Reachability of Black-Box Nonlinear Systems after Koopman Operator Linearization

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Presenter: Kostiantyn Potomkin
Presentation outline

- Motivation
- Koopman Operator
- Challenges
- Verifying linear systems with nonlinear observables
- Evaluation
- Conclusions
Motivation
From Newcastle. For the world.

The 7th IFAC Conference on Analysis and Design of Hybrid Systems, July 7-9, 2021
Koopman Operator
Example

Nonlinear dynamics:

\[ \begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= x_2 - x_1^2
\end{align*} \]
Example

Nonlinear dynamics:

\[ \dot{x}_1 = x_1 \]
\[ \dot{x}_2 = x_2 - x_1^2 \]

Substitution:

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3
\end{bmatrix} = \begin{bmatrix}
  x_1 \\
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\end{bmatrix}
\]
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\frac{d}{dt} y_3 = \frac{d}{dt} x_1^2 = 2 x_1 \dot{x}_1
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\begin{align*}
\dot{y}_1 &= y_1 \\
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\dot{y}_3 &= 2y_3
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Koopman linear system:

\[
\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \end{bmatrix}
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Koopman operator:
\[
\mathcal{K}_t = e^{\tilde{\mathcal{K}}t}
\]
Example

\[ y_1(0) \in [1, 3] \]
\[ y_2(0) \in [0, 2] \]
\[ y_3(0) = y_1(0)^2 \in [1, 9] \]
Koopman Operator

Red dots – linear system, green curves – trajectories of the original nonlinear system, blue sets – output of Flow*
Challenges
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• Obtain a Koopman linearized model of the nonlinear dynamics with a good approximation of the original system (ideally no approximation).
• Add a support of nonlinear initial state sets to state-of-the-art linear reachability algorithms.
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• Obtain a Koopman linearized model of the nonlinear dynamics with a good approximation of the original system (ideally no approximation).
• Add a support of nonlinear initial state sets to state-of-the-art linear reachability algorithms.
Verifying linear systems with nonlinear observables
Direct Encoding

\[ y_t = \mathcal{K}_t y \]

State space of the observables

\[ y = g(x_0) \]

Original state space

\[ x_t = My_t \]
\[ y_1(0) \in [1, 3] \]
\[ y_3(0) = y_1(0)^2 \in [1, 9] \]
Hyperplane Backpropagation

\[ q^T y \leq r \]
Hyperplane Backpropagation

\[ q^T y \leq r \]
\[ q^T K_t y \leq r \]
Hyperplane Backpropagation

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Hyperplane Backpropagation
Zonotope Domain Splitting

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Evaluation
Implementation

- SMT Solver: dReal
- Programming Language: Julia
- Koopman Linearization: DataDrivenDiffEq.jl - [github.com/SciML/DataDrivenDiffEq.jl](https://github.com/SciML/DataDrivenDiffEq.jl)
### Benchmarks


<table>
<thead>
<tr>
<th>Model Name</th>
<th>Number of original state variables</th>
<th>Number of observables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roessler</td>
<td>3</td>
<td>70</td>
</tr>
<tr>
<td>Steam</td>
<td>3</td>
<td>71</td>
</tr>
<tr>
<td>Coupled Van der Pol oscillator</td>
<td>4</td>
<td>131</td>
</tr>
<tr>
<td>Biological</td>
<td>7</td>
<td>104</td>
</tr>
</tbody>
</table>
Computational time (seconds) comparing Flow*, Direct Encoding and the Zonotope Domain Splitting. The dReal tool timed out on all models.

- dReal TO’s on all original nonlinear models
- Flow* outperforms Direct Encoding on most of the instances.
- Zonotope Domain Splitting outperforms all other tools on most of the instances.

<table>
<thead>
<tr>
<th>Name</th>
<th>i</th>
<th>Flow*</th>
<th>Direct</th>
<th>Zono</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roessler</td>
<td>0</td>
<td>55.28</td>
<td>181.06</td>
<td>9.53</td>
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<td></td>
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<td>78.33</td>
<td>177.92</td>
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<td>20</td>
<td>55.29</td>
<td>174.63</td>
<td>3.5</td>
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<tr>
<td>Steam</td>
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<td>197.08</td>
<td>182.62</td>
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<td></td>
<td>5</td>
<td>285.20</td>
<td>59.53</td>
<td>37.27</td>
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<td></td>
<td>10</td>
<td>77.68</td>
<td>29.21</td>
<td>18.52</td>
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<tr>
<td>Coupled VP</td>
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<td>788.45</td>
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</tr>
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<td>18.52</td>
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<tr>
<td>Biological</td>
<td>1</td>
<td>260.69</td>
<td>470.59</td>
<td>0.59</td>
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<tr>
<td></td>
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<td>10</td>
<td>238.56</td>
<td>427.00</td>
<td>179.25</td>
</tr>
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</table>
Conclusions
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• We presented novel techniques to efficiently handle non-linear initial sets which demonstrate competitive results.
• Koopman operator can be used as part of reachability analysis workflow.