1 ORTHONORMAL BASIS FUNCTIONS FOR A SPHERICAL DOMAIN

An efficient way of representing functions defined on the sphere $S^2$ is through the use of orthonormal basis functions. As set of spherical orthonormal basis functions $\psi_{lm}(\theta, \phi)$ has the following important properties:

**Orthogonality:**

$$\int_0^{2\pi} \int_0^\pi \psi_{lm}(\theta, \phi) \psi_{l'm'}(\theta, \phi) \sin \theta \, d\theta \, d\phi = \delta_{ll'} \delta_{mm'}$$

(1)

where $\delta_{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$

**Normalization:**

$$\int_0^{2\pi} \int_0^\pi \|\psi_{lm}(\theta, \phi)\|^2 \sin \theta \, d\theta \, d\phi = 1$$

(2)

Here $l$ is the degree and $m$ is the order of the basis function. A function $f(\theta, \phi)$ defined over a spherical domain, such as the HRTF, can be expressed in an orthonormal basis as follows:

$$\tilde{f}(\theta, \phi) = \sum_{l=0}^n \sum_{m=-l}^l f_{lm} \psi_{lm}(\theta, \phi),$$

(3)

where the $f_{lm}$ are the basis coefficients up to order $n$, for a total of $(n + 1)^2$ coefficients. As $n \to \infty$, the function $\tilde{f} \to f$. These basis coefficients can be calculated by evaluating the integral:

$$f_{lm} = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) \psi_{lm}(\theta, \phi) \sin \theta \, d\theta \, d\phi.$$  

(4)

If two functions defined over the spherical domain, $f(\theta, \phi)$ and $g(\theta, \phi)$, are expressed in the same orthonormal basis and have coefficients $f_{lm}$ and $g_{lm}$, then the following relationship holds:

$$\int_0^{2\pi} \int_0^\pi f(\theta, \phi) g(\theta, \phi) \psi_{lm}(\theta, \phi) \sin \theta \, d\theta \, d\phi = \sum_{l=0}^n \sum_{m=-l}^l f_{lm} g_{lm}.$$  

(5)

This important property allows integrals of the above form to be evaluated using an efficient dot product of basis function coefficients. This property has been used extensively in graphics literature [4, 1, 3].

Two classes of orthonormal basis functions for spherical domains are the spherical harmonics (SH) and spherical wavelets (SW).

**Spherical Harmonics:** The real-valued spherical harmonic functions $Y_{lm}(\theta, \phi)$ are given by the expressions:

$$Y_{lm}(\theta, \phi) = \begin{cases} Y_{lm} = \Gamma_{\ell|m|} P_{\ell|m|}^{m}(\cos \theta) \cos (|m|\phi) & : m > 0, \\ Y_{00} = 1 & : m = 0, \\ Y_{lm} = \Gamma_{\ell|m|} P_{\ell|m|}^{m}(\cos \theta) \sin (|m|\phi) & : m < 0, \end{cases}$$

where $\Gamma_{\ell|m|} = \sqrt{\frac{(2\ell+1)(\ell-|m|)!}{4\pi(\ell+|m|)!}}$ is a normalization constant and the $P_{\ell|m|}$ are the associated Legendre polynomials [2].

Of particular interest are the zonal spherical harmonics, a special case of the spherical harmonics where there is rotational symmetry about the $z$ axis [4]. In this case, the SH coefficient $f_{lm} = 0$ for $m \neq 0$. In other words, the zonal harmonics are the functions $Y_{l0}(\theta, \phi)$ and can be described by just $n + 1$ zonal harmonic coefficients, $z_i$. The coefficients can be efficiently rotated to any direction $(\theta, \phi)$ using the following equation:

$$f_{lm} = \sqrt{\frac{4\pi}{2\ell+1}} z_l Y_{lm}(\theta, \phi).$$  

(6)

The output of the rotation process is a set of spherical harmonic coefficients that describe the function in the rotated space.

**REFERENCES**


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