Properties of Regular Languages

COMP 455 – 002, Spring 2019

What Languages Aren't Regular?

The tool used for proving that a language is *not* regular is the *Pumping Lemma*.

The Pumping Lemma

Theorem 4.1 (the *Pumping Lemma for regular languages*):

Let L be a regular language. Then there exists a constant n (which depends on L) such that for every $w \in L$, where $|w| \geq n$, there exists strings x, y, and z such that

i. $w = xyz$,

ii. $|xy| \leq n$,

iii. $|y| \geq 1$, and

iv. for all $k \geq 0$, $xy^k z \in L$.

Proof of the Pumping Lemma

Theorem 4.1: Let L be a regular language. Then there exists a constant n (which depends on L) such that for every $w \in L$, where $|w| \geq n$, there exists strings x, y, and z such that **(i)** $w = xyz$, **(ii)** $|xy| \leq n$, **(iii)** $|y| \geq 1$, and **(iv)** for all $k \geq 0$, $xy^k z \in L$.

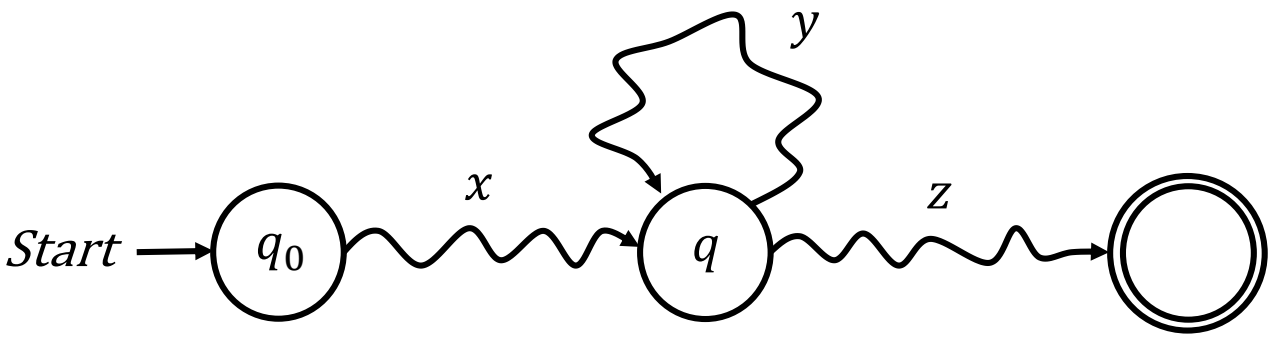
- Since L is regular, it is accepted by some DFA M.
- Let $n =$ the number of states in M.
- Pick any $w \in L$, where $|w| > n$.
- By the pigeonhole principle, M must repeat a state when processing the first n symbols in w.

Proof of the Pumping Lemma

Theorem 4.1: Let L be a regular language. Then there exists a constant n (which depends on L) such that for every $w \in L$, where $|w| \geq n$, there exists strings x, y, and z such that **(i)** $w = xyz$, **(ii)** $|xy| \leq n$, **(iii)** $|y| \geq 1$, and **(iv)** for all $k \geq 0$, $xy^k z \in L$.

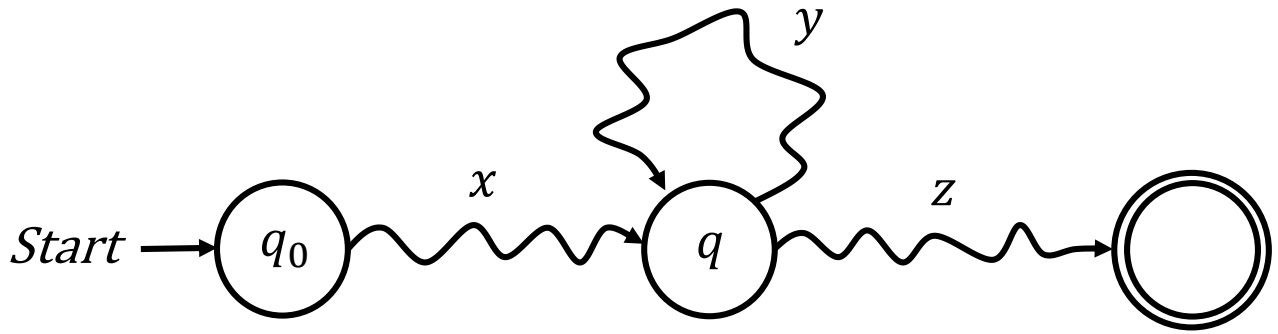
M must repeat a state, q, when processing the first n symbols of input w.

Define strings x, y, and z as in this figure:



Proof of the Pumping Lemma

Theorem 4.1: Let L be a regular language. Then there exists a constant n (which depends on L) such that for every $w \in L$, where $|w| \geq n$, there exists strings x, y, and z such that **(i)** $w = xyz$, **(ii)** $|xy| \leq n$, **(iii)** $|y| \geq 1$, and **(iv)** for all $k \geq 0$, $xy^k z \in L$.



The second occurrence of state q must occur within the first n symbols of w, so $|xy| \leq n$. Also, $|y| \geq 1$.

It must be possible to repeat the "y-loop" 0 or more times, and the resulting string will still be accepted.

Therefore, $xy^k z \in L$, for any $k \geq 0$.

Using the Pumping Lemma

- To show that a language L is *not* regular, show that the conditions of the Pumping Lemma do not hold.
- Formally, the Pumping Lemma says:

$$\exists n$$
$$:: \Big(\forall w : w \in L \wedge |w| \geq n$$
$$:: \Big(\exists x, y, z : xyz = w \wedge |xy| \leq n \wedge |y| \geq 1$$
$$:: \big(\forall k : k \geq 0 \ :: xy^k z \in L\big)\Big)\Big)$$

Using the Pumping Lemma

- To show that a language L is *not* regular, show that the conditions of the Pumping Lemma do not hold.
- The negation of the Pumping Lemma is:

$$\forall n$$
$$:: \Big(\exists w : w \in L \wedge |w| \geq n$$
$$:: \Big(\forall x, y, z : xyz = w \wedge |xy| \leq n \wedge |y| \geq 1$$
$$:: \big(\exists k : k \geq 0 \ :: xy^k z \notin L\big)\Big)\Big)$$

Using the Pumping Lemma

$$\forall n$$
$$:: \Big(\exists w : w \in L \wedge |w| \geq n$$
$$:: \Big(\forall x, y, z : xyz = w \wedge |xy| \leq n \wedge |y| \geq 1$$
$$:: \big(\exists k : k \geq 0 \ :: xy^k z \notin L\big)\Big)\Big)$$

No matter what n is…

We can find a string w longer than n in L…

Where no matter how w is divided into substrings x, y, and z (with some constraints)…

We can find a value k for which $xy^k z$ is not in L.

Show that this statement is true for L to show that L is *not* a regular language!

Using the Pumping Lemma: Example

- Consider the language 0^{i^2}, where $i \geq 1$.
- We want to prove that this language is *not* regular.
- We need to show that $\forall n :: \Big(\exists w : w \in L \wedge |w| \geq n \ :: \Big(\forall x, y, z : xyz = w \wedge |xy| \leq n \wedge |y| \geq 1 \ :: \big(\exists k : k \geq 0 \ :: xy^k z \notin L\big)\Big)\Big)$ is true for L.
- The proof is like a game with an adversary:
  - The adversary makes the "∀" choices.
  - We make the "∃" choices.

Using the Pumping Lemma: Example

$$\forall n :: \Big(\exists w : w \in L \wedge |w| \geq n :: \Big(\forall x, y, z : xyz = w \wedge |xy| \leq n \wedge |y| \geq 1 :: \big(\exists k : k \geq 0 :: xy^k z \notin L\big)\Big)\Big)$$

- $L = \{0^{i^2} \mid i \geq 1\}$
- Assume L is regular (this will be a proof by contradiction).
- Select any n.
- Let $w = 0^{n^2}$.
- Select any xyz where $w = xyz$, $|xy| \leq n$, and $|y| \geq 1$.
- This implies that $1 \leq |y| \leq n$.
- Let $k = 2$. This means that $xy^k z$ has $n^2 + |y|$ 0s.

Using the Pumping Lemma: Example

- (Reminder) $L = \{0^{i^2} \mid i \geq 1\}$
- Let $k = 2$. This means that xy^kz has $n^2 + |y|$ 0s.
- w has n^2 0s, and any entry in L longer than w must have at least $(n+1)^2$ 0s.
- $n^2 < n^2 + 1 \leq n^2 + |y| \leq n^2 + n < n^2 + 2n + 1$.

The length of w (in L)

The length of xy^2z if $|y| = 1$.

The length of xy^2z

The length of xy^2z if $|y| = n$.

$= (n+1)^2$, the length of the next string in L.

- So, $xy^kz \notin L$, contradicting the Pumping Lemma.

Using the Pumping Lemma: 2nd Example

- Consider the language $L =$ $\{x \mid x$ contains an equal number of 0s and 1s$\}$
- Is L regular?

Using the Pumping Lemma: 2nd Example

- $L = \{x \mid x$ contains an equal number of 0s and 1s$\}$
- Define $w = 0^n 1^n$ for an arbitrary n. $(w \in L)$
- No matter how w is divided into x, y, and z, y must consist solely of 0s because $|xy| \leq n$ and $y \neq \varepsilon$.
- Therefore, xy^0z has fewer 0s than 1s and is not in L.
  - In this case, $k = 0$.
  - This proof would also work with any $k \geq 2$, because xy^kz would have more 0s than 1s.

Closure Properties

- A **closure property** of regular languages is a property that, when applied to a regular language, results in another regular language.
  - Union and intersection are examples of closure properties.
- We will demonstrate several useful closure properties of regular languages.
- Closure properties can also be useful for proving that languages *aren't* regular.

Closure under Union

Theorem 4.4: If M and N are regular languages, then $M \cup N$ is a regular language.

Proof: Say that R and S are regular expressions where $L(R) = M$ and $L(S) = N$. Construct a regular expression $(R + S)$. This matches $M \cup N$. Since a regular expression exists for $M \cup N$, $M \cup N$ is a regular language.

Note: The proofs for concatenation and Kleene closure are similar.

Closure under Complementation

- If $L \subseteq \Sigma^*$, then the **complement** of L, denoted $\bar{L}$, is $\Sigma^* - L$.
- Theorem 4.5: If L is a regular language over Σ, then $\bar{L}$ is also a regular language.
- Proof sketch for Theorem 4.5:
  1. Construct a DFA for L
  2. This can be transformed into a DFA for $\bar{L}$ by making all accepting states non-accepting and vice versa.
  3. This can be proven correct by induction.

Closure under Intersection

Theorem 4.8: If M and N are regular languages, then $M \cap N$ is a regular language.

Proof: $M \cap N = \overline{\overline{M} \cup \overline{N}}$.

The book contains a more direct proof. The basic idea is to construct a DFA with states labeled $[p, q]$ where p tracks the state of a DFA for M and q tracks the state of a DFA for N.

Closure under Difference

- $M - N \equiv \{x \mid x \in M \wedge x \notin N\}$
- **Theorem 4.10**: If M and N are regular, then so is $M - N$.
- **Proof**: $M - N = M \cap \bar{N}$.

Closure under Reversal

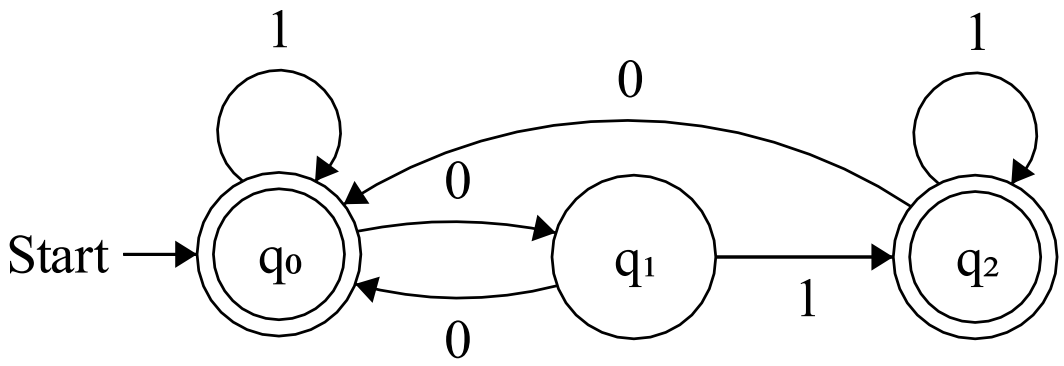
- The **reversal** of L, written L^R is $\{x \mid x^R \in L\}$ (x^R is the string x written backwards).
- **Theorem 4.11**: If L is regular, then so is L^R.

Proof Sketch for Theorem 4.11

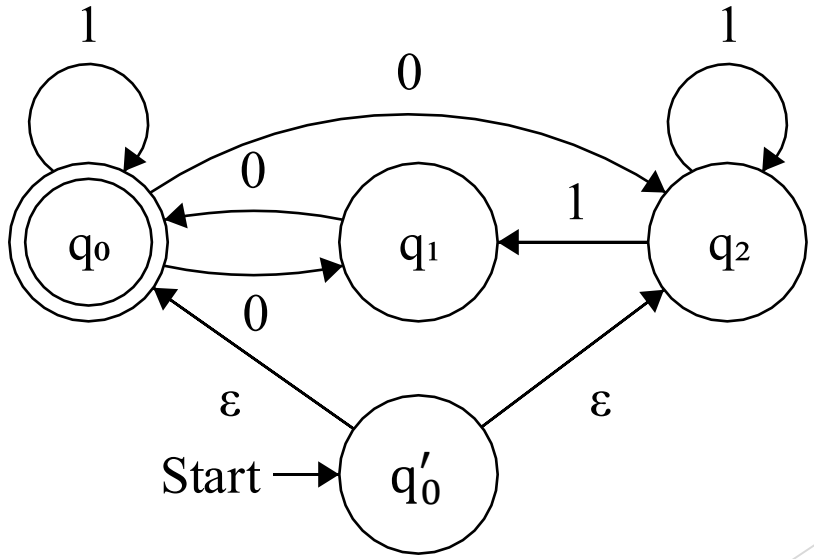
- Start with a DFA for L.
- Construct an ε-NFA for L^R as follows:
  1. Reverse all of the transitions in the DFA
  2. Make the DFA's start state the only accepting state.
  3. Create a new start state with ε-transitions to all of the original accepting states.
- This can be proven correct by induction.
- (The book proves Theorem 4.11 by reasoning about regular expressions.)

Example: Reversal of a DFA

- A DFA for L:



- An NFA with ε-transitions for L^R:



Homomorphisms

- A **homomorphism** maps *symbols* in an alphabet Σ to *strings* over a different alphabet, Δ.
- Example:
  - $\Sigma = \{0, 1\}, \Delta = \{\mathrm{a}, \mathrm{b}\}$
  - $h(0) = \mathrm{ab}, h(1) = \varepsilon$
- Homomorphisms can be extended to strings:
  - $h(\varepsilon) = \varepsilon$
  - $h(xa) = h(x)h(a)$, for string x and symbol a
- Homomorphisms can be extended to languages:
  - If L is a language, $h(L) = \bigcup_{x \in L} h(x)$

Homomorphism Example

- $\Sigma = \{0, 1\}, \Delta = \{\mathrm{a}, \mathrm{b}\}$
- $h(0) = \mathrm{ab}, h(1) = \varepsilon$
- Homomorphism of a string:
  - $h(0011) = \mathrm{abab}$
- Homomorphism of a language:
  - $h(\mathbf{10^*1}) = (\mathbf{ab})^*$

Closure under Homomorphism

Theorem 4.14: If L is a regular language over Σ, and $h: \Sigma \to \Delta^*$ is a homomorphism, then $h(L)$ is also a regular language.

Proof:

- Let $L = L(R)$ be the language defined by some regular expression R.
- Replace each symbol a in R by $h(a)$. Call the resulting regular expression $h(R)$.
- We will prove $L\big(h(R)\big) = h(L)$.

Proof that $L\big(h(R)\big) = h(L)$

- Let E be a subexpression of R.
- **Claim**: $L\big(h(E)\big) = h\big(L(E)\big)$.
- We will prove this by induction on the number of operators in E.
- **Base case**: E is ε, $\emptyset$, or **a**, where $a \in \Sigma$.
  - The only interesting case is where E is a single-character regular expression, so $h\big(L(E)\big) = \{h(a)\}$.
  - $h(E)$ is a regular expression for the same string $h(a)$, so $h\big(L(E)\big) = L\big(h(E)\big)$.

Proof that $L(h(R)) = h(L)$, continued

- Inductive step: E has at least one operator, and therefore has the form $F + G$, FG, or F^*.
- Proof for +:

$L(h(E))$

$= L(h(F) + h(G))$, by the definition of h for reg. exps

$= L(h(F)) \cup L(h(G))$, by the definition of the + operator.

$h(L(E))$

$= h(L(F) \cup L(G))$, since $L(E) = L(F) \cup L(G)$.

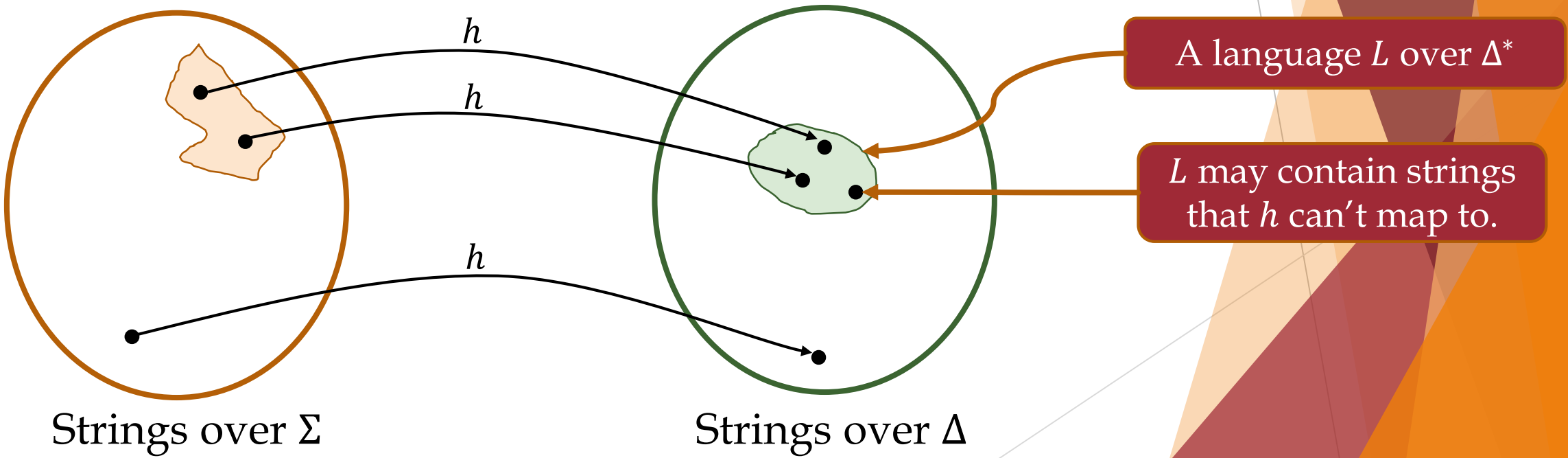
$= h(L(F)) \cup h(L(G))$, since h is applied to individual strings.

Proof that $L(h(R)) = h(L)$,continued

- We have shown the following properties are true for the + operator:
  - $h(L(E)) = h(L(F)) \cup h(L(G))$
  - $L(h(E)) = L(h(F)) \cup L(h(G))$
- To conclude the proof for the + operator, we note that $h(L(F)) = L(h(F))$ and $h(L(G)) = L(h(G))$ by the inductive hypothesis. So, $h(L(E)) = L(h(E))$.
- The proofs for concatenation and Kleene closure are similar.

Inverse Homomorphisms

- If h is a homomorphism from alphabet Σ to alphabet Δ, and L is a language in Δ, then $h^{-1}(L)$ is the set of strings w in Σ^* such that $h(w)$ is in L.



Inverse Homomorphism: Example

It can be difficult to determine $h^{-1}(L)$!

Example:

- $\Sigma = \{0,1\}$, $\Delta = \{\mathrm{a}, \mathrm{b}\}$.
- Let $h(0) = \mathrm{aa}$, $h(1) = \mathrm{aba}$.
- $L = (\mathbf{ab} + \mathbf{ba})^*\mathbf{a}$.

What is $h^{-1}(L)$?

What strings over $\{0, 1\}$ *can* h *map to a string in* L?

Inverse Homomorphism: Example

- $\Sigma = \{0,1\}$, $\Delta = \{a, b\}$
- $h(0) = aa$, $h(1) = aba$
- $L = (\mathbf{ab} + \mathbf{ba})^*\mathbf{a}$

What is $h^{-1}(L)$?

- No string in L begins with aa, so no string starting with 0 (over $\{0, 1\}$) can be mapped to a string in L.
- Strings starting with 1 will always be mapped to strings that start with a, because $h(1) = aba$.
- Strings in L that can start with a:
  - aba } 1 maps to this
  - abab... } No string starting with 1 maps to these
  - abba… } No string starting with 1 maps to these
  - a } No string starting with 1 maps to these
- So, $h^{-1}(L) = \{1\}$.

Closure Under Inverse Homomorphism

Theorem 4.16: If L is a regular language, then $h^{-1}(L)$ is also regular.

Proof:

- Let $M = (Q, \Delta, \delta, q_0, F)$ be a DFA accepting L.
- Let $h: \Sigma \to \Delta^*$.
- Define $M' = (Q, \Sigma, \delta', q_0, F)$ be a DFA accepting $h^{-1}(L)$.

Closure Under Inverse Homomorphism

$$M' = (Q, \Sigma, \delta', q_0, F)$$

- On an input $a \in \Sigma$, M' simulates the behavior of M on $h(a)$.
- Formally, $\delta'(q, a) = \hat{\delta}(q, h(a))$.

Symbol in Σ

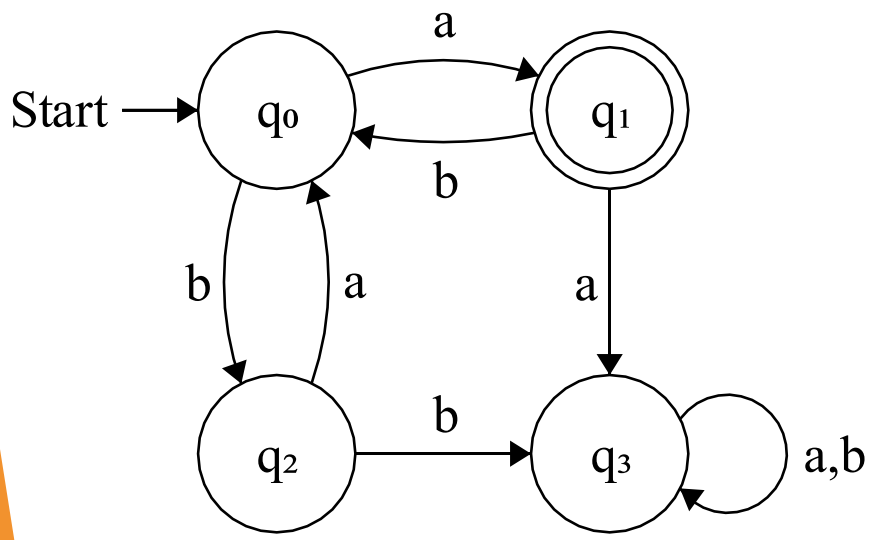
String over Δ

- Exercise: Prove by induction on $|x|$ that M' accepts x if and only if M accepts $h(x)$.

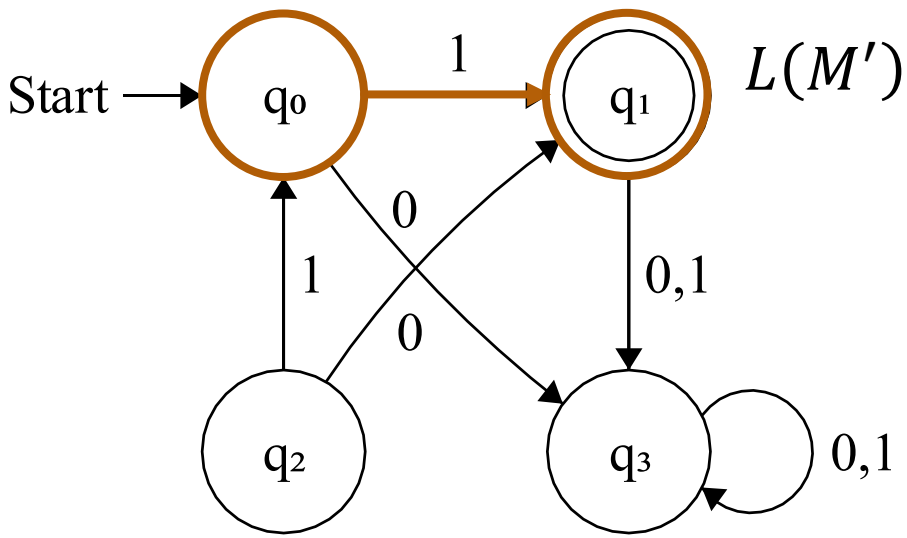
Closure Under Inverse Homomorphism

Example. $L = (\mathbf{ab} + \mathbf{ba})^*\mathbf{a}$, $h(0) = \text{aa}$, $h(1) = \text{aba}$.

M:



M':



$L(M') = \{1\}$

This can be a good strategy for figuring out $h^{-1}(L)$.

Decision Properties

- A **property** is a yes/no question about one or more languages. Some examples:
  - Is L empty?
  - Is L finite?
  - Are L_1 and L_2 equivalent?
- A property is a **decision property for regular languages** if an *algorithm* exists that can answer the question (for regular languages).

Decision Properties

- The book focuses largely on *efficiency issues* when discussing decision properties.
- Efficiency is more of a COMP 550 topic (not a prerequisite), so these slides approach decision properties slightly differently.
- We will consider the three properties from the previous slide: *emptiness*, *finiteness*, and *equivalence*.

Equivalence

- We want an algorithm that takes two languages, L_1 and L_2, and determines if they are the same.

The algorithm:

1. Convert L_1 and L_2 to DFAs.
2. Convert L_1 and L_2 to *minimal DFAs*. (See next slide)
3. Determine if the minimal DFAs are the same.

Minimal DFAs

- Section 4.4 of the textbook gives an algorithm that takes a DFA and outputs a DFA that accepts the same language, but has a minimal number of states.
- This *minimal DFA* is unique.
  - This means that if two different DFAs define the same language, both will be converted to the same minimal DFA.
- We will be skipping this algorithm—it takes a long time to explain and won't be used later in the class. You just need to know that it exists.

Emptiness and Finiteness

Theorem:

The set of strings accepted by a DFA M with n states is:

- nonempty if and only if M accepts a string of length less than n;
- infinite if and only if M accepts a string of length k, where $n \leq k < 2n$.

(This approach is different from the book.)

Proof of "Nonempty" Claim

"…nonempty if and only if M accepts a string of length less than n"

- "If": Obvious
- "Only if": Let w be the length of the shortest string accepted by M.
  - If $|w| < n$ we're done.
  - If $|w| \geq n$, then by the Pumping Lemma $w = xyz$, and $xz \in L(M)$. Contradiction.

We claimed that w was the shortest string accepted by M.

Proof of "Infinite" Claim

"…infinite if and only if M accepts a string of length k, where $n \leq k < 2n$"

- "If": If $w \in L(M)$ and $n \leq |n| < 2n$, then $L(M)$ is infinite by the Pumping Lemma.
- "Only if": Assume $L(M)$ is infinite.
  - Then, there exists some w where $|w| \geq n$. If such a string has a length under $2n$, we're done.
  - Otherwise…

Proof of "Infinite" Claim, continued

"…infinite if and only if M accepts a string of length k, where $n \leq k < 2n$"

…otherwise, all strings with a length greater than n are longer than $2n$. Assume this is the case.

Let w be the shortest such string. By the Pumping Lemma, $w = xyz$, $|y| \geq 1$, and $xz \in L(M)$.

- If $|xz| \geq 2n$, we've contradicted the assumption that w is the shortest string longer than $2n$.
- If $|xz| < 2n$, then $n \leq xz < 2n$. This contradicts the assumption that the shortest string longer than n is longer than $2n$.

Decision Algorithms

- Algorithm for "nonemptiness":
  - See if any string with length at most $|n|$ is in $L(M)$.
  - Can be done using breadth- or depth-first search to find paths from the start state to a final state.
- Algorithm for "infiniteness":
  - See if any string with length $|k|$, where $n \leq k < 2n$ is in $L(M)$.
  - More efficient to check for "reachable cycles".
  - Can use depth-first search, but it's less efficient.

Another Equivalence Algorithm

We can now build another algorithm for testing equivalence:

- Let $L_1 = L(M_1)$ and $L_2 = L(M_2)$
  - M_1 and M_2 are DFAs
- Create M_3, where $L(M_3) = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$
  - Note that $L(M_3)$ is nonempty if and only if $L_1 \neq L_2$.
- Test whether L_3 is empty using the "nonemptiness" algorithm.