Volume Rendering Digest (for NeRF)

Andrea Tagliasacchi\textsuperscript{1,2} Ben Mildenhall\textsuperscript{1}

\textsuperscript{1}Google Research \textsuperscript{2}Simon Fraser University

Neural Radiance Fields \cite{NeRF} employ simple volume rendering as a way to overcome the challenges of differentiating through ray-triangle intersections by leveraging a probabilistic notion of visibility. This is achieved by assuming the scene is composed by a cloud of light-emitting particles whose density changes in space (in the terminology of physically-based rendering, this would be described as a volume with absorption and emission but no scattering \cite[Sec 11.1]{PBR_book}). In what follows, for the sake of exposition simplicity, and without loss of generality, we assume the emitted light does not change as a function of view-direction. This technical report is a condensed version of previous reports \cite{Tagliasacchi2019, Mildenhall2021}, but rewritten in the context of NeRF, and adopting its commonly used notation\footnote{If you are interested in borrowing the LaTeX notation, please refer to: https://www.overleaf.com/read/fkhpkzzhnywa}.

**Transmittance.** Let the density field $\sigma(x)$, with $x \in \mathbb{R}^3$, indicate the differential likelihood of a ray hitting a particle (i.e. the probability of hitting a particle while travelling an infinitesimal distance). We reparameterize the density along a given ray $r = (o, d)$ as a scalar function $\sigma(t)$, since any point $x$ along the ray can be written as $r(t) = o + td$. Density is closely tied to the transmittance function $T(t)$, which indicates the probability of a ray traveling over the interval $[0, t)$ without hitting any particles. Then the probability $T(t+dt)$ of not hitting a particle when taking a differential step $dt$ is equal to $T(t)$, the likelihood of the ray reaching $t$, times $(1 - dt \cdot \sigma(t))$, the probability of not hitting anything during the step:

$$T(t+dt) = T(t) \cdot (1 - dt \cdot \sigma(t)) \tag{1}$$

This is a classical differential equation that can be solved as follows:

$$T'(t) = -T(t) \cdot \sigma(t) \tag{3}$$

$$\frac{T'(t)}{T(t)} = -\sigma(t) \tag{4}$$

$$\int_a^b \frac{T'(t)}{T(t)} \, dt = - \int_a^b \sigma(t) \, dt \tag{5}$$

$$\log T(t) |_a^b = - \int_a^b \sigma(t) \, dt \tag{6}$$

$$T(a \to b) \equiv \frac{T(b)}{T(a)} = \exp \left( - \int_a^b \sigma(t) \, dt \right) \tag{7}$$

where we define $T(a \to b)$ as the probability that the ray travels from distance $a$ to $b$ along the ray without hitting a particle, which is related to the previous notation by $T(t) = T(0 \to t)$. 

\[\]

\[\]

\[\]
**Probabilistic interpretation.** Note that we can also interpret the function $1 - T(t)$ (often called opacity) as a cumulative distribution function (CDF) indicating the probability that the ray does hit a particle sometime before reaching distance $t$. Then $T(t) \cdot \sigma(t)$ is the corresponding probability density function (PDF), giving the likelihood that the ray stops precisely at distance $t$.

**Volume rendering.** We can now calculate the expected value of the light emitted by the particles in the volume as the ray travels from $t=0$ to $D$, composited on top of a background color. Since the probability density for stopping at $t$ is $T(t) \cdot \sigma(t)$, the expected color is

$$C = \int_0^D T(t) \cdot \sigma(t) \cdot c(t) \, dt + T(D) \cdot c_{bg}$$

where $c_{bg}$ is a background color that is composited with the foreground scene according to the residual transmittance $T(D)$. Without loss of generality, we omit the background term in what follows.

**Homogeneous media.** We can calculate the color of some homogeneous volumetric media with constant color $c_a$ and density $\sigma_a$ over a ray segment $[a, b]$ by integration:

$$C(a \rightarrow b) = \int_a^b T(a \rightarrow t) \cdot \sigma(t) \cdot c(t) \, dt$$

$$= \sigma \cdot c_a \int_a^b T(a \rightarrow t) \, dt$$

$$= \sigma \cdot c_a \int_a^b \exp \left( - \int_a^t \sigma(u) \, du \right) \, dt$$

$$= \sigma \cdot c_a \int_a^b \exp \left( - \sigma_a u \right) \, dt$$

$$= \sigma \cdot c_a \int_a^b \exp \left( - \sigma_a (t - a) \right) \, dt$$

$$= \sigma \cdot c_a \left. \exp \left( - \sigma_a (t - a) \right) \right|_a^b$$

$$= c_a \cdot (1 - \exp (- \sigma_a (b - a)))$$

**Transmittance is multiplicative.** Note that transmittance factorizes as follows:

$$T(a \rightarrow c) = \exp \left( - \left[ \int_a^b \sigma(t) \, dt + \int_b^c \sigma(t) \, dt \right] \right)$$

$$= \exp \left( - \int_a^b \sigma(t) \, dt \right) \exp \left( - \int_b^c \sigma(t) \, dt \right)$$

$$= T(a \rightarrow b) \cdot T(b \rightarrow c)$$

This also follows from the probabilistic interpretation of $T$, since the probability that the ray does not hit any particles within $[a, c]$ is the product of the probabilities of the two independent events that it does not hit any particles within $[a, b]$ or within $[b, c]$.

**Transmittance for piecewise constant data.** Given a set of intervals $\{[t_n, t_{n+1}]\}_{n=1}^N$ with constant density $\sigma_n$ within the $n$-th segment, and with $t_1=0$ and $\delta_n = t_{n+1} - t_n$, transmittance is equal to:

$$T_n = T(t_n) = T(0 \rightarrow t_n) = \exp \left( - \int_0^{t_n} \sigma(t) \, dt \right) = \exp \left( \sum_{k=1}^{n-1} -\sigma_k \delta_k \right)$$
Volume rendering of piecewise constant data. Combining the above, we can evaluate the volume rendering integral through a medium with piecewise constant color and density:

\[ C(t_{N+1}) = \sum_{n=1}^{N} \int_{t_n}^{t_{n+1}} T(t) \cdot \sigma_n \cdot c_n \, dt \]

piecewise constant \hfill (20)

\[ = \sum_{n=1}^{N} \int_{t_n}^{t_{n+1}} T(0 \rightarrow t_n) \cdot T(t_n \rightarrow t) \cdot \sigma_n \cdot c_n \, dt \]

from \(18\) \hfill (21)

\[ = \sum_{n=1}^{N} \int_{t_n}^{t_{n+1}} T(t_n \rightarrow t) \cdot \sigma_n \cdot c_n \, dt \]

constant \hfill (22)

\[ = \sum_{n=1}^{N} T(0 \rightarrow t_n) \cdot (1 - \exp(-\sigma_n(t_{n+1} - t_n))) \cdot c_n \]

from \(15\) \hfill (23)

This leads to the volume rendering equations from NeRF \([3, \text{ Eq.}3]:\)

\[ C(t_{N+1}) = \sum_{n=1}^{N} T_n \cdot (1 - \exp(-\sigma_n \delta_n)) \cdot c_n, \quad \text{where} \quad T_n = \exp(\sum_{k=1}^{n-1} -\sigma_k \delta_k) \]

\hfill (24)

Finally, rather than writing these expressions in terms of volumetric density, we can re-express them in terms of alpha-compositing weights \(\alpha_n \equiv 1 - \exp(-\sigma_n \delta_n),\) and by noting that \(\prod_i \exp x_i = \exp (\sum_i x_i)\) in \((19):\)

\[ C(t_{N+1}) = \sum_{n=1}^{N} T_n \cdot \alpha_n \cdot c_n, \quad \text{where} \quad T_n = \prod_{n=1}^{N-1} (1 - \alpha_n) \]

\hfill (25)

Alternate derivation. By making use of the earlier connection between CDF and PDF that \((1-T)' = T \sigma,\) and by assuming constant color \(c_a\) along an interval \([a, b]:\)

\[ \int_a^b T(t) \cdot \sigma(t) \cdot c(t) \, dt = c_a \int_a^b (1 - T)'(t) \, dt \]

\hfill (26)

\[ = c_a \cdot (1 - \sigma(a - b)) \]

\hfill (27)

\[ = c_a \cdot T(a) - T(b) \]

\hfill (28)

\[ = c_a \cdot T(a) \cdot (1 - T(a \rightarrow b)) \]

\hfill (29)

Combined with constant per-interval density, this identity yields the same expression for color as \((24).\)

References


Acknowledgements

Thanks to Daniel Rebain, Soroosh Yazdani and Rif A. Saurous for a careful proofread.