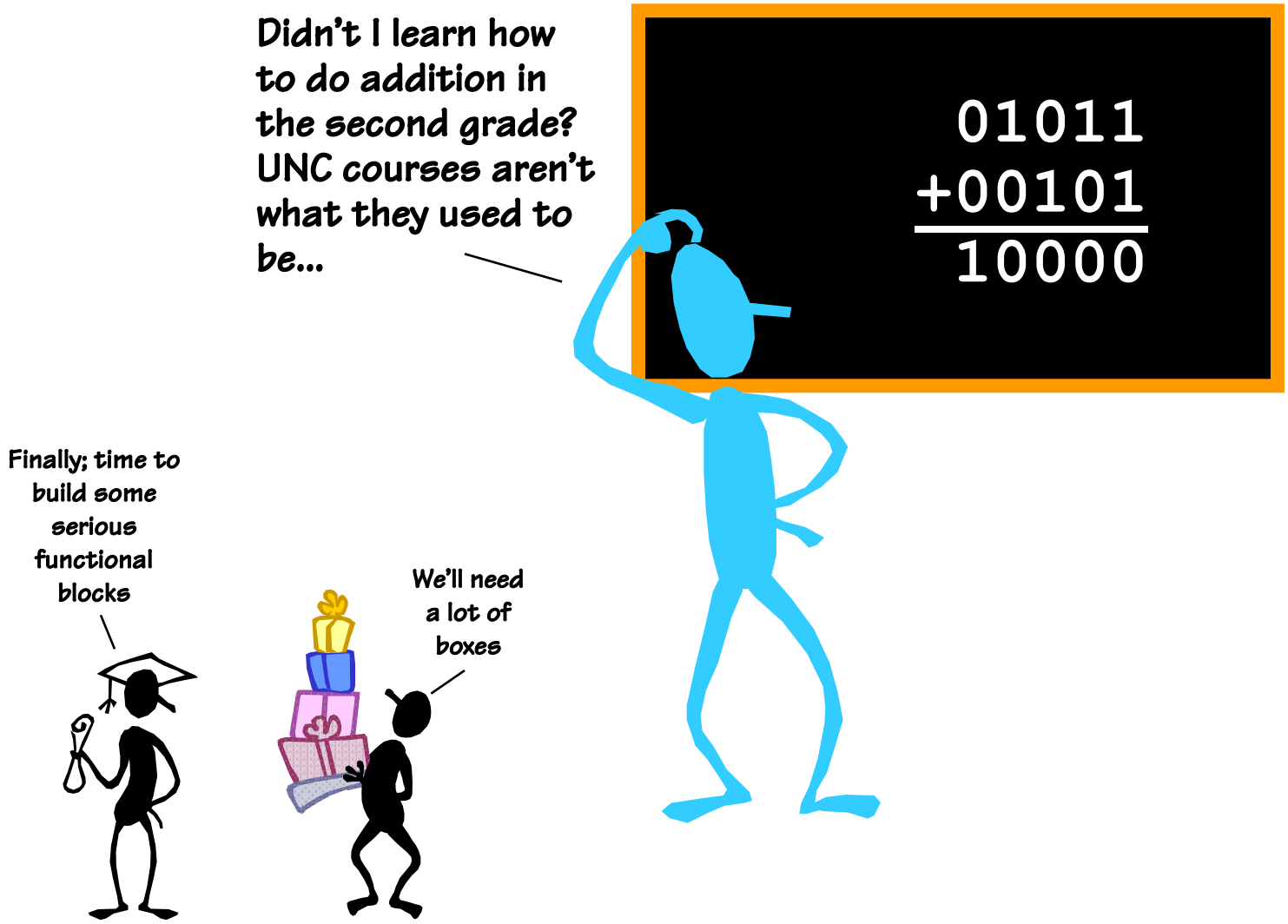
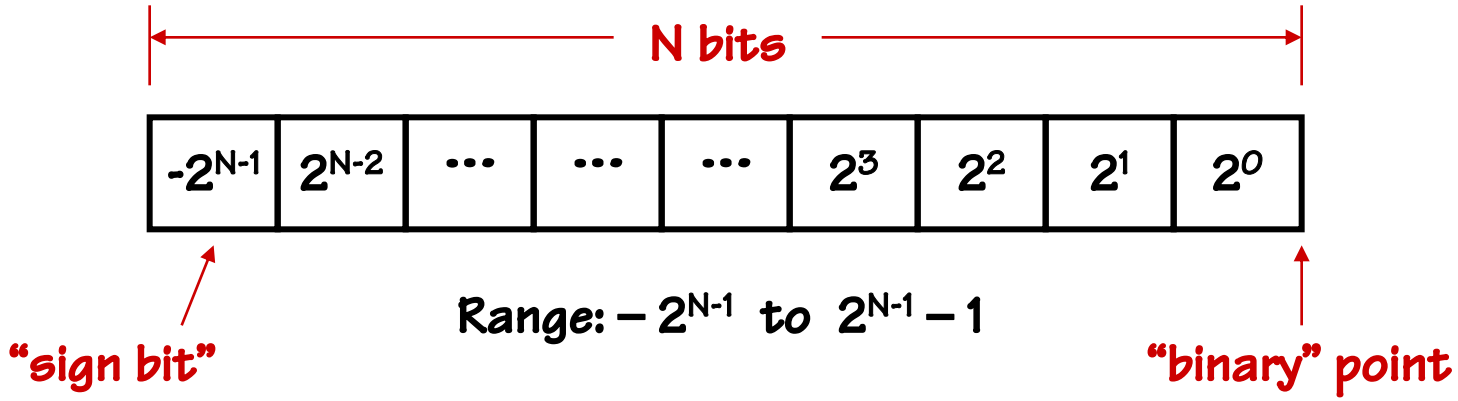
Arithmetic Circuits



Review: 2's Complement



8-bit 2's complement example:

$11010110 = -2^7 + 2^6 + 2^4 + 2^2 + 2^1 = -128 + 64 + 16 + 4 + 2 = -42$

If we use a two's-complement representation for signed integers, the same binary addition procedure will work for adding both signed and unsigned numbers.

By moving the implicit "binary" point, we can represent fractions too:

$1101.0110 = -2^3 + 2^2 + 2^0 + 2^{-2} + 2^{-3} = -8 + 4 + 1 + 0.25 + 0.125 = -2.625$

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```
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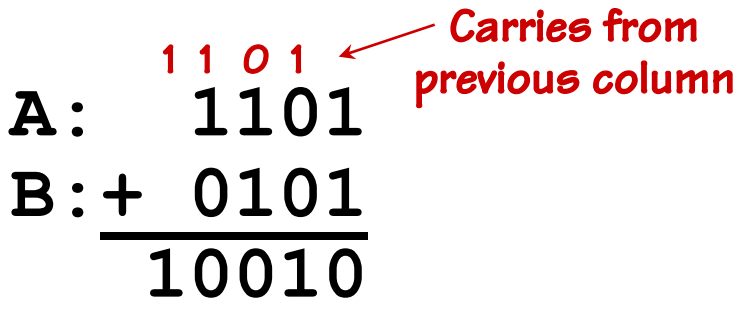

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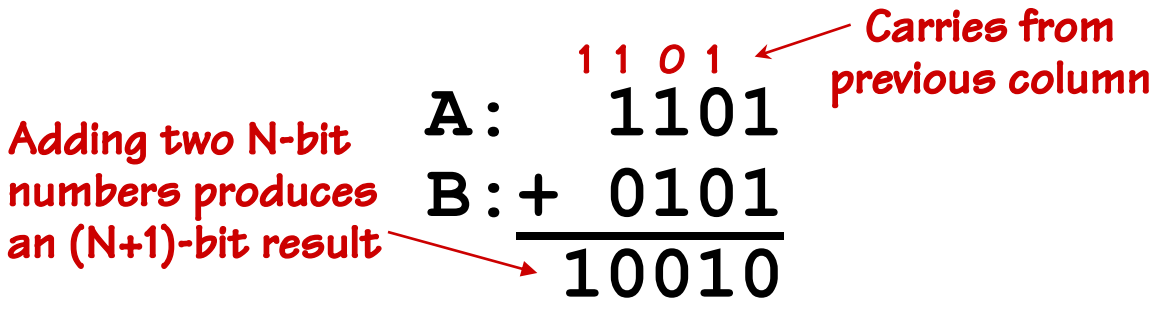

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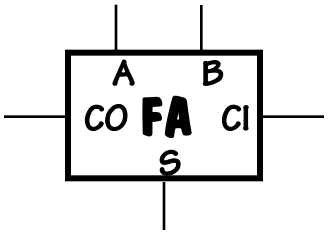
Adding two N-bit numbers produces an (N+1)-bit result

Let's start by building a block that adds one column:



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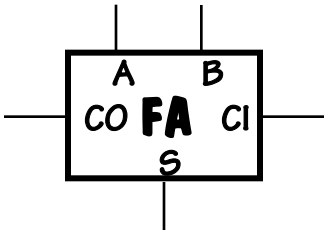

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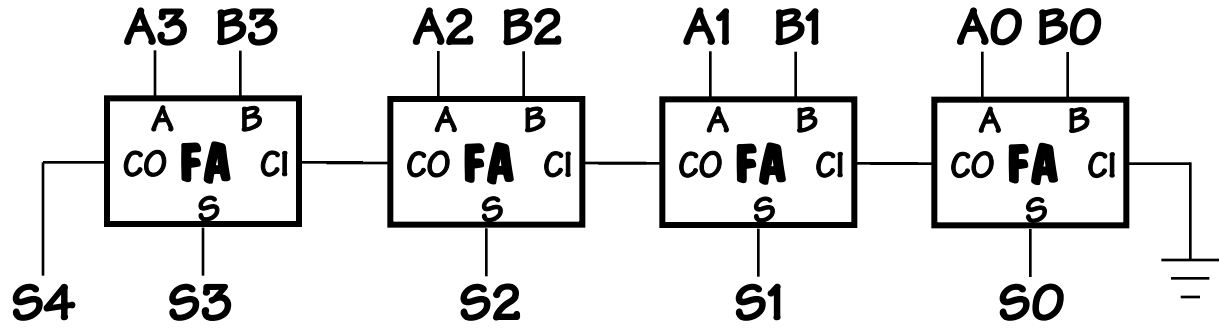
Then we can cascade them to add two numbers of any size…



Designing a "Full Adder": From Last Time

1) Start with a truth table:

C_i	A	B	C_o	S
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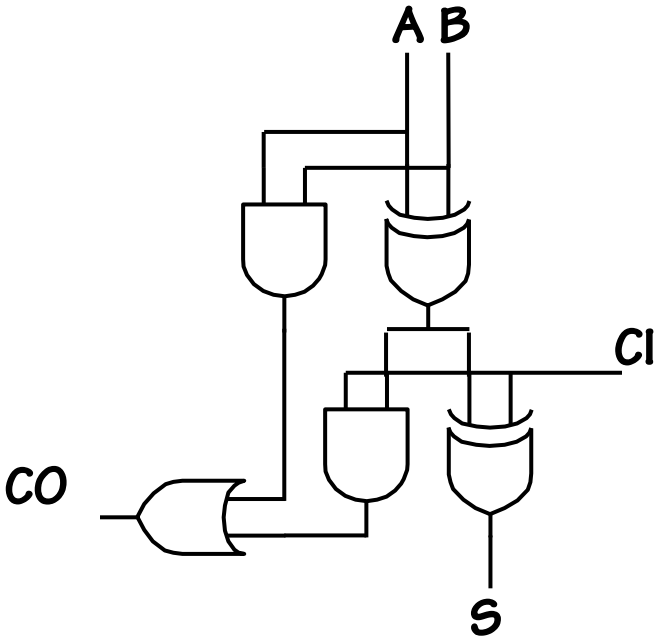

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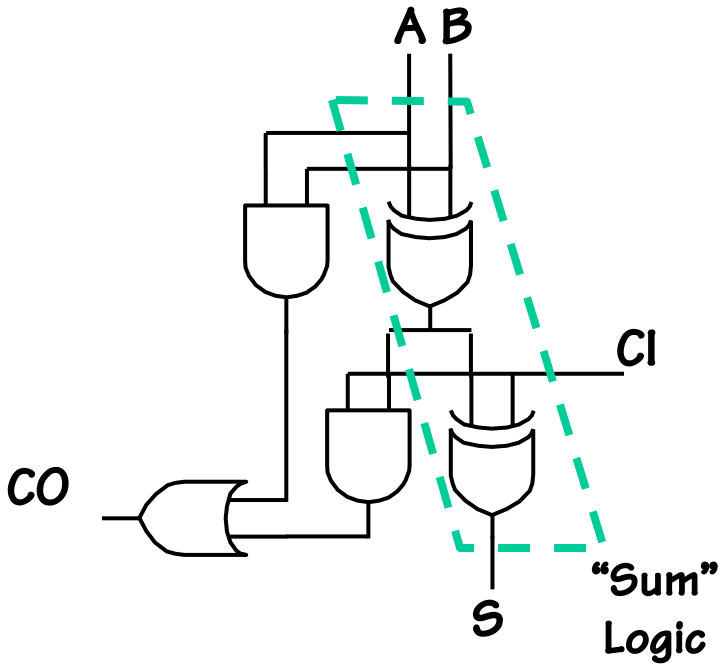

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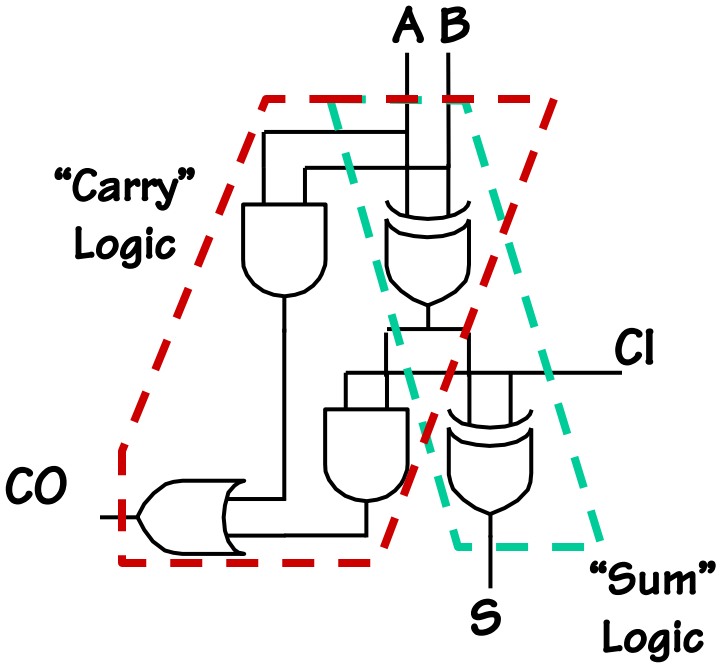

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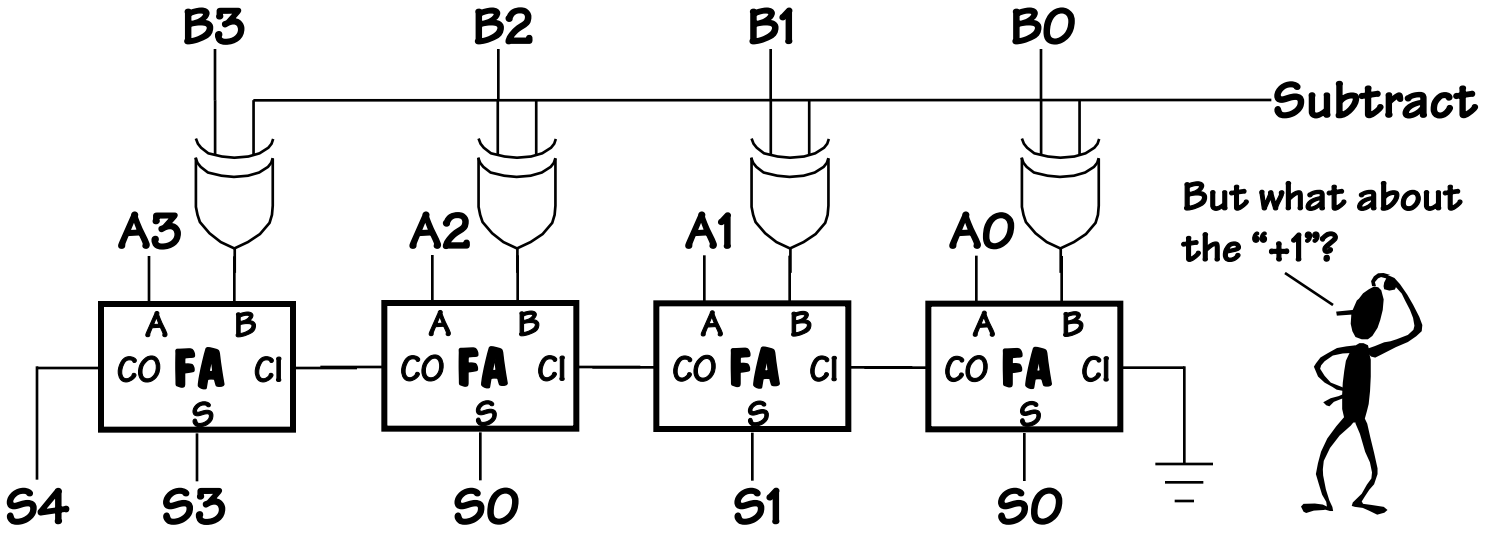

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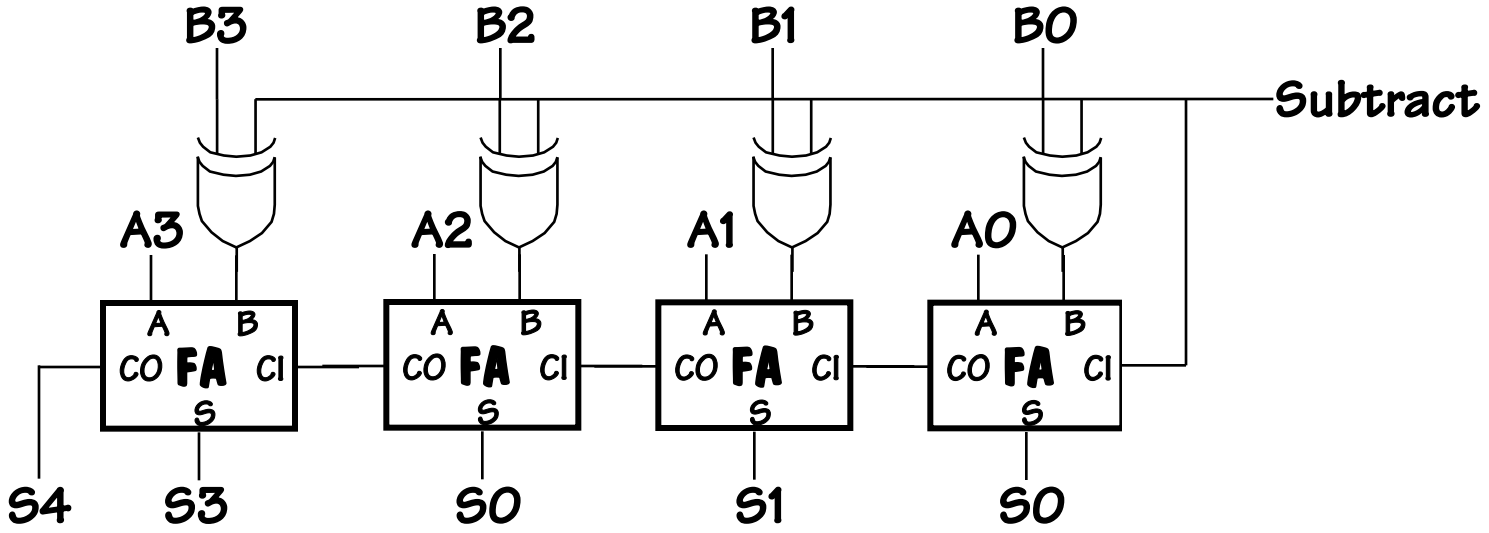

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To compare A and B, perform A–B and use condition codes:

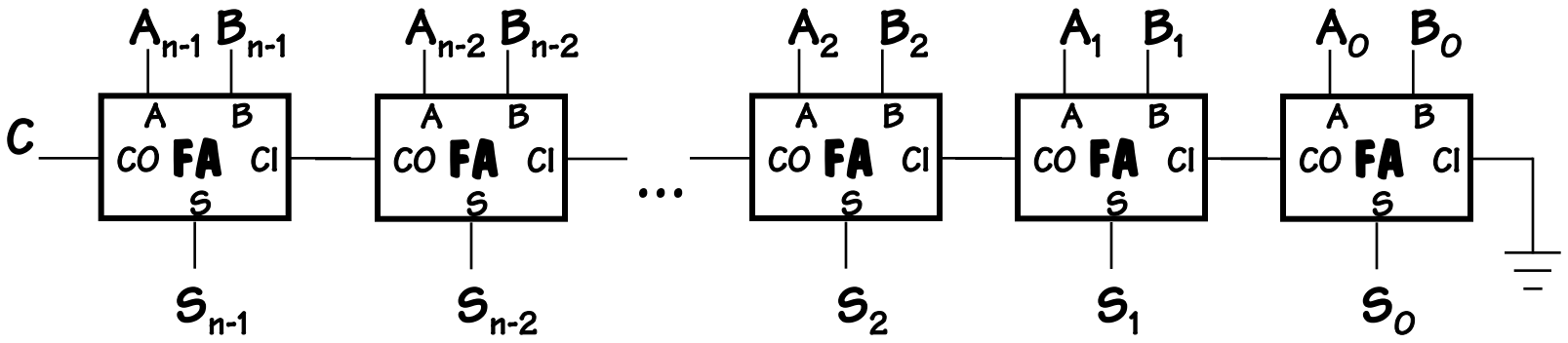
Signed comparison:

LT	N⊕V
LE	Z+ (N⊕V)
EQ	Z
NE	~Z
GE	~ (N⊕V)
GT	~ (Z+ (N⊕V))

Unsigned comparison:

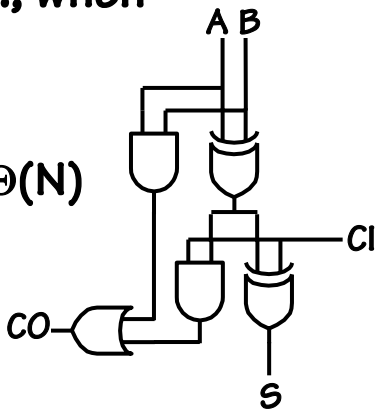
LTU	C
LEU	C+Z
GEU	~C
GTU	~ (C+Z)

T_{PD} of Ripple-Carry Adder



Worse-case path: carry propagation from LSB to MSB, e.g., when adding 11…111 to 00…001.

$$t_{PD} = \underbrace{(t_{PD,XOR} + t_{PD,AND} + t_{PD,OR})}_{\text{A,B to CO}} + \underbrace{(N-2)*(t_{PD,OR} + t_{PD,AND})}_{\text{CI to CO}} + \underbrace{t_{PD,XOR}}_{CI_{N-1} \text{ to } S_{N-1}} \approx \Theta(N)$$



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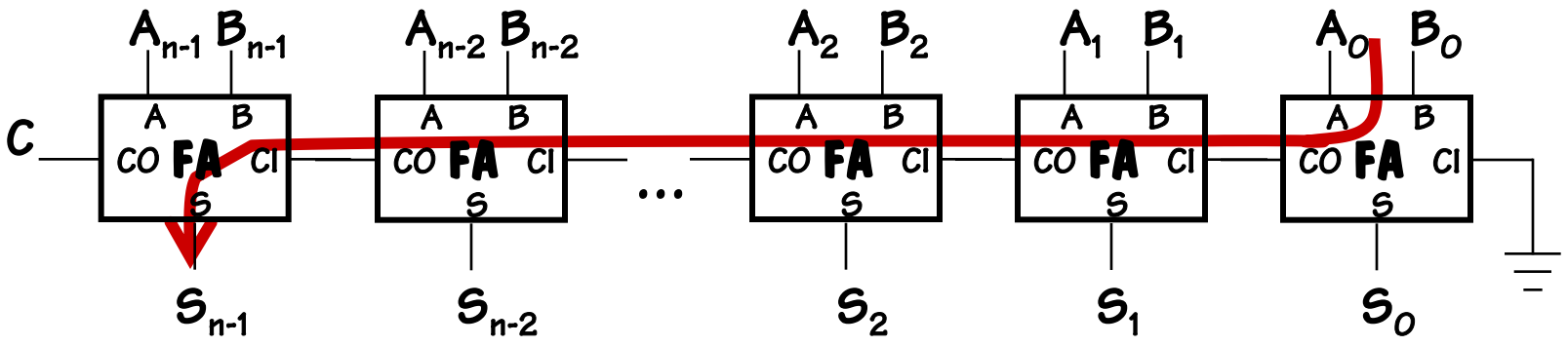

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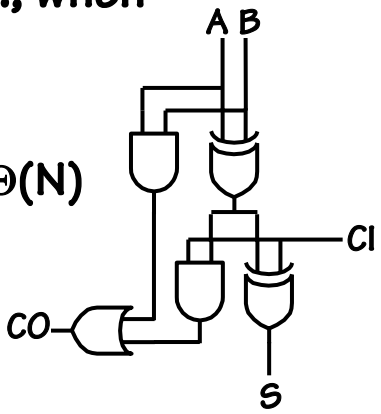

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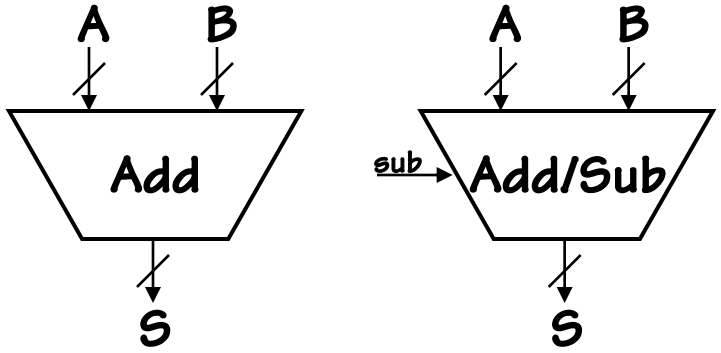
Faster Carry Logic

- Carry-Lookahead Adders (CLA)
- Carry-Skip Adders
- Carry-Select Adders

Adder Summary

Adding is not only common, but it is also tends to be one of the most time-critical of operations. As a result, a wide range of adder architectures have been developed that allow a designer to tradeoff complexity (in terms of the number of gates) for performance.

At this point we'll define a high-level functional unit for an adder, and specify the details of the implementation as necessary.



Shifting Logic

Shifting is a common operation that is applied to groups of bits. Shifting can be used for alignment, as well as for arithmetic operations.

X << 1 is approx the same as 2*X

X >> 1 can be the same as X/2

For example:

$$X = 20_{10} = 00010100_2$$

Left Shift:

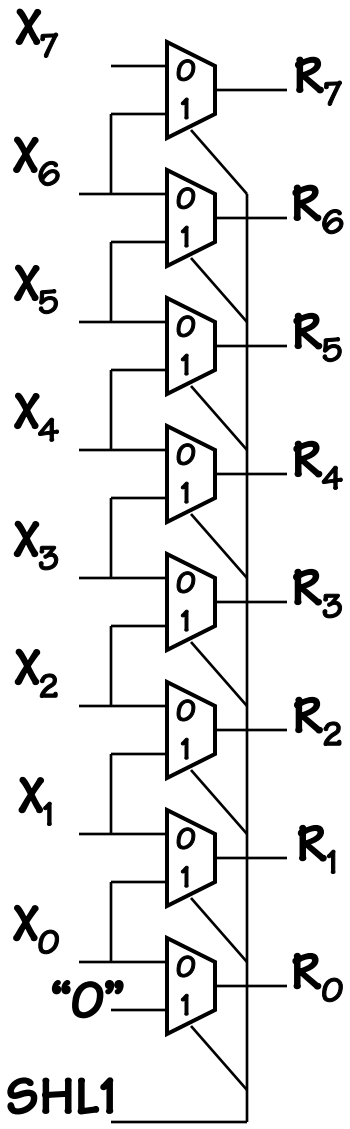
$(X << 1) = 00101000_2 = 40_{10}$

Right Shift:

$(X >> 1) = 00001010_2 = 10_{10}$

Signed or "Arithmetic" Right Shift:

$(-X >> 1) = (11101100_2 >> 1) = 11110110_2 = -10_{10}$



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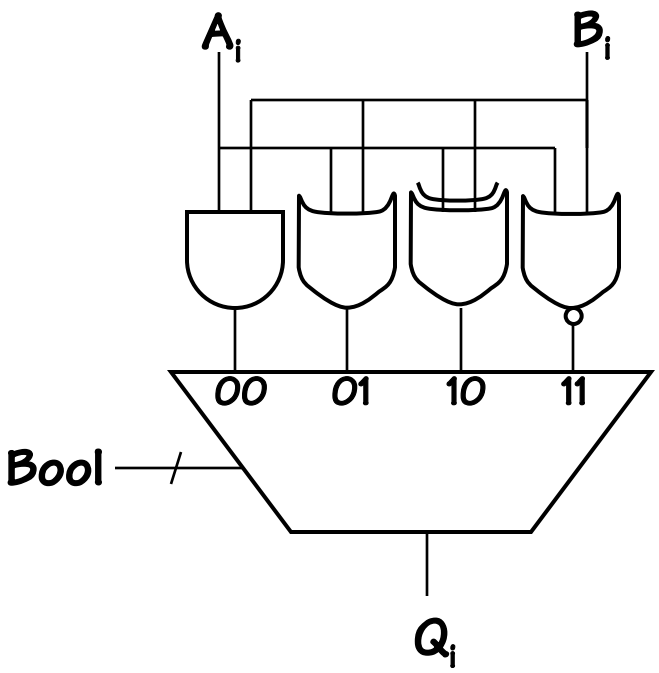
NORing is useful.. Uhm, because John Hennessy says it is!
ex. 10101110 # 00001111 = 01010000 (0's complement, 1's clear)

Boolean Unit

It is simple to build up a Boolean unit using primitive gates and a mux to select the function.

Since there is no interconnection between bits, this unit can be simply replicated at each position. The cost is about 7 gates per bit. One for each primitive function, and approx 3 for the 4-input mux.



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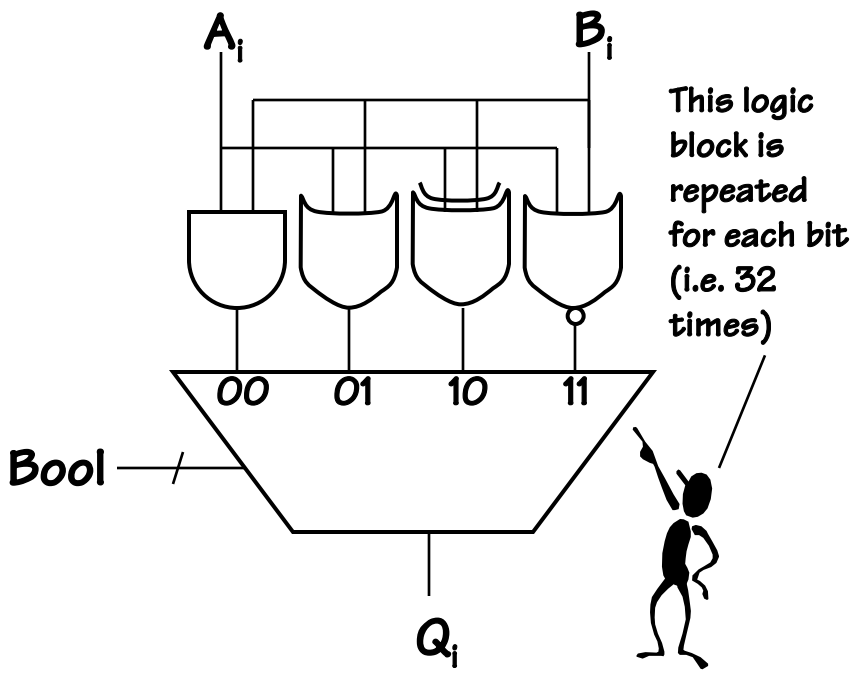

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An ALU, at Last

Now we're ready for a big one! An Arithmetic Logic Unit.