Newcastle University

Reachability of Black-Box Nonlinear Systems after Koopman Operator Linearization

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Presenter:

Kostiantyn Potomkin

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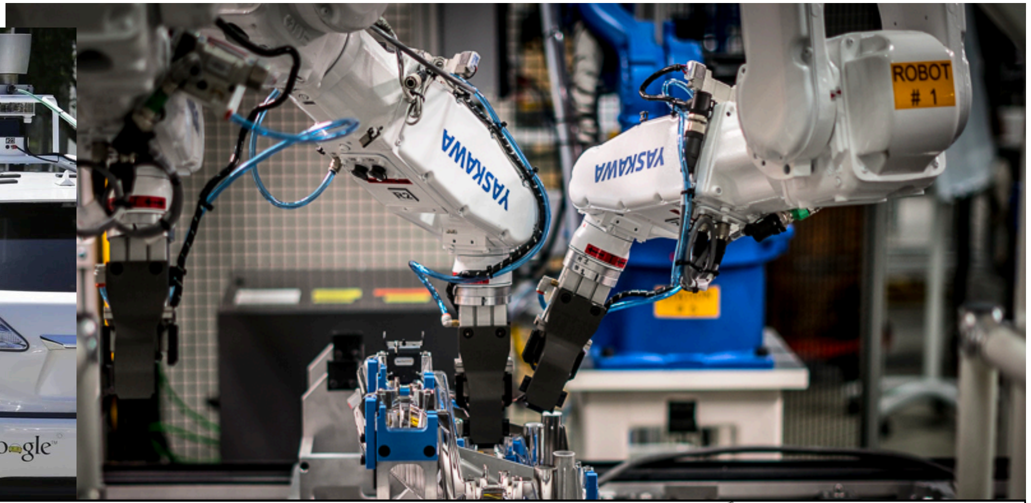
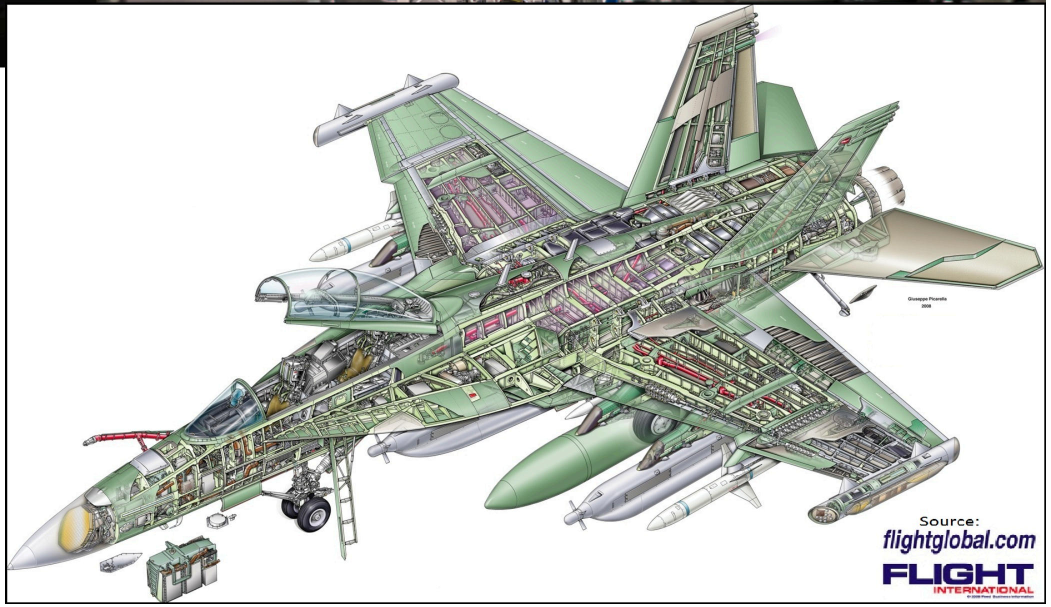
Presentation outline

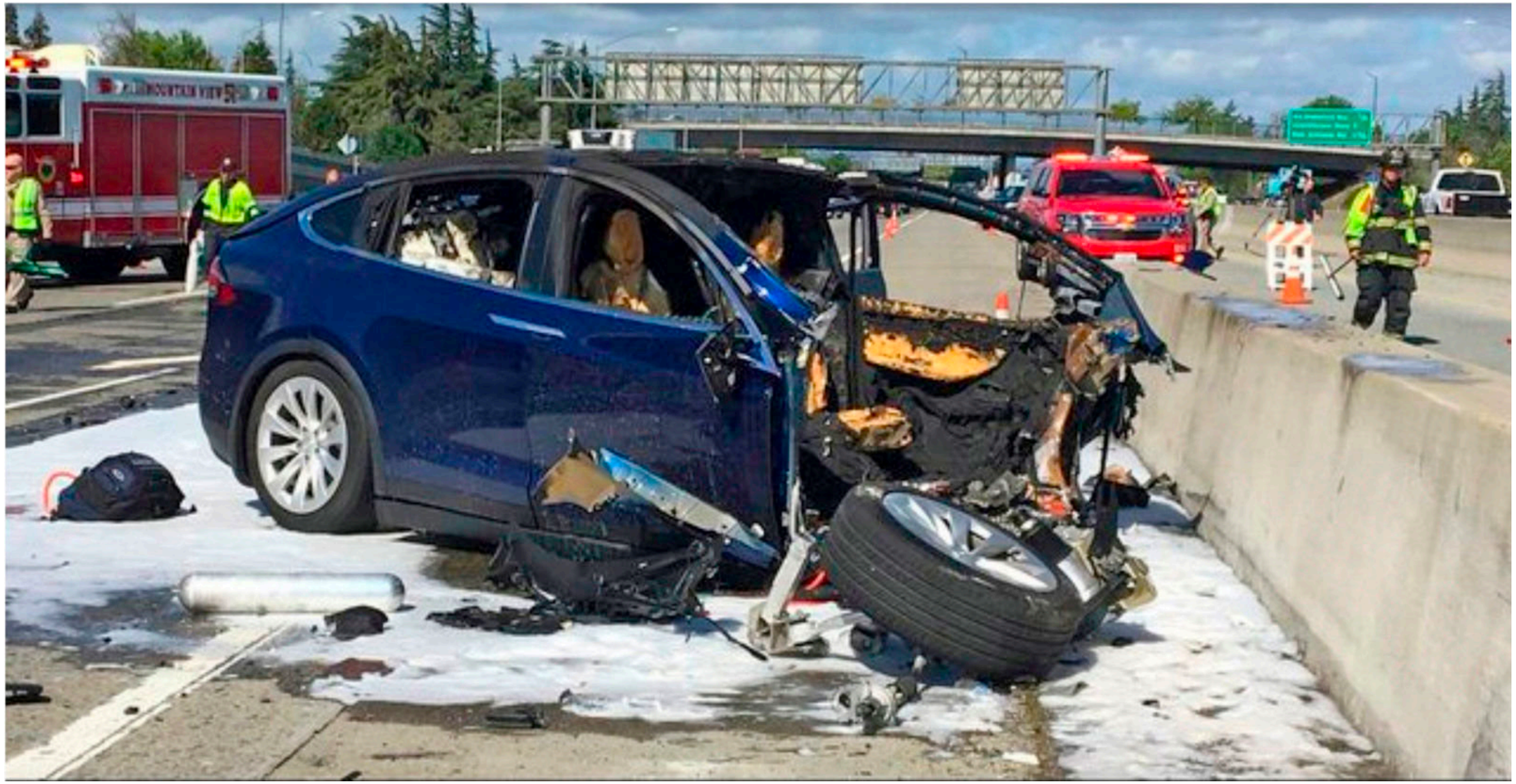
- Motivation
- Koopman Operator
- Challenges
- Verifying linear systems with nonlinear observables
- Evaluation
- Conclusions

Motivation

Reliability and Safety







Koopman Operator

Example

Nonlinear dynamics:

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2 - x_1^2$$

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$$\dot{y}_1 = y_1$$
$$\dot{y}_2 = y_2 - y_3$$
$$\dot{y}_3 = 2y_3$$

Example

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Koopman linear system:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \; for \; \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \end{bmatrix}$$

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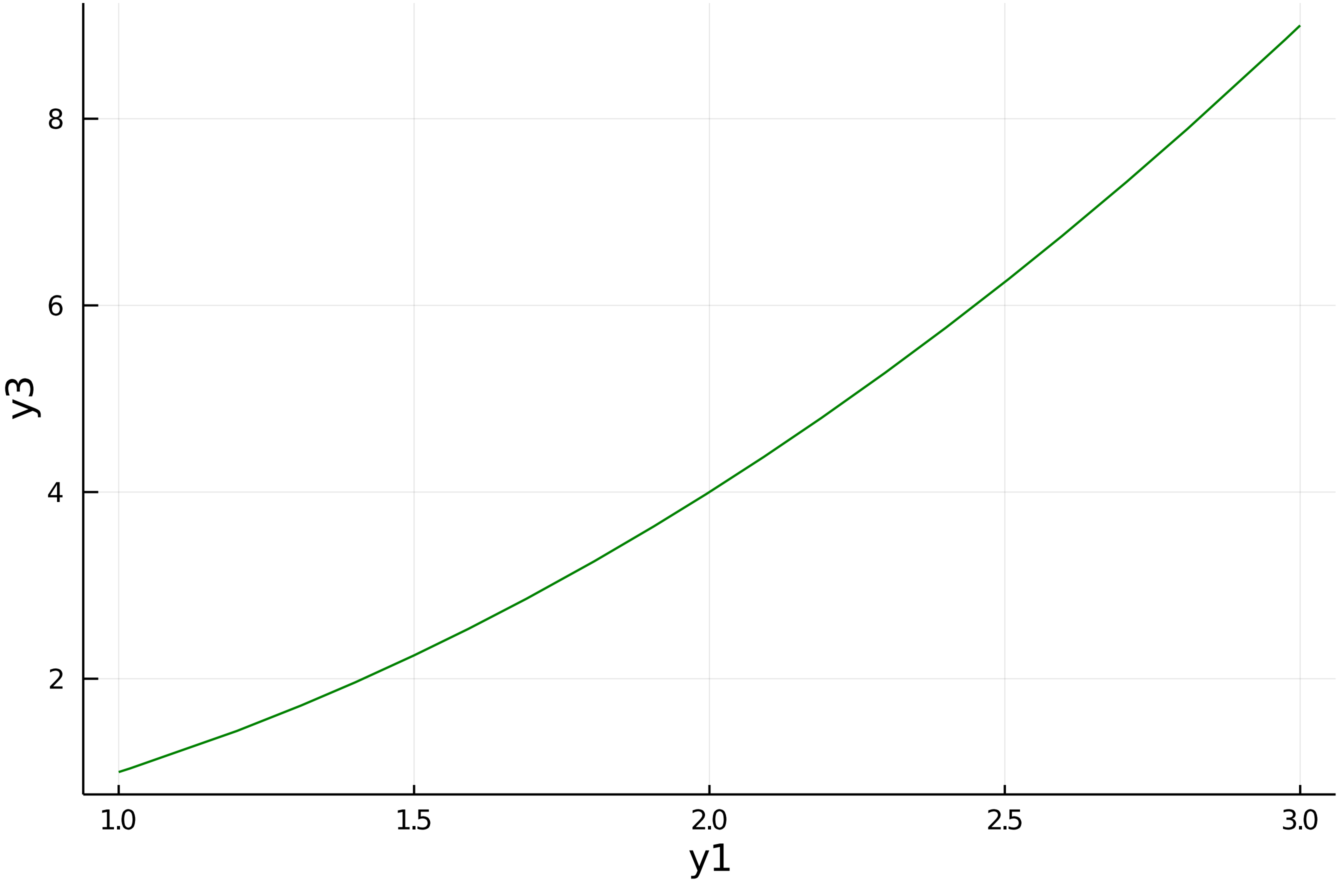
Koopman operator:

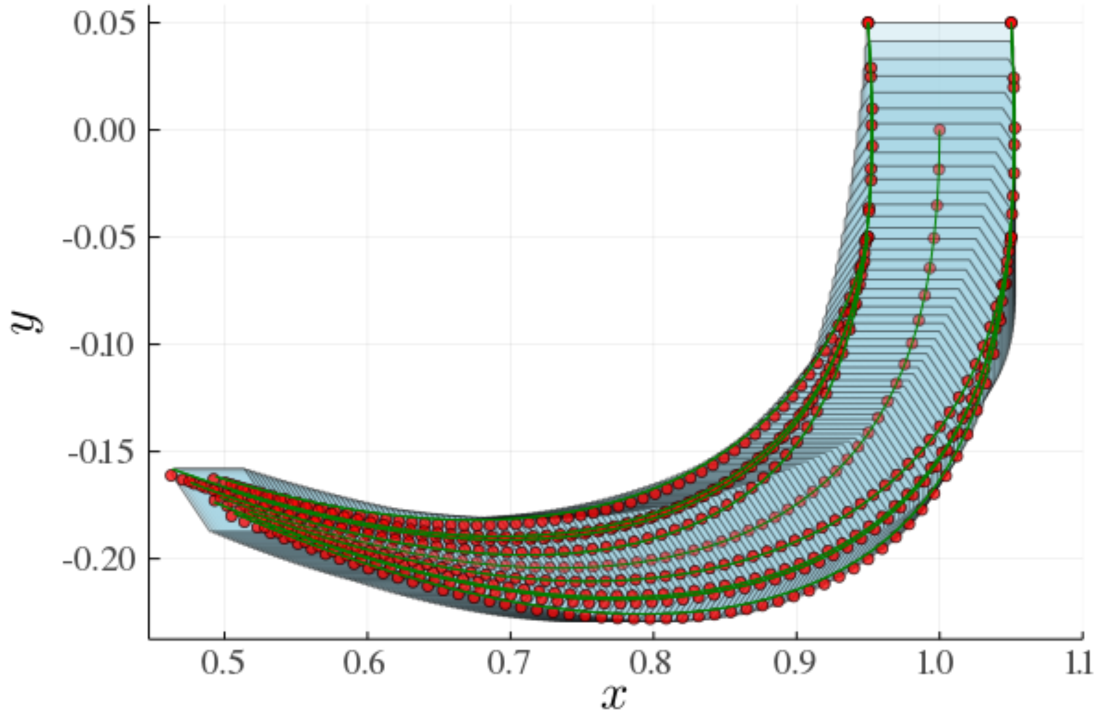
$$\mathcal{K}_t = e^{\tilde{\mathcal{K}} t}$$

Example

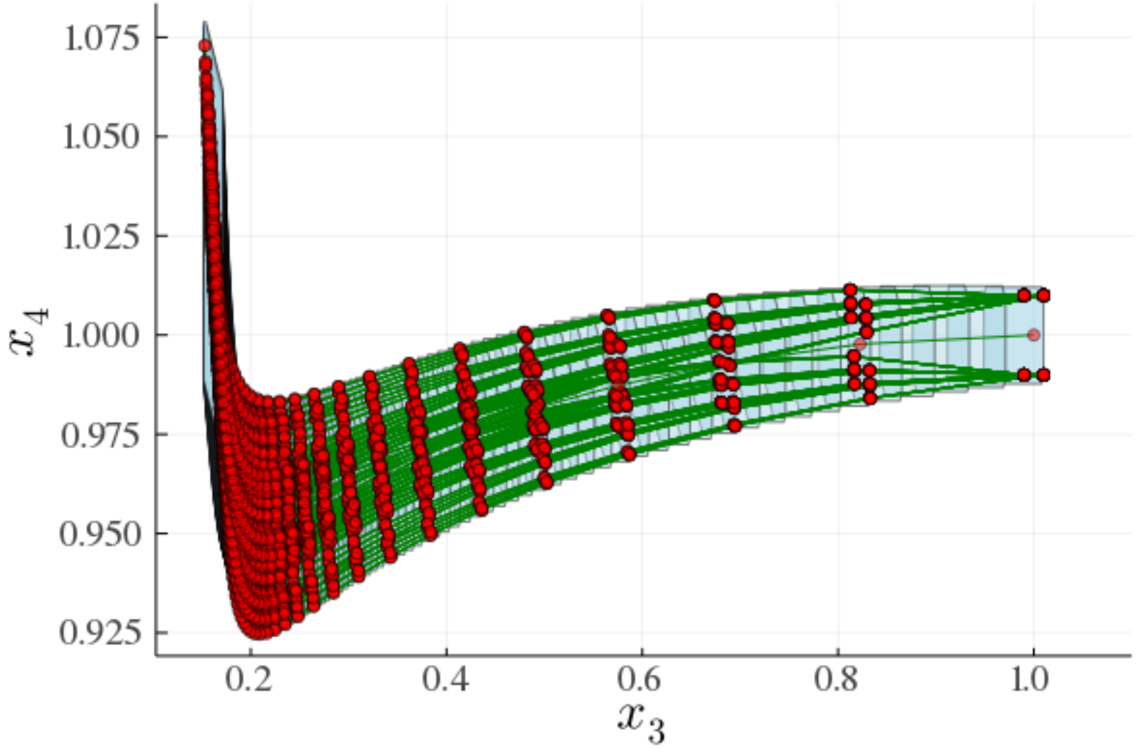
$$y_1(0) \in [1,3]$$
$$y_2(0) \in [0,2]$$
$$y_3(0) = y_1(0)^2 \in [1,9]$$



Red dots – linear system, **green curves – trajectories of the original nonlinear system**, **blue sets – output of Flow***



Steam model



Biological model

Challenges

- Obtain a Koopman linearized model of the nonlinear dynamics with a good approximation of the original system (ideally no approximation).
- Add a support of nonlinear initial state sets to state-of-the-art linear reachability algorithms.

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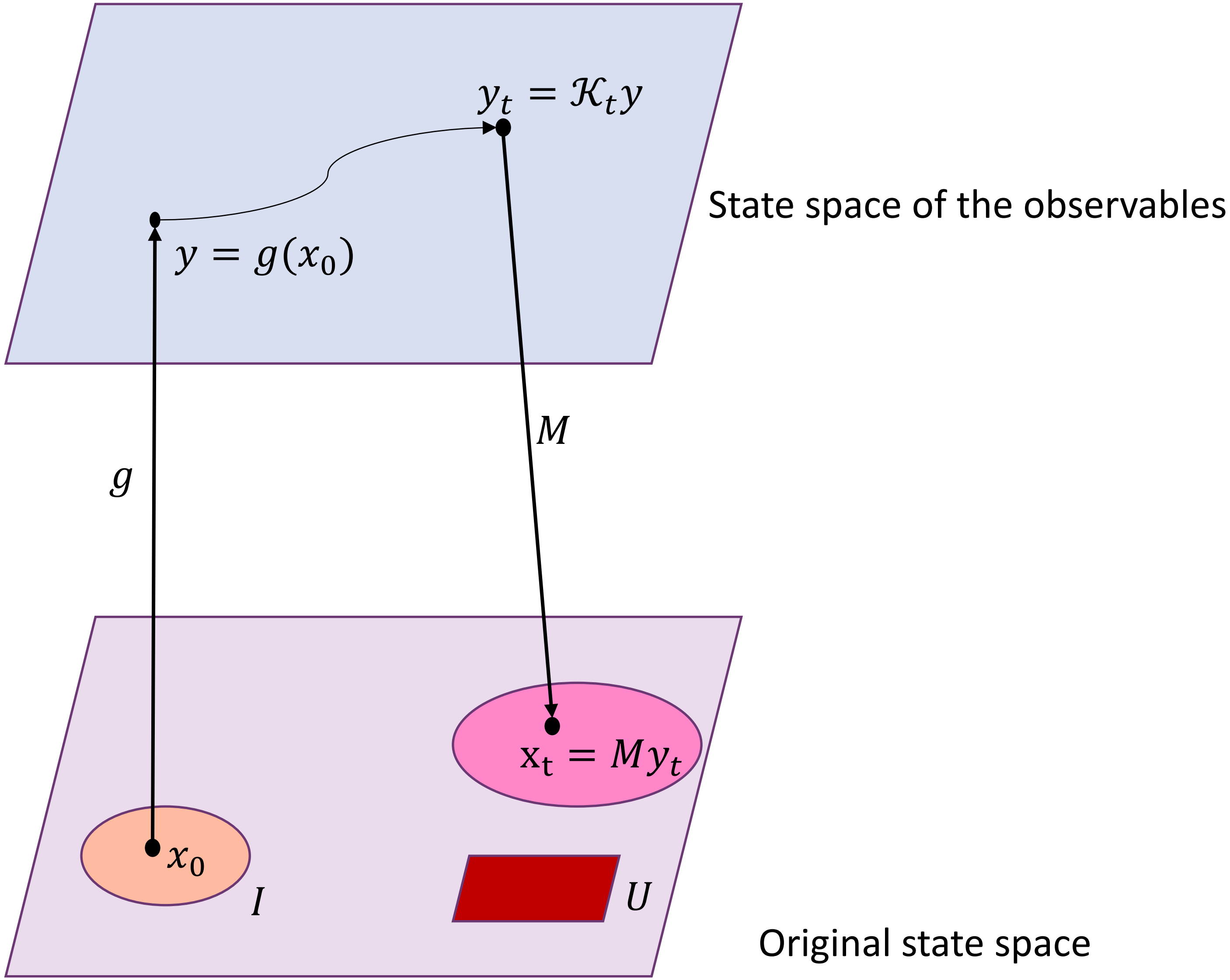
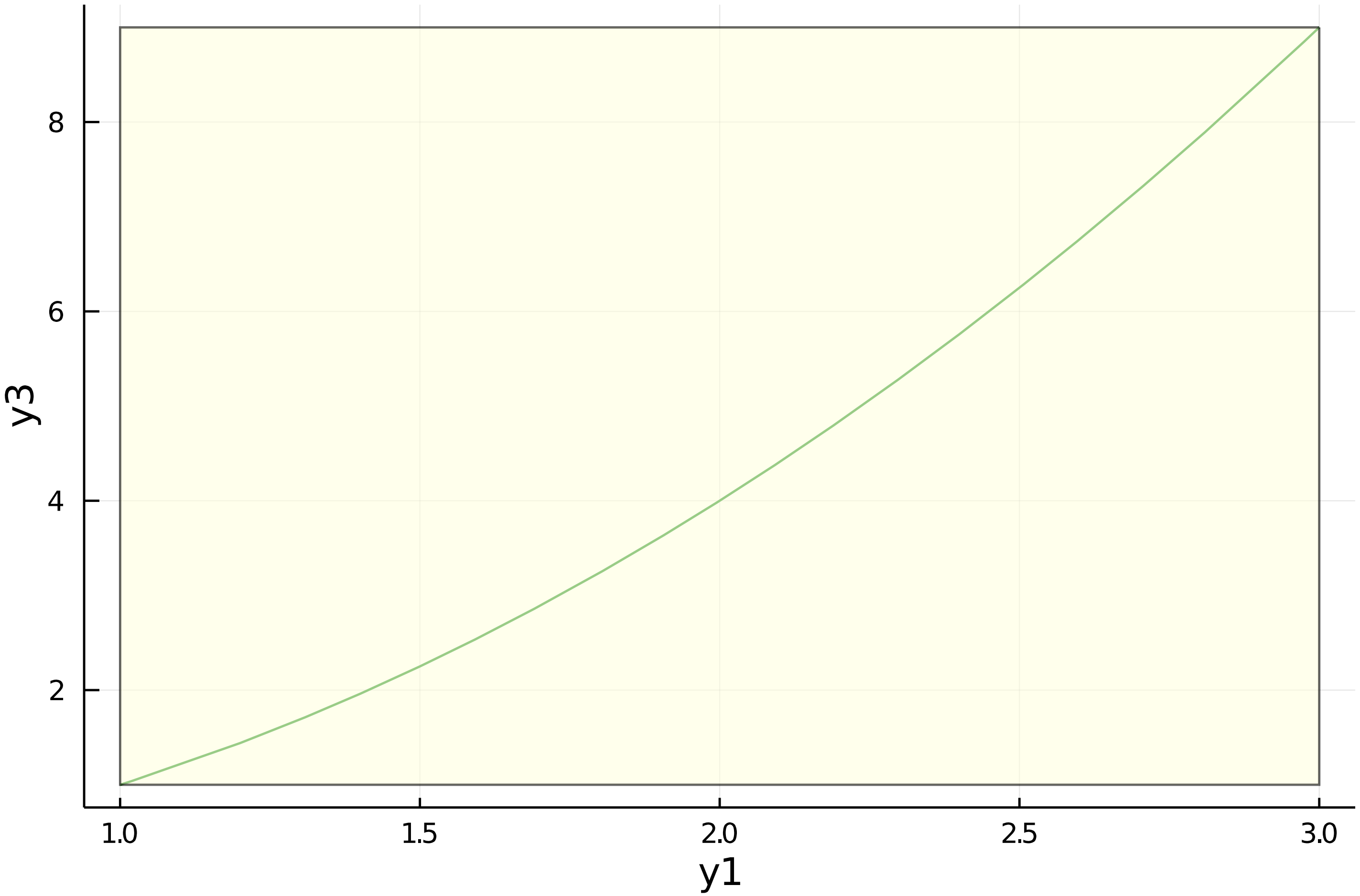
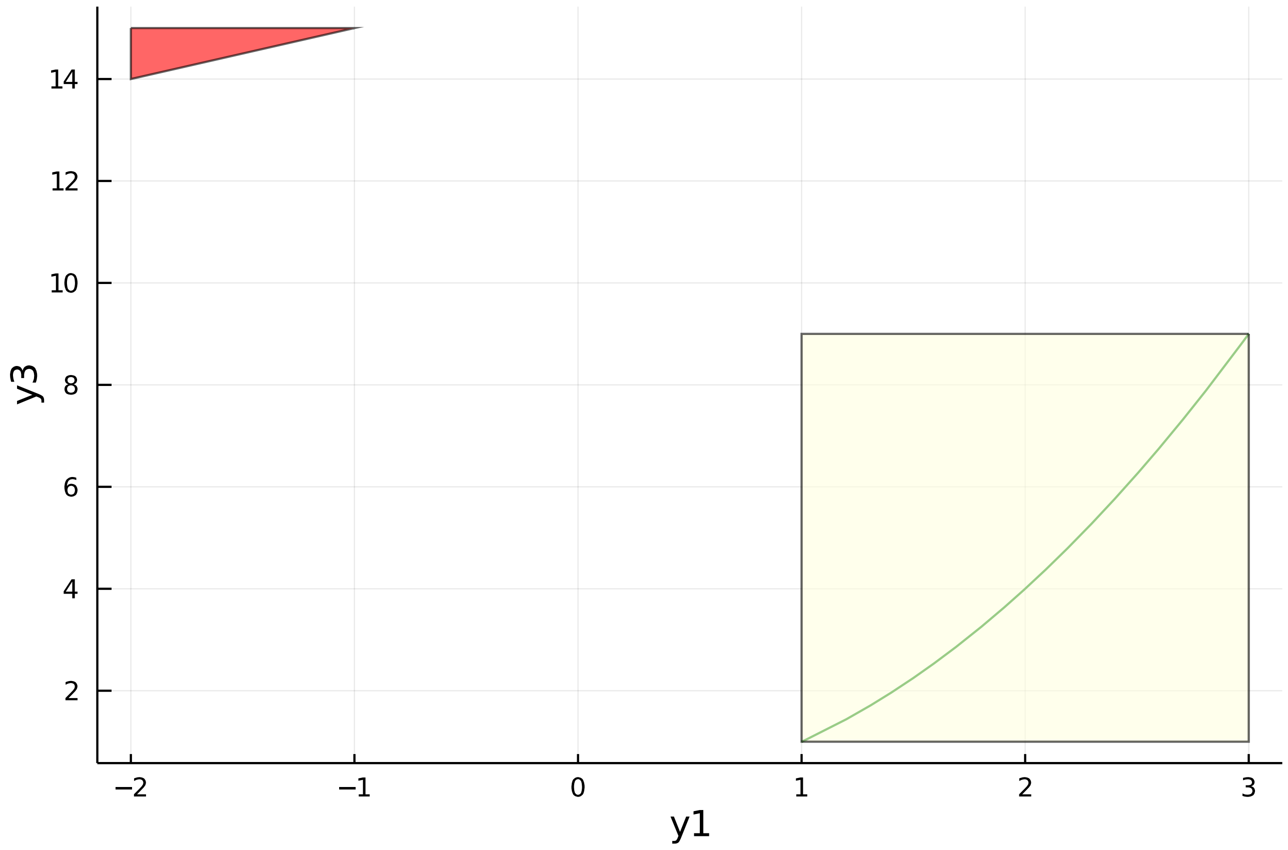
Verifying linear systems with nonlinear observables

Direct Encoding



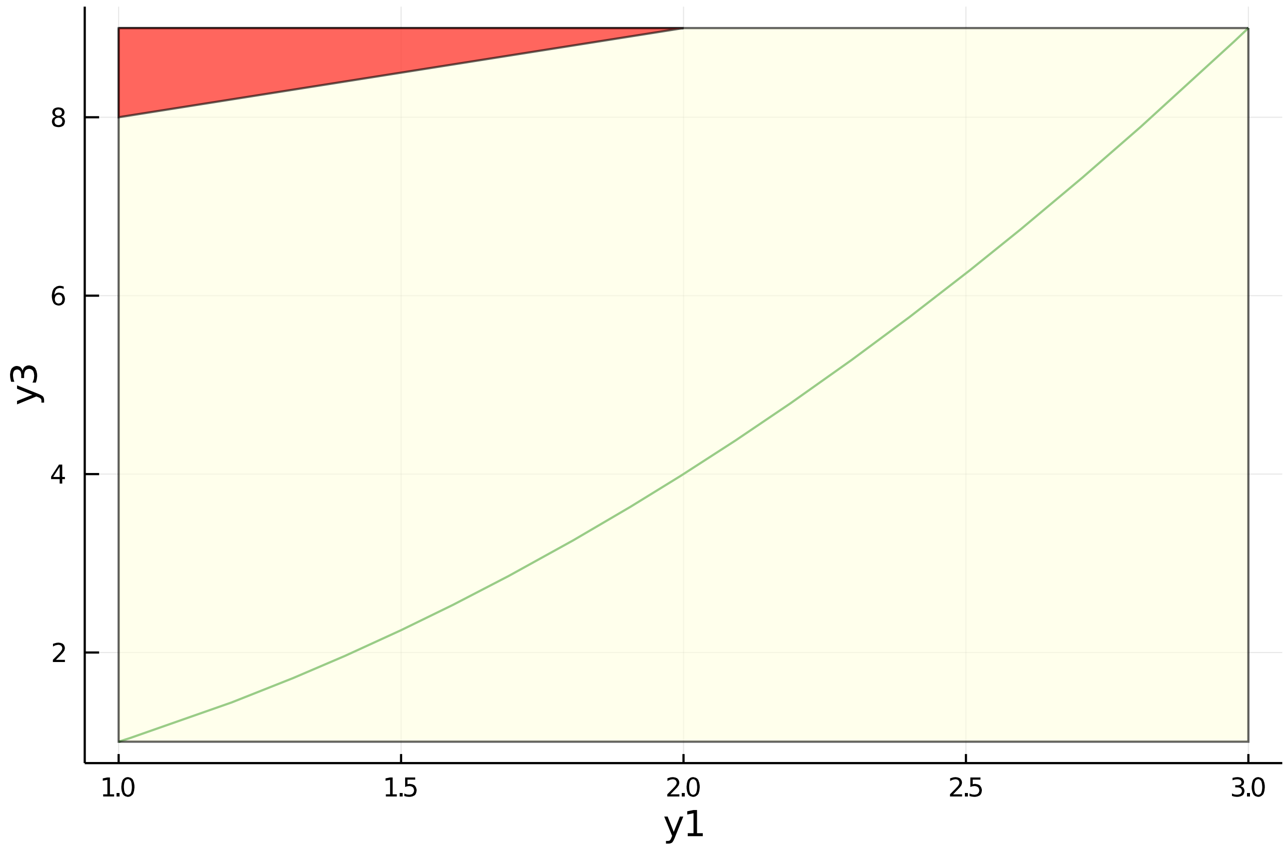


$$y_1(0) \in [1, 3]$$
$$y_3(0) = y_1(0)^2 \in [1,9]$$

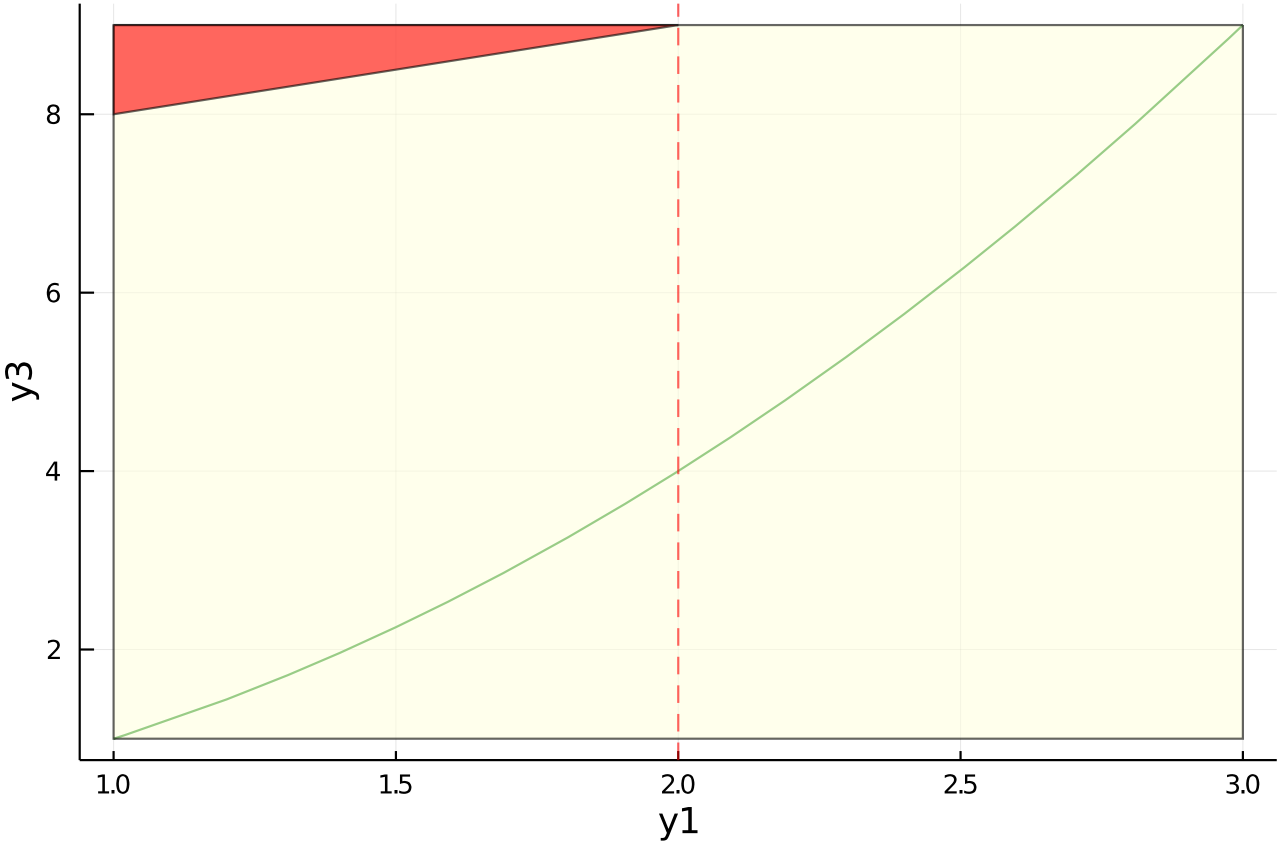


$$q^T y \leq r$$

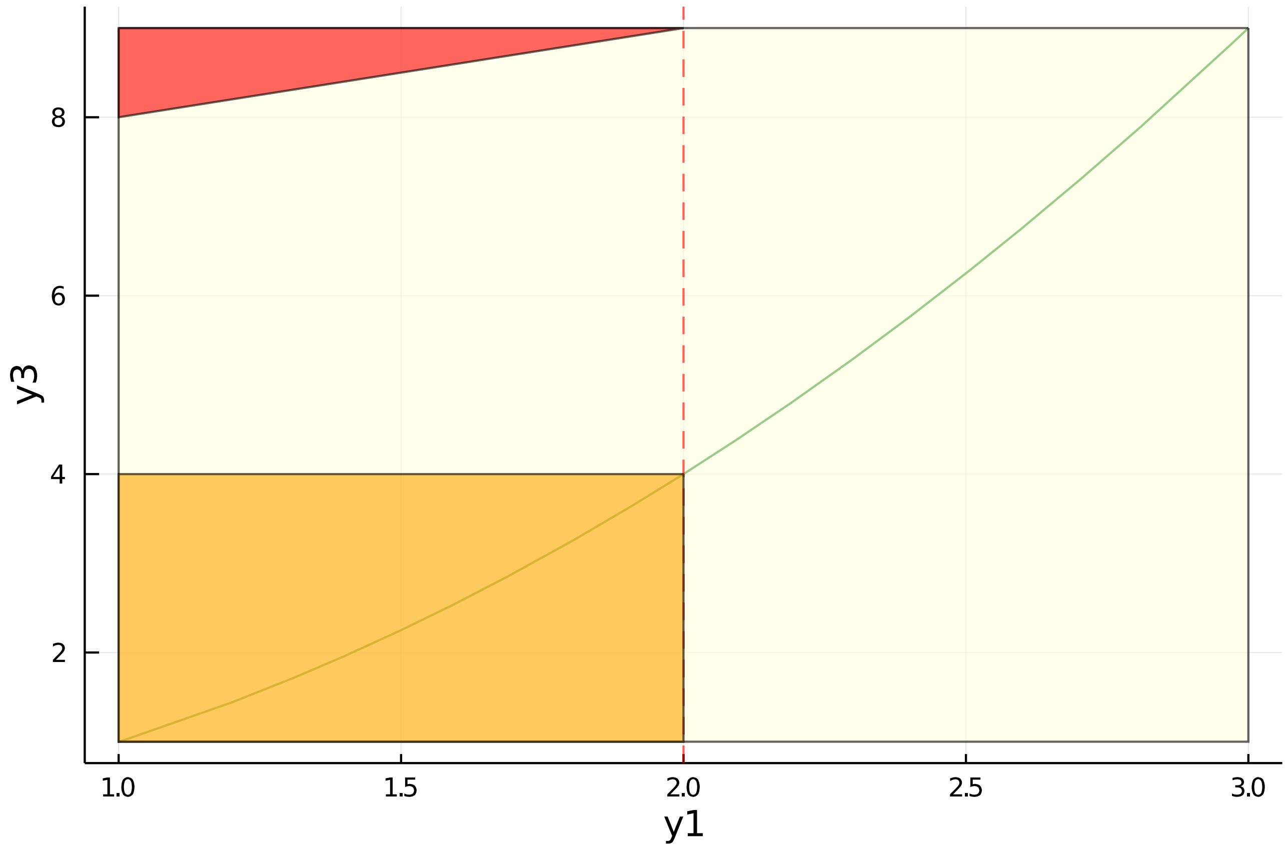
Hyperplane Backpropagation



$$q^T y \leq r$$
$$q^T \mathcal{K}_t y \leq r$$


8
6
4
2
y3
1.0
1.5
2.0
2.5
3.0
y1

Hyperplane Backpropagation



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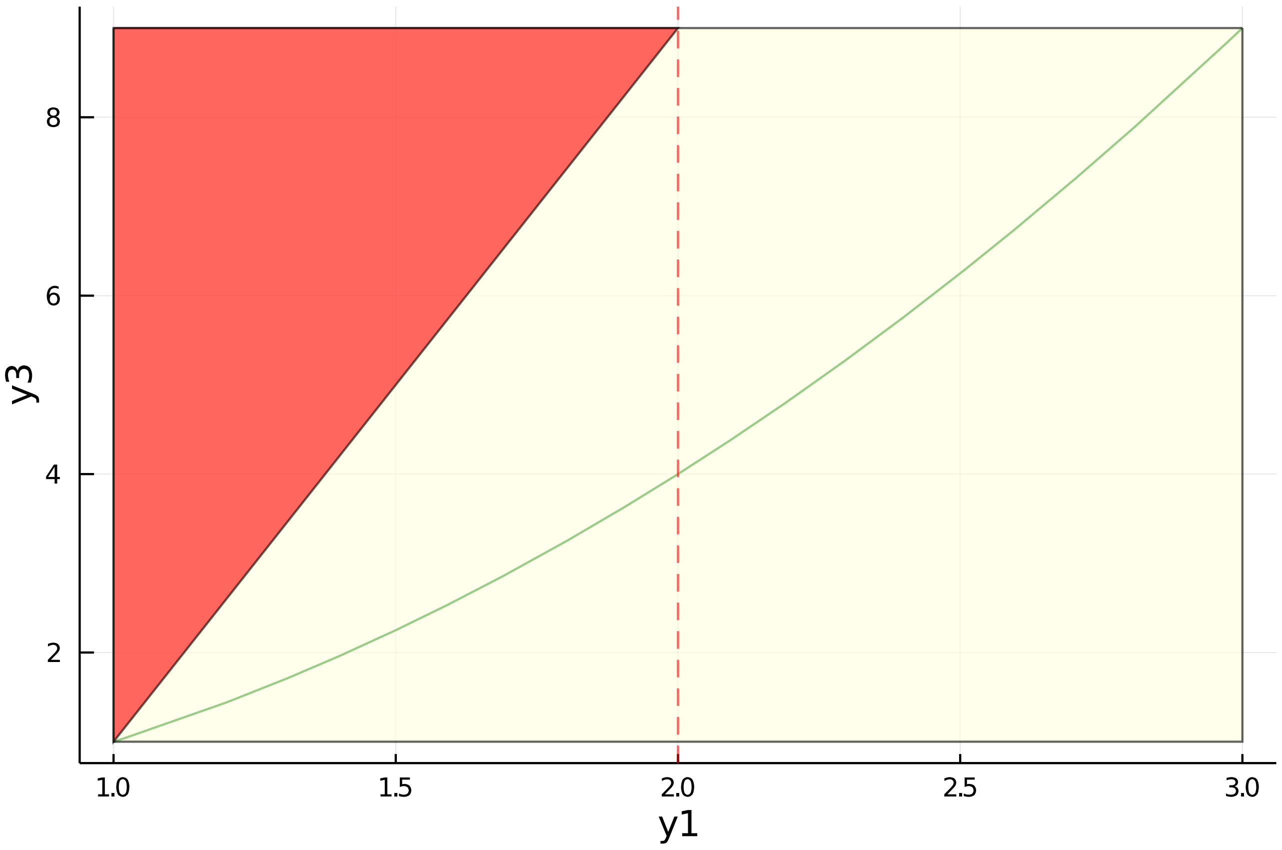

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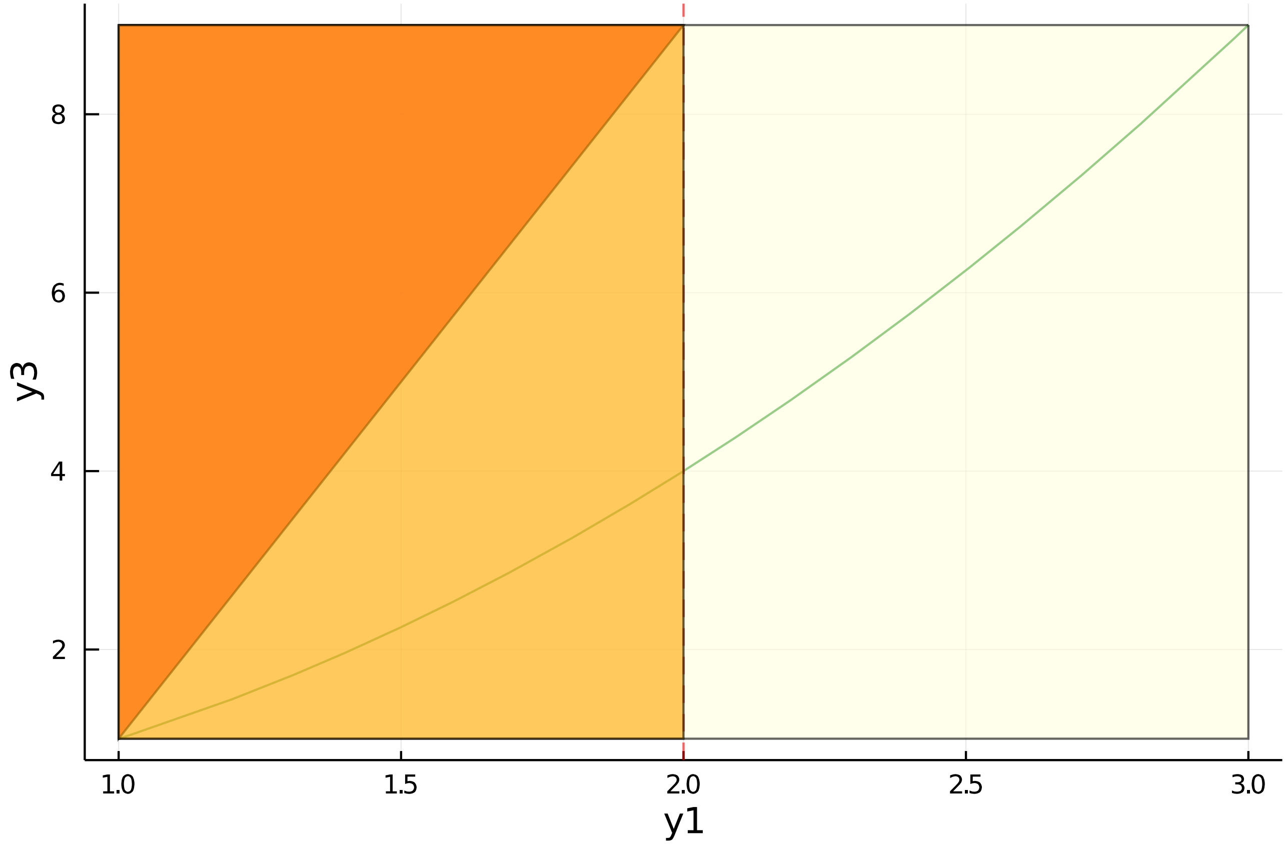

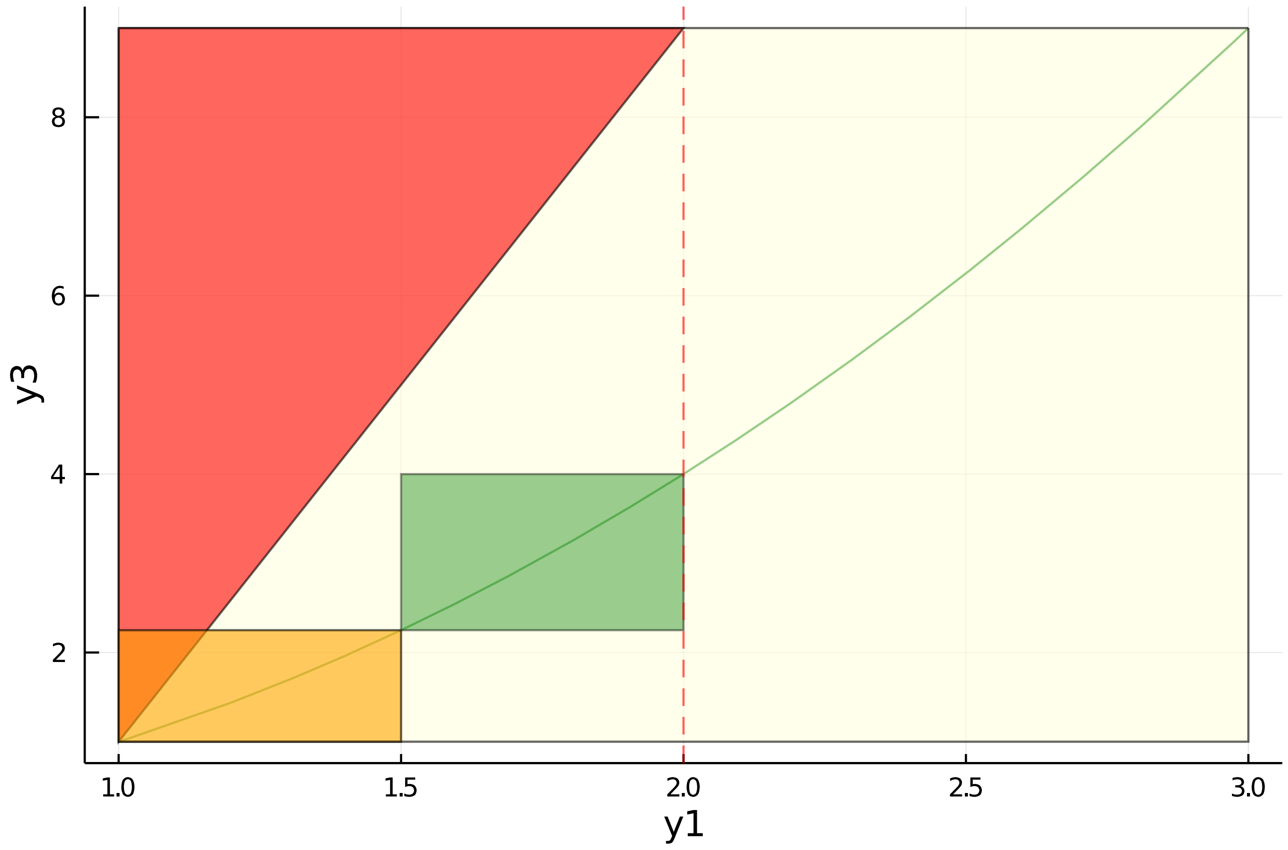
8
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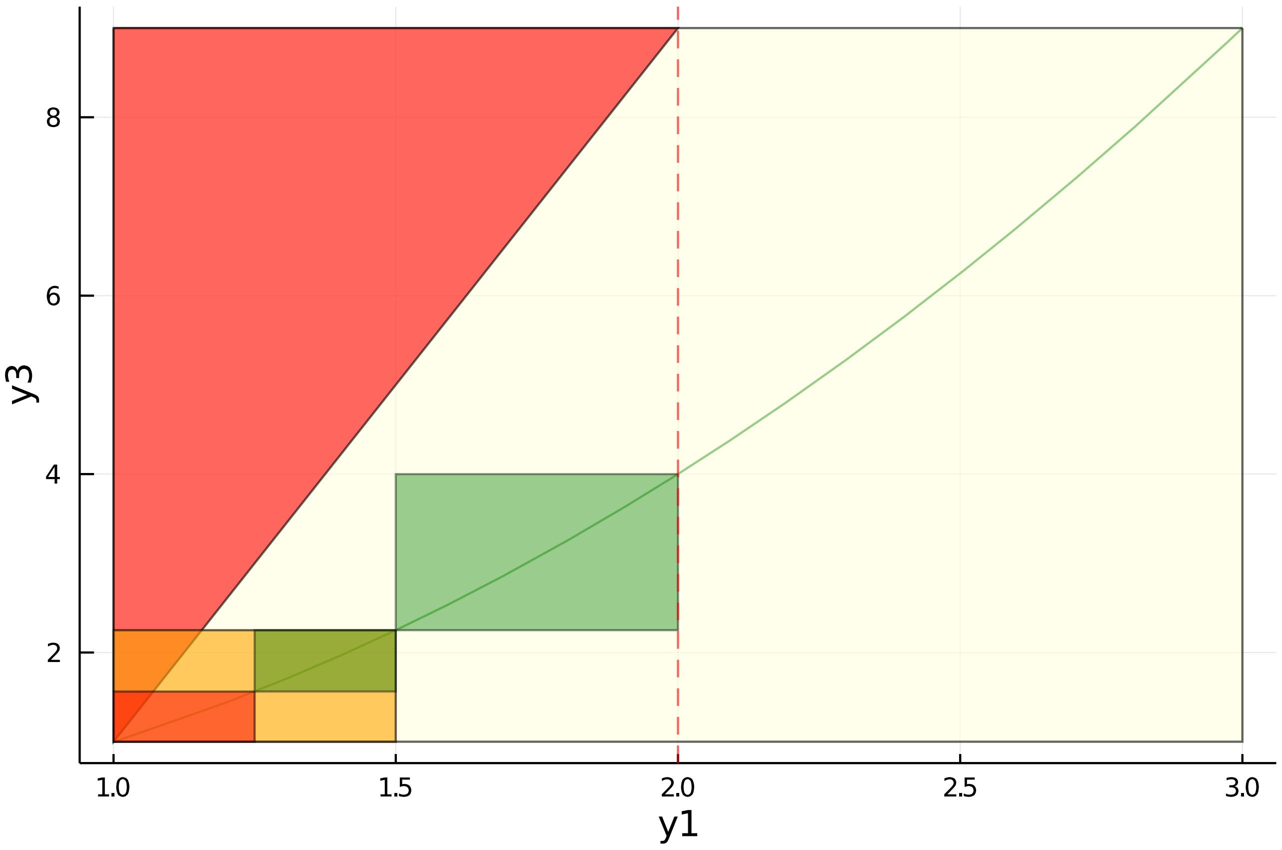
Zonotope Domain Splitting



Evaluation

- SMT Solver: dReal
- Programming Language: Julia
- Koopman Linearization: DataDrivenDiffEq.jl - github.com/SciML/DataDrivenDiffEq.jl

Model Name	Number of original state variables	Number of observables
Roessler	3	70
Steam	3	71
Coupled Van der Pol oscillator	4	131
Biological	7	104

From HyPro benchmark repository: https://ths.rwth-aachen.de/research/projects/hypro/benchmarks-of-continuous-and-hybrid-systems/

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Computational time (seconds) comparing Flow*, Direct Encoding and the Zonotope Domain Splitting. The dReal tool timed out on all models.

Name	i	Flow*	Direct	Zono
Roessler	0	55.28	181.06	9.53
	10	78.33	177.92	5.01
	20	55.29	174.63	3.5
Steam	0	61.06	197.08	182.62
	5	285.20	59.53	37.27
	10	77.68	29.21	18.52
Coupled VP	1	251.11	788.45	0.57
	8	497.61	680.61	53.91
	16	1665.16	557.24	18.52
Biological	1	260.69	470.59	0.59
	5	250.26	426.37	49.41
	10	238.56	427.00	179.25

- dReal TO's on all original nonlinear models
- Flow* outperforms Direct Encoding on most of the instances.
- Zonotope Domain Splitting outperforms all other tools on most of the instances.

Conclusions

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- We presented novel techniques to efficiently handle non-linear initial sets which demonstrate competitive results.
- Koopman operator can be used as part of reachability analysis workflow.