Traffic and Related Self-Driven Many-Particle Systems

Presentation by David Wilkie

March 24, 2009

Motivation

- ”The volume of vehicular traffic in the past several years has rapidly outstripped the capacities of the nation’s highways. It has become increasingly necessary to understand the dynamics of traffic flow and obtain a mathematical description of the process.” –Greensberg, *1959*

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- ...and the situation hasn’t improved.

Can science help?

Some fairly wild claims.

Research Timeline

- 1935 – Early research on vehicular traffic by Greenshields.
- 1950s – Research activity on traffic in operations research and engineering.
- 1992 – Physicists take notice, starting with Biham, Nagel and Schreckenberg, and Kerner and Konhauser.
- And an avalanche of research followed, including in computer science.
- These papers view traffic as a *self-driven nonequilibrium system*.

Nonequilibrium Systems

- Some systems are not closed. They exchange
  - Energy,
  - Particles, or
  - Information

  with the surrounding environment.
- These systems are called *Nonequilibrium* systems.
- They often show complex behavior, and no general results exist as they do for gasses, liquids and solids in equilibrium.

Particle Pair Interactions

- Consider the equations of motion from classical mechanics for a particle α subject to *pair interactions* with objects β.

$$m_\alpha \ddot{\mathbf{x}}_\alpha(t) = \sum_{\beta(\neq\alpha)} \mathbf{F}_{\alpha\beta}(t) \qquad (1)$$

- These interaction forces usually depend on the distance $\mathbf{d}_{\alpha\beta} = (\mathbf{x}_\beta - \mathbf{x}_\alpha)$, but can also depend on other properties, like the velocities.

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- How would a celestial system be characterized?

Driven Many-Particle Systems

- A many-body system subject to additional interactions with the environment is said to be *driven*.
- Examples include fluids subject to boundary forces or sand subject to vibration.
- Additional forces need to be added, including
  - $\mathbf{F}_0(\mathbf{x}, t)$ – forces due to boundary interactions and force fields,
  - $\mathbf{F}_{fr}(t)$ – frictional forces, and
  - $\zeta_a(t)$ – individual fluctuations due to thermal interactions or variation in particle surface structure.
- Combining this with Eq. (1), we have

$$m_a \ddot{\mathbf{x}}_a(t) = \mathbf{F}_0(\mathbf{x}, t) - \mathbf{F}_{fr}(t) + \sum_{\beta(\neq a)} \mathbf{F}_{a\beta}(t) + \zeta_a(t). \qquad (2)$$

Self-Driven Many-Particle Systems

- To model living systems of cells, animals or even humans, we can use the simple abstraction of the *self-driven* particle.
- These systems are driven systems, but the driving forces are self-produced.
- We can modify Eq. (2) by changing $\mathbf{F}_0(\mathbf{x}, t)$, the external driving force to, $\mathbf{F}_0(\mathbf{x}, t)$, a self-produced driving force.
- Additionally, *Newton's third law, (action=reaction)* does not necessarily apply.

Self-Driven Many-Particle Systems, 2

- Let us express a particle's acceleration in terms of scaled quantities.
- Then our driving force is $\mathbf{F}_\alpha^0(t) = \gamma_\alpha v_\alpha^0(t)\mathbf{e}_\alpha^0(t)$, where $\gamma_\alpha = m_\alpha/\tau_\alpha$.
- In these systems, the idea of *mass* is not always well defined. So we'll define $\mathbf{F}_{\alpha\beta}(t) = m_\alpha \mathbf{f}_{\alpha\beta}$ and $\zeta_\alpha(t) = \gamma_\alpha \xi_\alpha(t)$.

Self-Driven Many-Particle Systems, 3

- Using these, we can define our particle's acceleration as

$$\frac{d\mathbf{v}_\alpha(t)}{dt} = \frac{v_\alpha^0(t)\mathbf{e}_\alpha^0(t) + \xi_\alpha(t) - \mathbf{v}_\alpha(t)}{\tau_\alpha} + \sum_{\beta(\neq\alpha)} \mathbf{f}_{\alpha\beta}(t). \tag{3}$$

- We see that, with a relaxation time τ_α, the particle will adopt the desired speed $v_\alpha^0(t)$ and direction $\mathbf{e}_\alpha^0(t)$.
- This desired velocity is perturbed by the fluctuations and interaction forces.

Self-Driven Many-Particle Systems, 4

- We can further simplify this equation by assuming instaneous relaxation, $\tau_\alpha \approx 0$.
- This yields an equation for the velocity of particle α of

$$\mathbf{v}_\alpha(t) = v_\alpha^0(t)\mathbf{e}_\alpha^0(t) + \sum_{\beta(\neq\alpha)} \mathbf{v}_{\alpha\beta}(t) + \xi_\alpha(t), \tag{4}$$

where $\mathbf{v} = \tau_\alpha \mathbf{f}_{\alpha\beta}(t)$.

Can we really simulate humans as particles?

- Human behavior appears chaotic, irregular, and unpredictable, so when can we use the above equations?
- We can use them in situations in which
  - There is movement in a continuous space (possibly an abstract space), and
  - Most of the movement is due to deterministic processes we can model.
- Traffic seems to meet these requirements: we drive in a continuous space and most actions can be considered *automatic*, such as turning, accelerating, and changing lanes.

Methods of Measurement

- Previous empirical data gathering techniques include
  - aerial photography,
  - equipment in cars, and
  - detectors at road cross sections x, the most widely used technique.
- An example is the single-loop induction detector, below.

Detector Measurements

- As detectors are the most widely used form of data gathering, let's investigate what they measure.
- Over a time interval ΔT, a detector can measure
  - ΔN – the number of vehicles a that cross the detector,
  - t_a^0 and t_a^1 – the times at which a vehicle reaches the detector and leaves the detector,
  - v_a – a vehicle's velocity, and
  - l_a – a vehicle's length.
- From these, we can calculate

$$Q(x,t) = \frac{\Delta N}{\Delta T}, \tag{5}$$

the *vehicle flow*, as well as the mean velocity,

$$V(x,t) = \langle v \rangle = \frac{1}{\Delta N} \sum_a v_a. \tag{6}$$

Density and Measurement Issues

- Using Eqs. 5 and 6, we can define density using the fluid-dynamic equation,

$$\rho(x,t) = Q(x,t)/V(x,t). \tag{7}$$

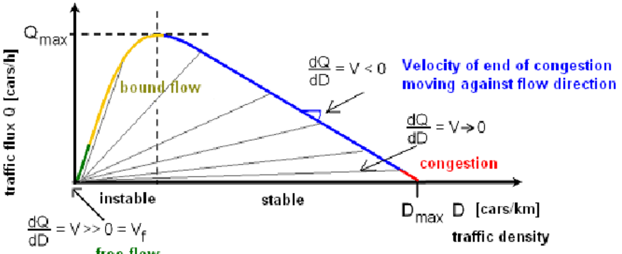
- However, there is a problem: the velocity distribution depends on how it's measured.
  - If the velocity distribution is measured over a length ΔX, the result will count fast moving cars more often than slow moving cars.
  - This is not true if the distribution is measured over a time interval, ΔT.
- Our density equation mixes a temporal measurement, Q, with a spatial, V, which causes a slight bias.

The Fundamental Diagram

- The relations between flow, average velocity, and density have long held academic interest.
- We can develop an empirical flow-density relation, shown below, called the *fundamental diagram*,

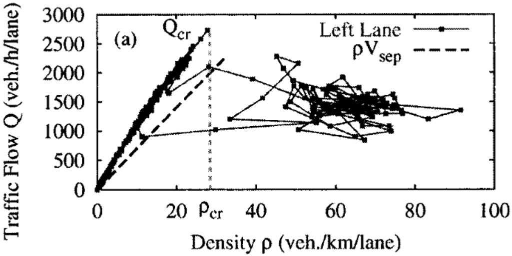
$$Q_e(\rho) = \rho V_e(\rho), \tag{8}$$

where V_e and Q_e are empirically gathered.



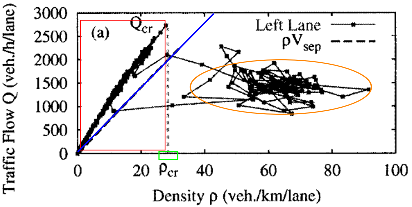
Fundamental Diagram Without Jams

- By removing data of cars in *wide moving jams*, we get a different flow-density relation, which calls into question the fundamental diagram.
- Here, the congested traffic is termed *synchronized flow* as the velocities of the cars tend to be the same.



Observations of the Fundamental Diagram

- At low densities, $\exists$ an almost 1D relation between flow and density. (Red)
- We can approximately divide *free* traffic from *congested* traffic at ρV_{sep}. (Blue)
- There is a critical density region ρ_{cr} where we have either free or congested traffic. (Green)
- The flow for *synchronized traffic* is scattered in a wide region. (Orange)



Further Empirical Observations

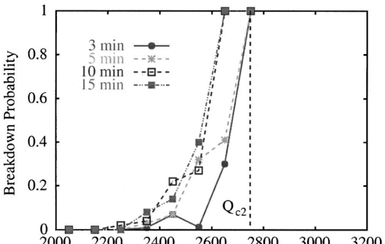
- Driver behavior appears to be influenced by the clearance to the next car and by relative velocity.
- Different lanes have different average velocities, but the difference decreases almost linearly with density.
- Even in congested traffic, when velocities are nearly synchronized, the average speed for cars is faster than that of trucks.

Correlations

- At low densities, there is a strong positive correlation between flow and density.
- At high density, there is a strong negative correlation between velocity and density.
- The velocity average and variance are correlated.
- In congested traffic, the average velocity of neighboring lanes are synchronized, and
- density changes in neighboring lanes are correlated.
- In free traffic, velocities of successive cars are seemingly independent, but
- successive cars have long range velocity correlation in synchronized flow traffic, which is interpreted as *platoons* of cars.

Stop-and-go Waves

- A phenomenon of congested traffic is stop-and-go waves.
- The velocity of the *fluent stage* does not depend on the flow, but the oscillation frequency does.
- The duration of a wave is between 4 and 20 minutes, and the average wavelength is between 2.5 and 5 km.
- The probability of a *fluent stage* breaking down into congestion is dependent on the waiting time.



Phantom Traffic Jams

- Many traffic jams seem to have no cause - no accident or bottleneck.
- However, aerial photography has traced these jams to a vehicle changing lanes in front of a chain of closely following cars.
- A small disturbance like this can cause a large jam formation.

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- However, aerial photography has traced these jams to a vehicle changing lanes in front of a chain of closely following cars.
- A small disturbance like this can cause a large jam formation.
- Phantom traffic jam → some jerk's traffic jam

Wide Moving Jams

- Traffic jams can propagate upstream, against the flow of traffic.
- The speed at which they propagate C is roughly constant at 15 ± 5km/h.
- Additional "universal" characteristics include
  - the density ρ_{jam} inside jams,
  - the average velocity and flow within jams (~ 0),
  - the outflow of jams (approximately 2/3 of free flow Q_{max}), and
  - the density ρ_{out} downstream of jams.

Extended Congested Traffic – or Synchronized Flow

- The most common form of jam is the type that forms every rush hour.
- These jams are caused by some bottleneck ∈ –on-ramps, lane number reductions, accidents, speed limits, road works, gradients, curves, road conditions, visibility conditions, etc.˝.
- The congested velocity drops, but is finite.
- The flow drops, but less so than the velocity. It also exhibits a near linear flow-density relation.
- The front of this type of jam is fixed at the bottleneck.
- The end of the jam will move depending on the incoming flow:
  - If the incoming flow is greater than the bottleneck capacity Q_{bot}, the end moves upstream,
  - Otherwise it moves downstream.

Theories of Jam Formation

- Kerner and Rehborn pointed out that the transition from free to congested traffic appears similar to the phase transition from supersaturated water to vapor.
  - It is often triggered by a small overcritical peak in traffic flow.
  - This perturbation grows and moves upstream.
- Stop-and-go traffic has been questioned by Kerner and Daganzo, who suggest a mechanism for jam formation:
  - In synchronized flow, upstream of the bottleneck, there is a *pinch region*.
  - Within this region, there are spontaneous births of small density clusters.
  - Wide moving jams can form from the merging of these clusters.

Timeline of Modeling Approach Invention

- 1950s – Microscopic (follow-the-leader) models
- 1950s – Macroscopic (fluid-dynamic) models
- 1960s – Mesoscopic (gas-kinetic) models
- 1990s – Cellular Automata models
- Over 100 models total from engineering, mathematics, operations research, physics, *and computer science*.

Criteria for Good Models

- Models should have only a few, intuitive parameters and variables.
- Variables should be easy to measure.
- Models should reproduce all known traffic phenomena.
- Models should be theoretically consistent and make new predictions.
- Models should not lead to collisions (unless that is the intent..).
- Models should allow for fast numerical simulation.

Microscopic Framework

- Microscopic models assume the acceleration of car a is dependent on neighboring vehicles.
- The primary influence is the *leading vehicle*, $a-1$, i.e. the car ahead.
- The model of behavior is then

$$\frac{dv_a(t)}{dt} = \frac{v_a^0 + \xi_a(t) - v_a(t)}{\tau_a} + f_{a,a-1}(t), \tag{9}$$

- where $f_{a,a-1}$ describes the effect of $a-1$ on a, and is generally a function of
  - relative velocity, $\Delta v_a(t)$,
  - the velocity of a, and
  - the headway, $d_a(t) = x_{a-1}(t) - x_a(t)$, or clearance, $s_a(t) = d_a(t) - \text{length}_{a-1}$.

Microscopic Framework Simplification

- Our interaction function is then $f(s_a(t), v_a(t), \Delta v_a(t))$.
- Let us also define a traffic-dependent velocity as $v^e(s_a, v_a, \Delta v_a) = v_0 + \tau f(s_a, v_a, \Delta v_a)$.
- Ignoring fluctuations, we can rewrite our model as

$$\frac{dv_a}{dt} = \frac{v^e(s_a, v_a, \Delta v_a)}{\tau} \tag{10}$$

Noninteger Car-following Model, 1

- The simplest assumption for a *follow the leader* model is that the clearance is equal to the velocity-dependent safe distance, or

$$s_a(t) = s^*(v_a(t)) = s' + Tv_a, \tag{11}$$

  where T is the safe time clearance.
- Differentiation yields

$$\frac{dv_a(t)}{dt} = \frac{v_{a-1} - v_a}{T}. \tag{12}$$

- However, this model does not have the empirically observed stop-and-go waves.
- To produce these, the model is modified into a *delay differential equation* by adding a time delay,

$$\frac{dv_a(t + \Delta t)}{dt} = \frac{v_{a-1} - v_a}{T}. \tag{13}$$

Noninteger Car-following Model, 2

- The time delay does yield stop-and-go traffic, but it also causes cars to collide.
- To remedy this, and other issues, a *generalized sensitivity factor* was introduced,

$$\frac{1}{T} = \frac{1}{T_0} \frac{(v_a(t+\Delta t))^{m_1}}{(x_{a-1}(t) - x_a(t))^{m_2}}. \qquad (14)$$

- Plugging this equation into Eq. (13) and simplifying yields

$$f_{m_1}(v_a(t+\Delta t)) = c_0 + c_1 f_{m_2}(d_a(t)), \qquad (15)$$

- with $f_k(z) = z^{1-k}$ if $k \neq 1$ and $\ln(z)$ otherwise, and c_1, c_0 are integration constants.

Newell Model

- A flaw in the separation distance model above is that cannot describe the behavior of a single vehicle, i.e. for $d_a \to \inf$.
- In this case, the car a should adapt to a desired velocity v_a^0.
- Generally, we want a to adapt to a distance-dependent "optimal" velocity $v_e'(d_a)$, so that safety is taken into account.
- One model, by Newell of this assumes a delay,

$$v_a(t + \Delta t) = v_e'(d_a(t)) = v_e(s_a(t)). \tag{16}$$

Optimal Velocity Model

- Alternatively Bando *et al.* suggest the velocity

$$v_e'(d) = (v_0/2)(\tanh(d - d_c) + \tanh d_c) \tag{17}$$

with constants v_0, d_c in the optimal velocity model,

$$\frac{dv_\alpha(t)}{dt} = \frac{v_e'(d_\alpha(t)) - v_\alpha(t)}{\tau}. \tag{18}$$

- This latter equation can model the amplification of a small perturbation into a traffic jam if
  - the relaxation time τ is large or
  - the change in $v_e(s_\alpha)$ with clearance s_α is large.

Intelligent Driver Model, 1

- A problem with the above model is that it does not consider relative velocity, Δv.
- It thus inaccurately models the distance real drivers keep from each other at high Δv,
- And cars with high Δv can collide.
- The *intelligent driver model* is an example of a model meant to more accurately capture how drivers behave.
- In this case, the acceleration of a is a continuous function of s_a, Δv, and v_a:

$$\frac{dv_a}{dt} = a_a \left[1 - \left(\frac{v_a}{v_a^0} \right) - \left(\frac{s_a^*(v_a, \Delta v_a)}{s_a} \right) \right]. \tag{19}$$

Intelligent Driver Model, 2

- This model is a superposition of an acceleration tendency, $a_a(1-(v_a/v_a^0)^\delta)$, and a deceleration tendency $f_{a,a-1} = -a_a(s_a^*(v_a, \Delta v_a)/s_a)^2$.
- The parameter δ allows the acceleration to be fit:
  - $\delta = 1$ corresponds to an exponential-in-time acceleration;
  - $\delta \to \inf$ corresponds to constant acceleration of a_a.
- Deceleration depends on the ratio of the desired clearance to the actual clearance, where the desired clearance is given as

$$s_a^*(v_a, \Delta v_a) = s_a' + s_a'' \sqrt{\frac{v_a}{v_a^0}} + Tv_a + \frac{v_a \Delta v_a}{2\sqrt{a_a b_a}}. \tag{20}$$

- The model parameters are then

desired velocity v_a^0,	safe time clearance T,
max acceleration a_a,	max deceleration b_a,
acceleration exponent δ,	and jam lengths s' and s''

Features of Cellular Automata

- Cellular automata models are interesting for their speed and their complex dynamic behavior.
- Their speed comes from their
  - uniform discretization of space,
  - finite number of possible states,
  - parallel, uniform time update,
  - global update rules, and
  - short range of interaction.
- And they've been shown exhibiting dynamics such as
  - self-organized criticality,
  - formation of spirals, and
  - oscillatory or chaotic sequences of states.

Cellular Automata Traffic Modeling

- Cellular automata models are less detailed than the follow the leader models above.
- A basic approach is to
  - divide the road into cells of equal length, Δx,
  - divide the time into intervals of equal duration, Δt,
  - allow each cell to be either occupied or vacant,
  - set each car's speed to $v_i = \hat{v}_i \frac{\Delta x}{\Delta t}$, where $\hat{v}_i$ is an integer $\leq v_{\hat{\text{max}}}$.
- Each timestep, the state of the cells, occupied or vacant, changes based on a set of rules.

Nagel-Schreckenberg Model

- In a model proposed by Nagel and Schreckenberg (92), for every rule, each car
  - *Motion:* moves forward by $\hat{v}_i$ cells;
  - *Acceleration:* accelerates by 1 if $\hat{v}_i < \hat{v_{\text{max}}}$;
  - *Deceleration:* adopts a new velocity $v_i'' = (\hat{d}_i - 1)$ if $\hat{d}_i \leq \hat{v}_i'$;
  - *Randomization:* slows by 1 with probability p.
- To summarize, the rules amount to choosing a new velocity based on

$$\hat{v}_{i++} = \max(0, \min(\hat{v}_{\text{max}}, \hat{d}_i - 1, \hat{v}_i + 1) - \xi_i^{(p)}), \qquad (21)$$

where d_i is the clearance, v is velocity, and $\xi_i^{(p)}$ is 1 with probability p and 0 otherwise.

Randomization in the Nagel-Schreckenberg Model

- The randomized slowdown effect models delayed acceleration or other imperfect driving and is needed for the model to exhibit stop and go waves, shown below.

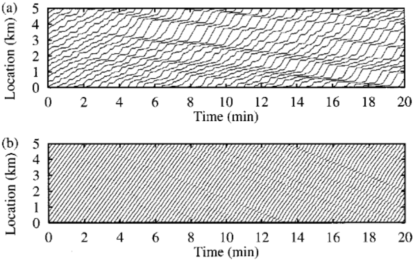

FIG. 28. Trajectories of each tenth vehicle for unstable traffic in the Nagel-Schreckenberg model, if the slowdown probability is (a) $p = 0.5$ and (b) $p = 0.001$. In the limit $p \rightarrow 0$, the jam amplitude goes to zero.

Variations on the Basic Model

- One variant of this model features *cruise control*, meaning that the randomized slow down does not occur when the agent is moving at maximum speed.
- Another variant includes randomized acceleration.
- A *slow-to-start* rule is included in one variant. This variant replaces *acceleration* by the rule
  - accelerate by 1 with probability $q = (1 - p)$ if the car is not moving and there is exactly one empty cell ahead,
  - otherwise, accelerate by 1 deterministically.

Macrosopic Model Overview

- Unlike Microscopic models, Macroscopic models only deal with collections of vehicles.
- The calculations are done in terms of descriptions of these collectives:
  - spatial vehicle density $\rho(x, t)$,
  - average velocity $V(x.t)$, and
  - traffic flow or flux $Q(x, t) = \rho(x, t)V(x, t)$.
- Macroscopic models are computationally more efficient than microscopic models, but less so than cellular automata.

Features of Macroscopic Models

- Though slower than cellular automata models, macroscopic models are ofter preferred for their
  - good agreement with empirical data,
  - suitability for analytical investigations,
  - simple treatment of inflows from ramps, and
  - ability to simulate multi-lane traffic using a collection of one-lane models.

Lighthill and Whitham Model

- The oldest and still most popular macroscopic model is by Lighthill and Whitham.
- This model is based on the observation that, away from ramps and other roads, the number of cars within a road is conserved.
- This leads to a *continuity equation*,

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(x.t)}{dx} = 0. \qquad (22)$$

Flow Equation

- A difficulty in using this model is specifying the flow, $Q(x,t)$.
- One approach is to assume the flow is a function of density,

$$Q(x,t) = Q_e(\rho(x,t)) = \rho V_e(\rho(x,t)) \geq 0 \tag{23}$$

where $V_e(\rho(x,t))$ is a function fit to empirical data.
- We can substitute this into Eq. (22), which gives us

$$\frac{\partial \rho}{\partial t} + C(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{24}$$

where $C(\rho)$ is given as

$$C(\rho) = V_e(\rho) + \rho \frac{dV_e}{d\rho}. \tag{25}$$

Velocity Equation

- In the above formula, Eq. (25), V_e is a function fit to the emipirical velocity data.
- One model if this is from Greenshields (1935), who suggested a linear relation

$$V_e(\rho) = V_0(1 - \rho/\rho_{\text{jam}}), \tag{26}$$

in which V_0 is a preferred velocity and ρ_{jam} is the density at which free traffic changes to congested traffic.

Shock Waves

- The density waves of the Lighthill-Whitham model tend to form shock fronts over time.
- This feature makes the model difficult to integrate.

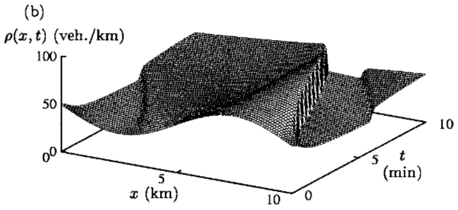

FIG. 30. Formation of shock fronts: (a) Trajectories of each tenth vehicle and (b) corresponding spatio-temporal density plot illustrating the formation of a shock wave on a circular road according to the Lighthill-Whitham model. Although the initial condition is a smooth sinusoidal wave, the upstream and downstream fronts grow steeper and steeper, eventually producing discontinuous jumps in the density profile, which propagate with constant speeds. The wave amplitude remains

Diffusion Term

- To avoid the development of shock waves, a diffusion term can be added to smooth the wave fronts.
- One such term is $Q = Q_e(\rho) - D\partial\rho/\partial x$, or equivalently

$$V(x,t) = V_e(\rho(x,t)) - \frac{D}{\rho(x,t)}\frac{\partial\rho(x,t)}{\partial x}, \tag{27}$$

- where D is the diffusion constant.

Burger's Equation

- Eq. (27) above can be integrated into our continuity equation,

$$\frac{\partial \rho}{\partial t} + \left(V_e(\rho) + \rho \frac{dV_e}{d\rho} \right) \frac{\partial \rho}{\partial x} = 0, \tag{28}$$

  to yield

$$\frac{\partial \rho}{\partial t} + \left[V_e(\rho) + \rho \frac{dV_e}{d\rho} \right] \frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x} \left(D \frac{\partial \rho}{\partial x} \right). \tag{29}$$

- Letting the wave propagation speed be $C(x,t) = V_0[1 - 1\rho(x,t)/\rho_{\text{jam}}]$, the above equation can be transformed into *Burger's Equation*,

$$\frac{\partial C(x,t)}{\partial t} + C(x,t) \frac{\partial C(x,t)}{\partial x} = D \frac{\partial^2 C(x,t)}{\partial x^2}, \tag{30}$$

  which can be solved exactly due to its similarity with the linear heat equation.

Payne's Velocity Equation

- The Lighthill-Whitham model cannot capture the stop-and-go behavior of traffic.
- To achieve this, Payne derived a velocity equation based on the optimal velocity microscopic model,

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = \frac{1}{\Delta t}\left[V_e(\rho) - V - \frac{D(\rho)}{\rho}\frac{\partial \rho}{\partial x}\right], \tag{31}$$

  where $V_e(\rho) - V$ is the *relaxation term* and $-[D(\rho)/\rho]$ is the *anticipation term*.
- The relaxation term describes the adaptation of the average velocity V to the density-dependent equilibrium velocity $V_e(\rho)$.
- The anticipation term describes the reaction of drivers to their surroundings.
- This model has the instability of stop-and-go waves.

Phase-Space Density

- The gas-kinetic models of traffic are based on the idea of phase-space density,

$$\tilde{\rho}(x, v, t) = \rho(x, t)\tilde{P}(v; x, t), \tag{32}$$

where $\rho(x, t)$ is the vehicle density and $\tilde{P}(v; x, t)$ is the distribution of velocities at x, t.

Continuity Equation for Phase-Space Density

- As vehicles are still conserved, we can write a kind of continuity equation,

$$\frac{\partial\tilde{\rho}}{\partial t} + v\frac{\partial\tilde{\rho}}{\partial x} = \left(\frac{d\tilde{\rho}}{dt}\right)_{\text{acc}} + \left(\frac{d\tilde{\rho}}{dt}\right)_{\text{int}} . \tag{33}$$

- The right hand side is not 0 as vehicles can change their speed.
- The two terms represent the changes in the phase-space density due to *acceleration* and *interactions*.

Acceleration Term

- The acceleration term is defined as

$$\left(\frac{d\tilde{\rho}}{dt}\right)_{\text{acc}} = \rho(x,t)\frac{\tilde{P}_0(v) - \tilde{P}(v;x,t)}{\tau(\rho(x,t))}, \tag{34}$$

or a relaxation of the current velocity distribution $\tilde{P}(v;x,t)$ to some desired distribution $\tilde{P}(v)$.

Interaction Term

- The interaction term is defined as the Boltzmann-like equation

$$\left(\frac{d\tilde{\rho}}{dt}\right)_{\text{int}} = \int_{w>v} (1 - \hat{p}(\rho))|w - v|\tilde{\rho}(x, w, t)\tilde{\rho}(x, v, t)dw \qquad (35)$$
$$- \int_{w<v} (1 - \hat{p}(\rho))|v - w|\tilde{\rho}(x, w, t)\tilde{\rho}(x, v, t)dw.$$

- The idea is that vehicles with velocity w are either faster or slower than vehicles with velocity v.
- Vehicles with velocity w will interact with vehicles with velocity v at a rate of $|w - v|\tilde{\rho}(x, w, t)\tilde{\rho}(x, v, t)$, which describes how often vehicles with velocities w and v meet at place x.
- A faster vehicle w can overtake a slower vehicle with probability $\tilde{p}(\rho)$, so it will have to slow down win probability $(1 - \tilde{p}(\rho))$, increasing the phase space density.

The Density and Velocity Equations

- As the density is given by

$$\rho(x,t) = \int \tilde{\rho}(x,v,t)dv, \tag{36}$$

we can integrate the phase-space density continuity equation.

- Doing so produces a density equation and velocity equation
  - Density Equation:

$$\frac{d\rho_v}{dt} = -\rho\frac{\partial V}{\partial x} \tag{37}$$

  - Velocity Equation:

$$\frac{dV_v}{dt} = -\frac{1}{\rho}\frac{d\rho\Theta}{d\rho}\frac{\partial\rho}{\partial x} + \frac{1}{\tau}(V_e - V), \tag{38}$$

where $V_e = V_0 - \tau(\rho)[1 - \hat{p}(p)]\rho\Theta$, Θ is the velocity variance, and $\hat{p}(\rho)$ is the probability of overtaking.

Computational Efficiency

- Broadly, the computational costs of the models can be divided up from fastest to slowest as:
  - Cellular Automata
  - Macroscopic
  - Mesoscopic
  - Microscopic
- The accuracy of the models is harder to discuss, but it can be considered roughly the reverse of the above list.

Considerations in Choosing a Model

- While the cellular automata model is the fastest computationally, its parameters and mechanisms can bee seen as unrealistic from a physical point of view.
- Macroscopic models are faster than microscopic and their parameters are empirically measurable, but it can be difficult to integrate heterogeneous agents and to generalize to road topologies other than a straight highway.
- While microscopic offers the finest level of detail, there is still no guarantee that your result will actually be more accurate than a macroscopic simulation.

Qualitative Performance

- Models from all the groups can capture phenomena such as
  - Stop-and-Go Traffic,
  - ”constants” of traffic including the fundamental diagram,
  - synchronized flow, and
  - wide moving jams.
- Though different models have different mechanisms and tuning requirements to achieve these phenomena.