Lecture 13: Two-view Geometry

COMP 590/776: Computer Vision

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Course Website:
Scan Me!

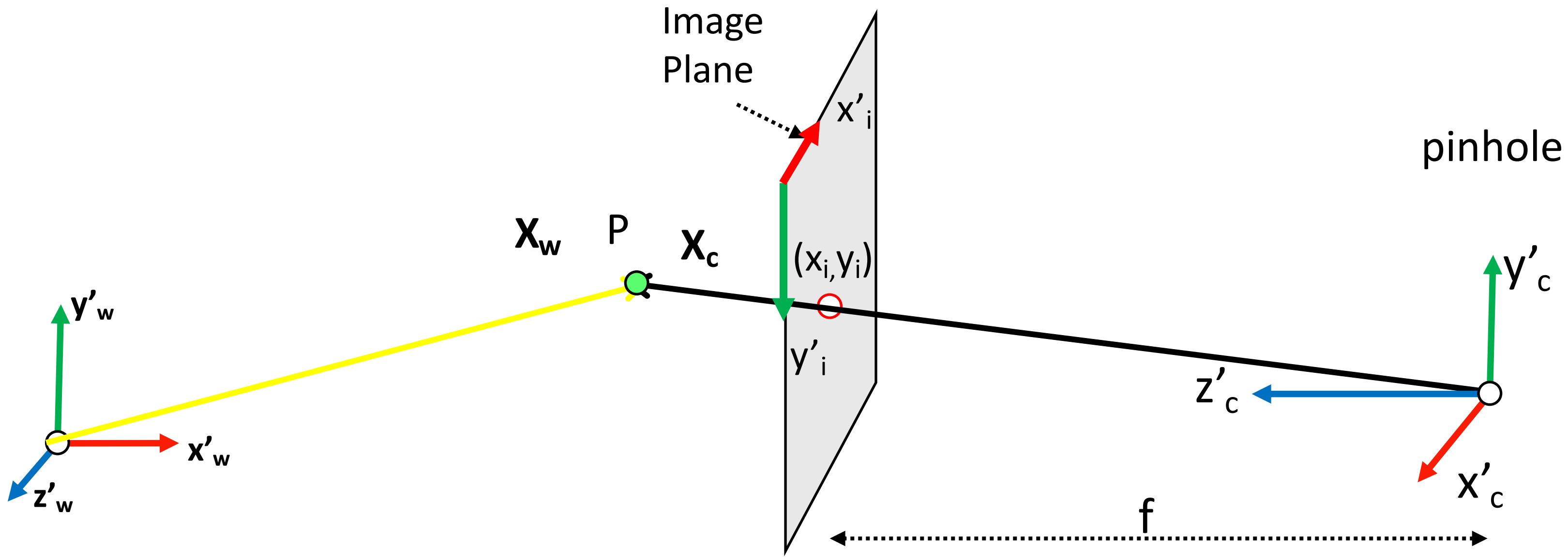


Image Coordinates

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

Perspective Projection

$$\begin{bmatrix} f_x & 0 & o_x & 0 \\ 0 & f_y & o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Intrinsics

Camera Coordinates

$$\mathbf{X}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

Coordinate Transformation

$$\begin{bmatrix} R_{3\times 3} & \mathbf{t} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

Extrinsics

World Coordinates

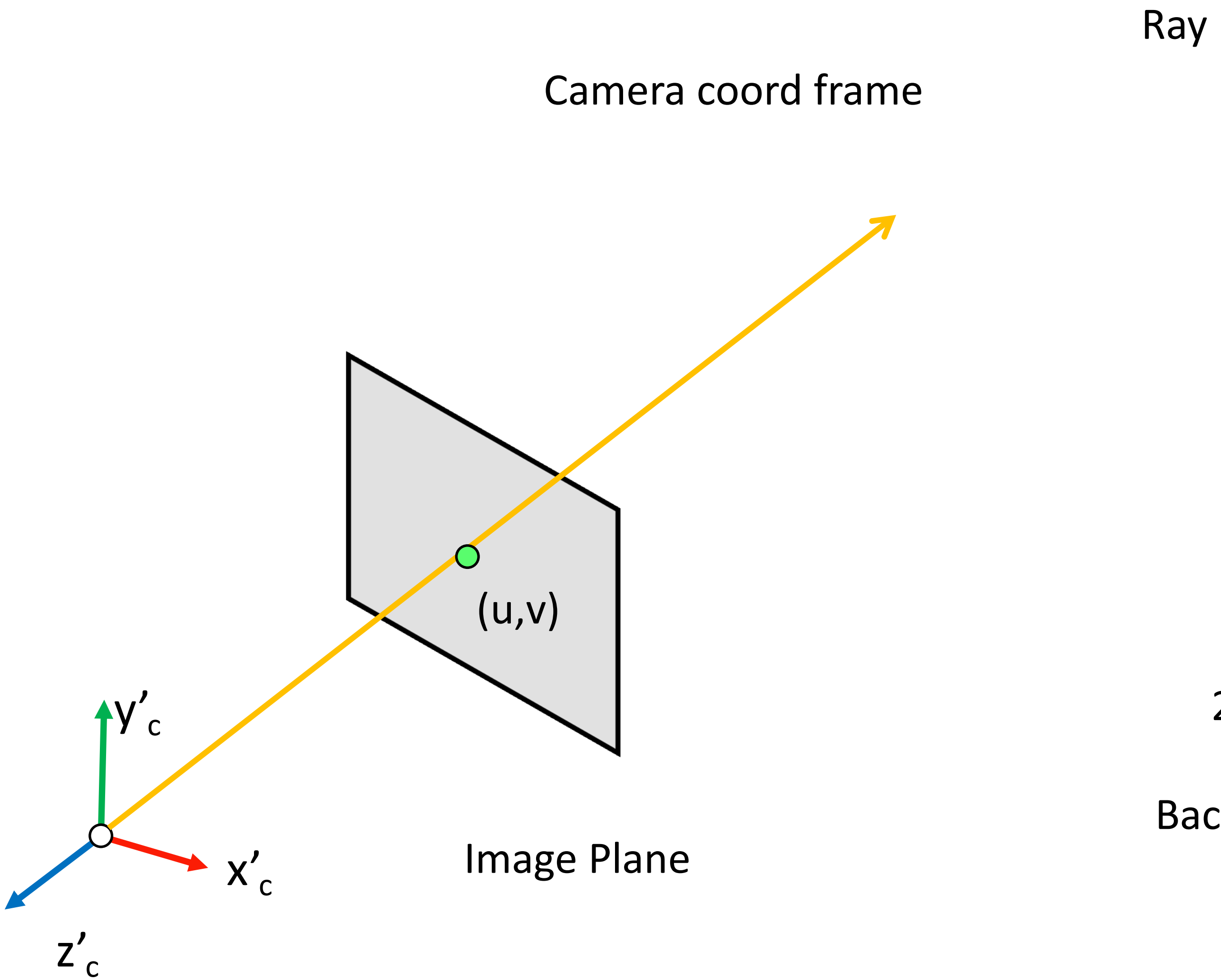
$$\mathbf{X}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$$

Last lecture:
How to calibrate the camera?

Slide inspired by Shree Nayar

Now that our cameras are calibrated, can we find the 3D scene point of a pixel?

You know we can't, but we know it'll be...
on the ray!



3D to 2D:
(point)

$$u = f_x \frac{x_c}{z_c} + o_x$$
$$v = f_y \frac{y_c}{z_c} + o_y$$

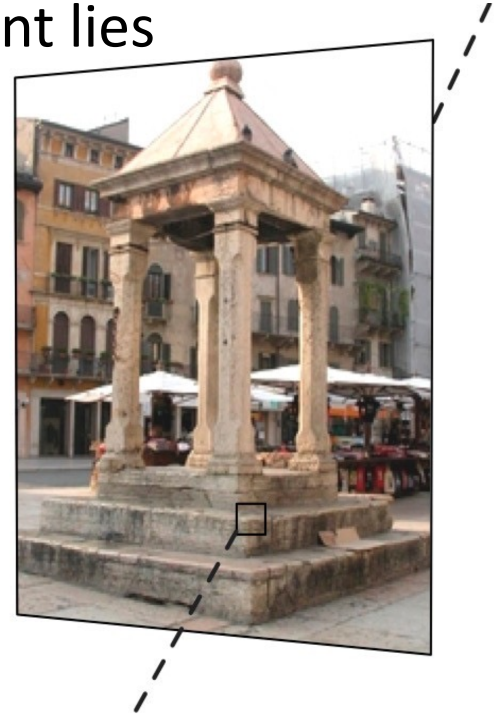
2D to 3D:
(ray)
Back projection

$$x = \frac{z}{f_x}(u - o_x)$$
$$y = \frac{z}{f_y}(v - o_y)$$
$$z > 0$$

Slide credit: Shree Nayar

Our goal: Develop theories and study how a 3D point and its projection in 2 images are related to each other!

From a single image you can only back project a pixel to obtain a ray on which the actual 3D point lies



To find the actual location of the 3D point, you need:

an additional image captured from another viewpoint.

Today’s class

- Epipolar Geometry
- Essential Matrix
- Fundamental Matrix
- 8-point Algorithm
- Triangulation

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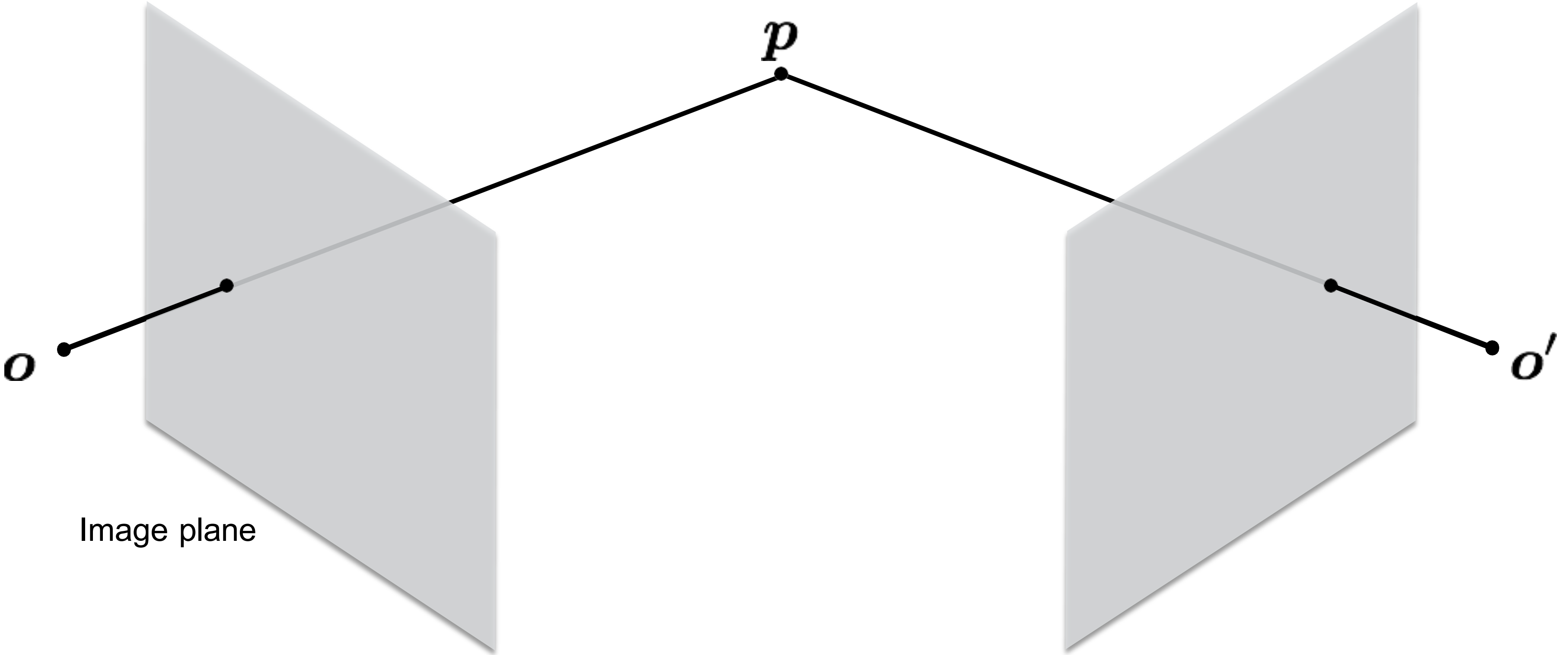
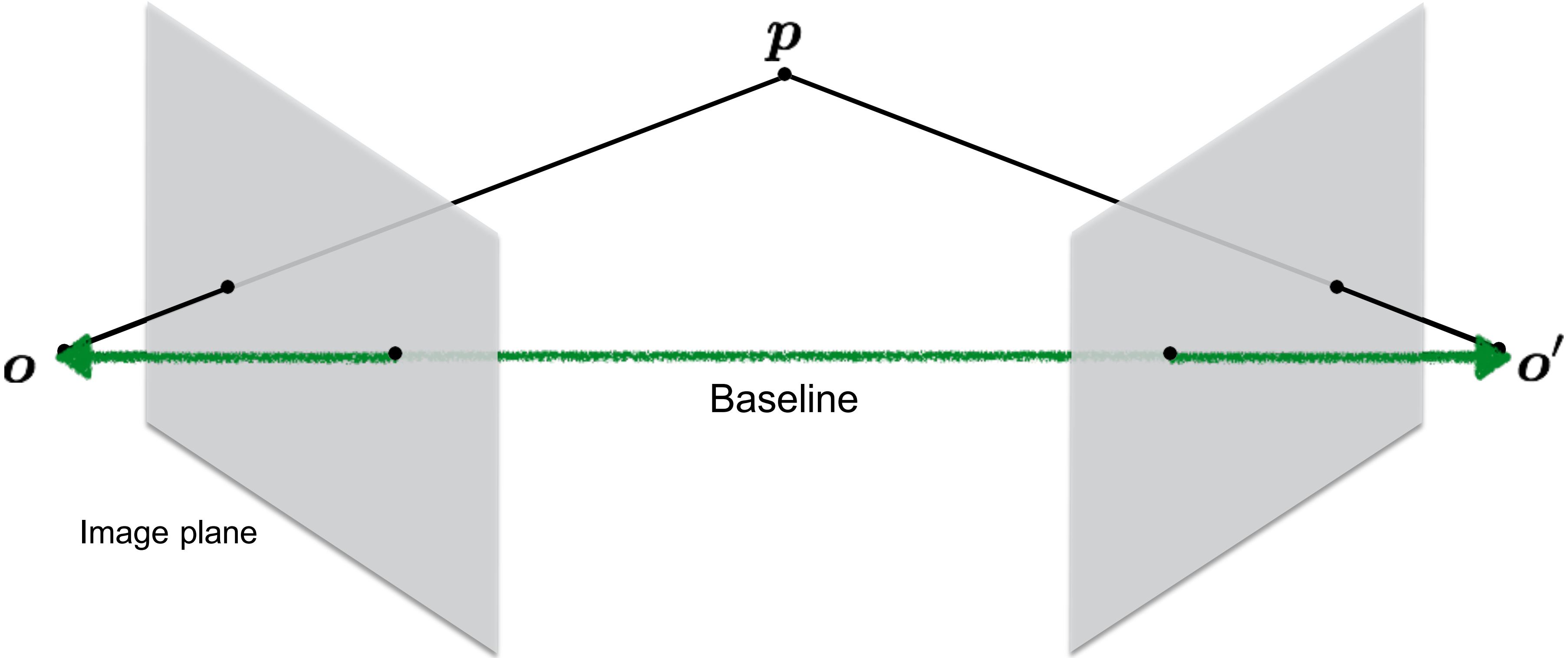
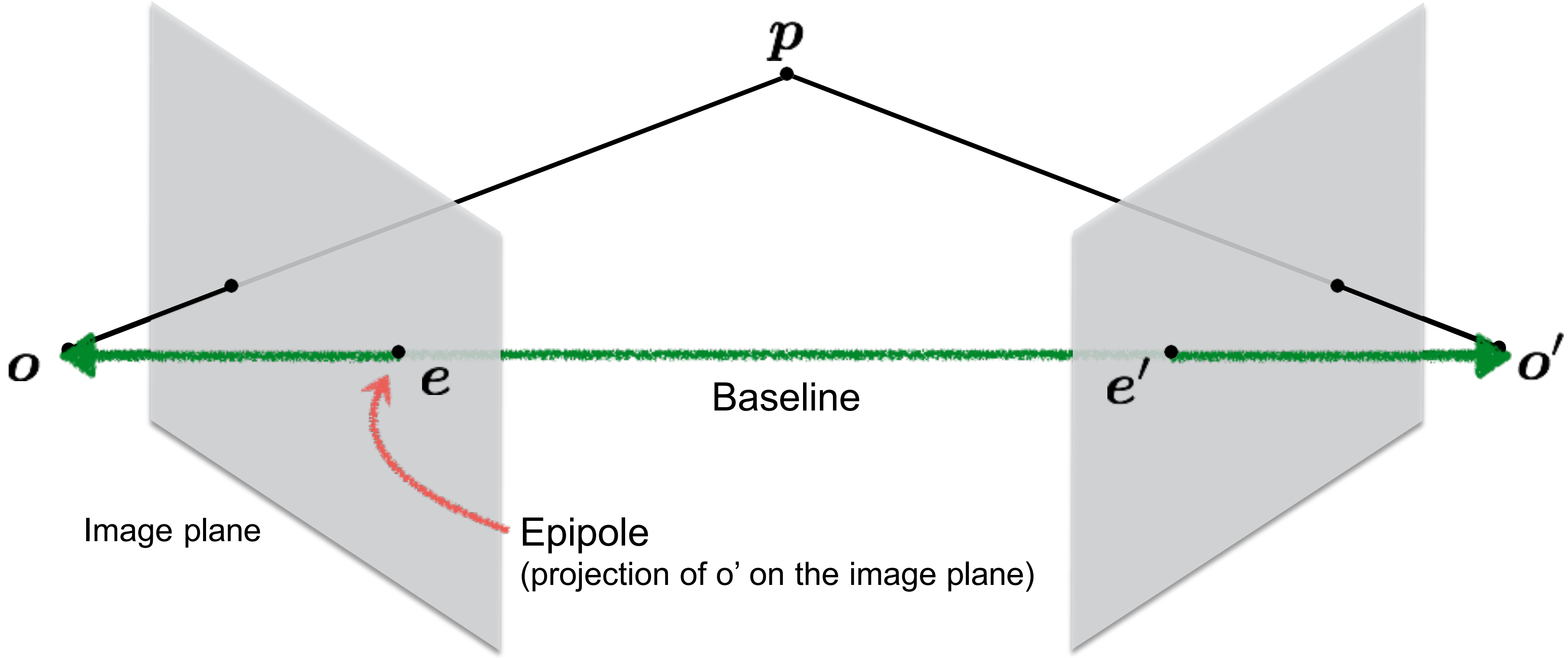
Epipolar geometry



Epipolar geometry



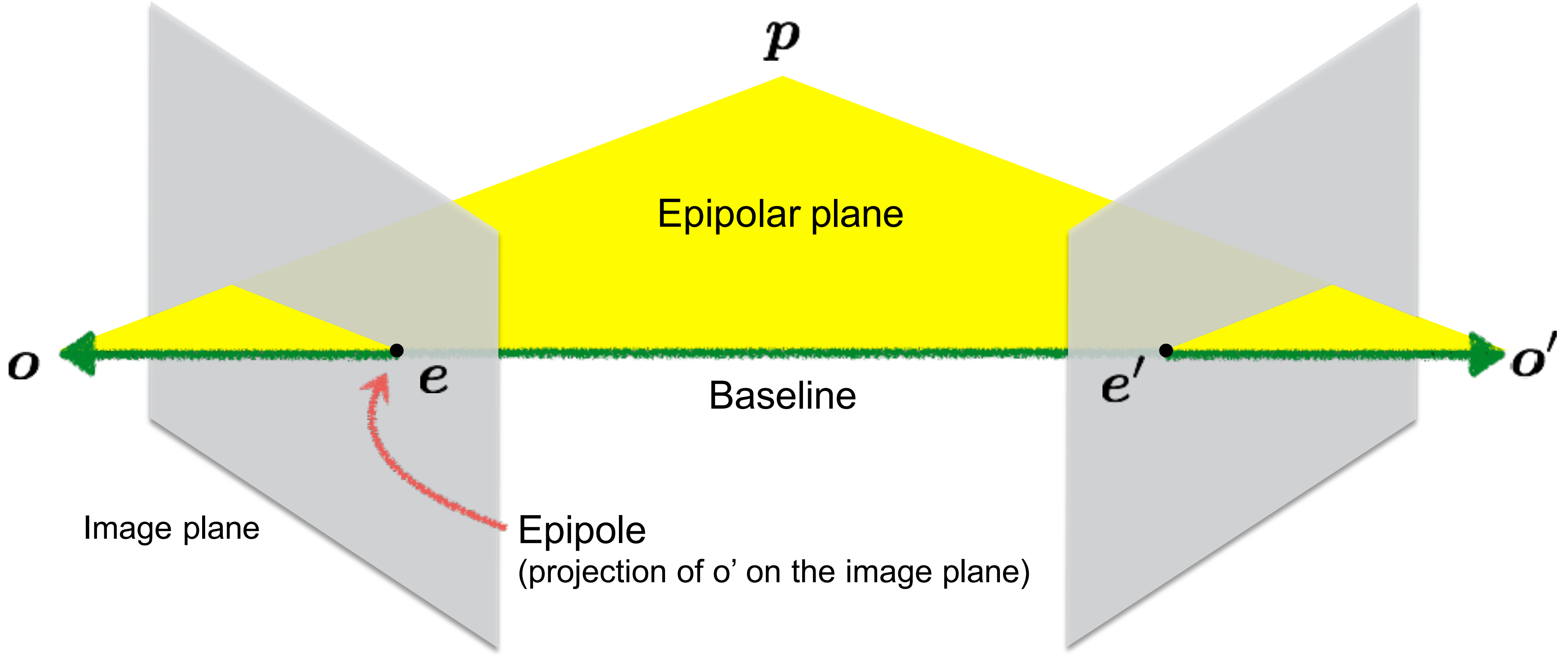
Epipolar geometry



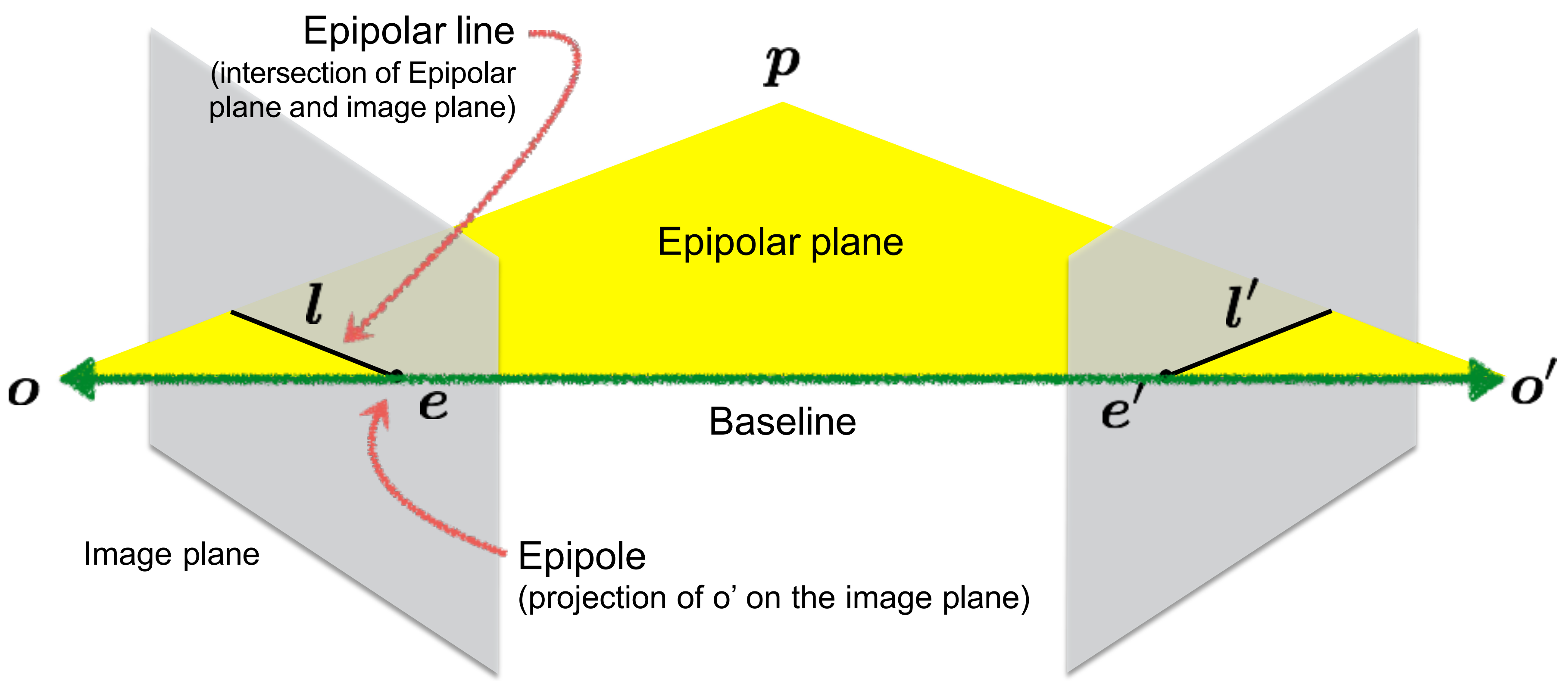
The Epipole

Photo by Frank Dellaert

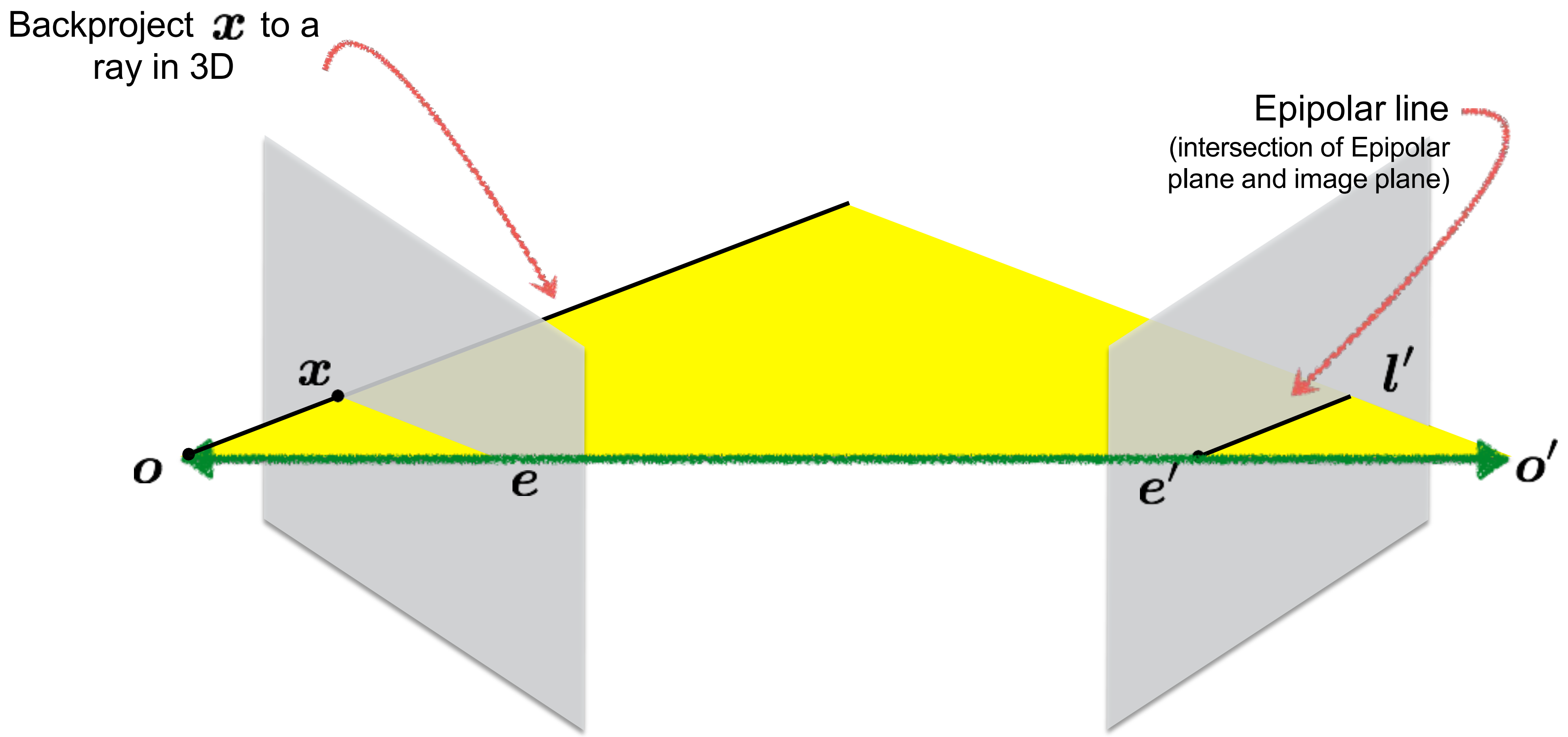
Epipolar geometry



Epipolar geometry

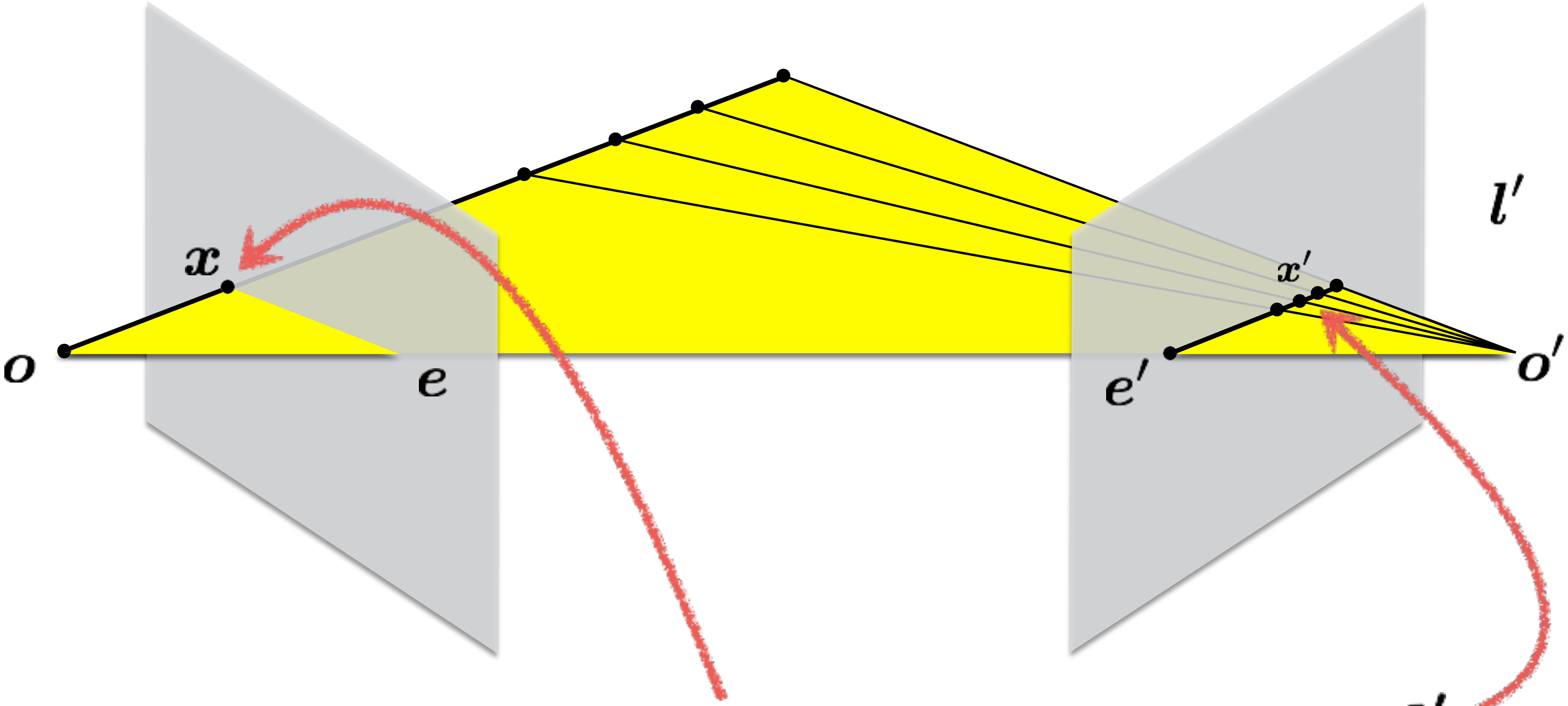


Epipolar constraint



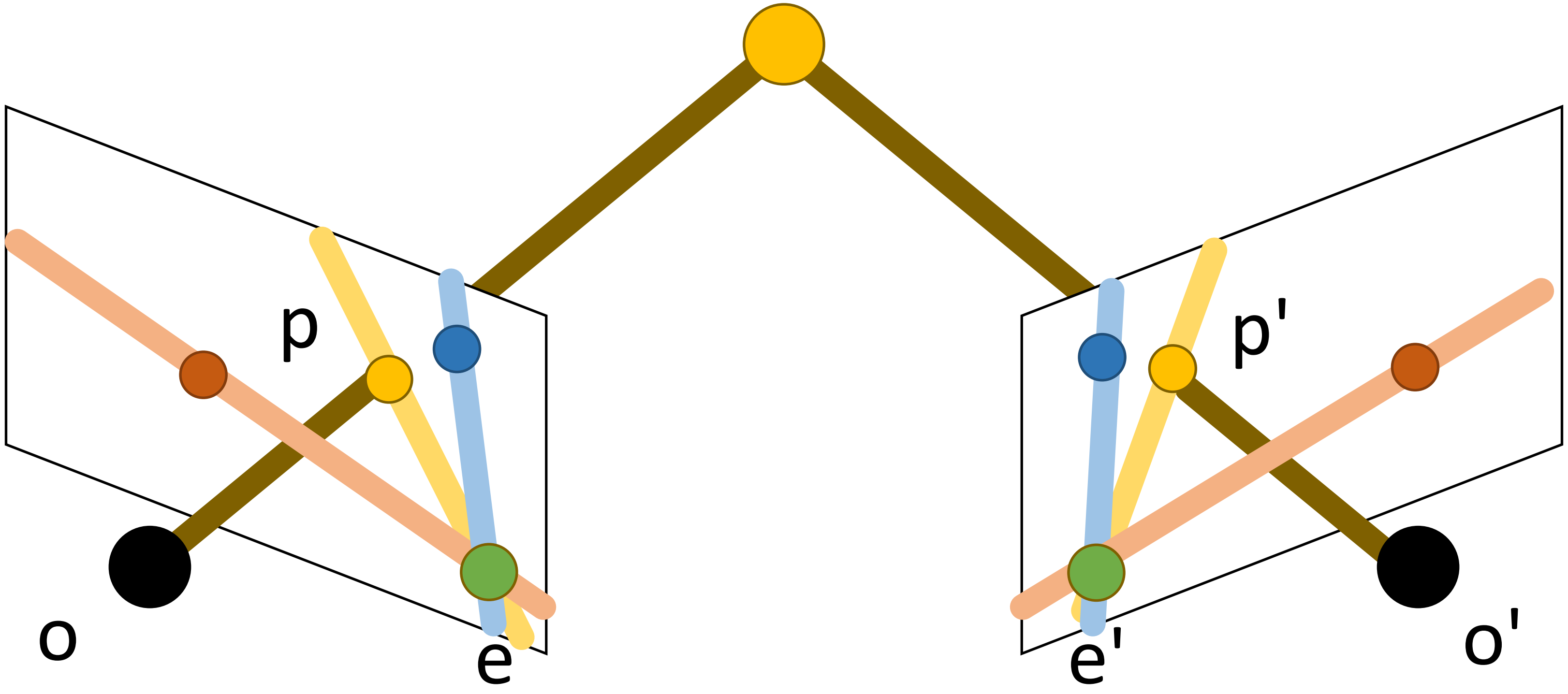
Another way to construct the epipolar plane, this time given x

Epipolar constraint



Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l}'$

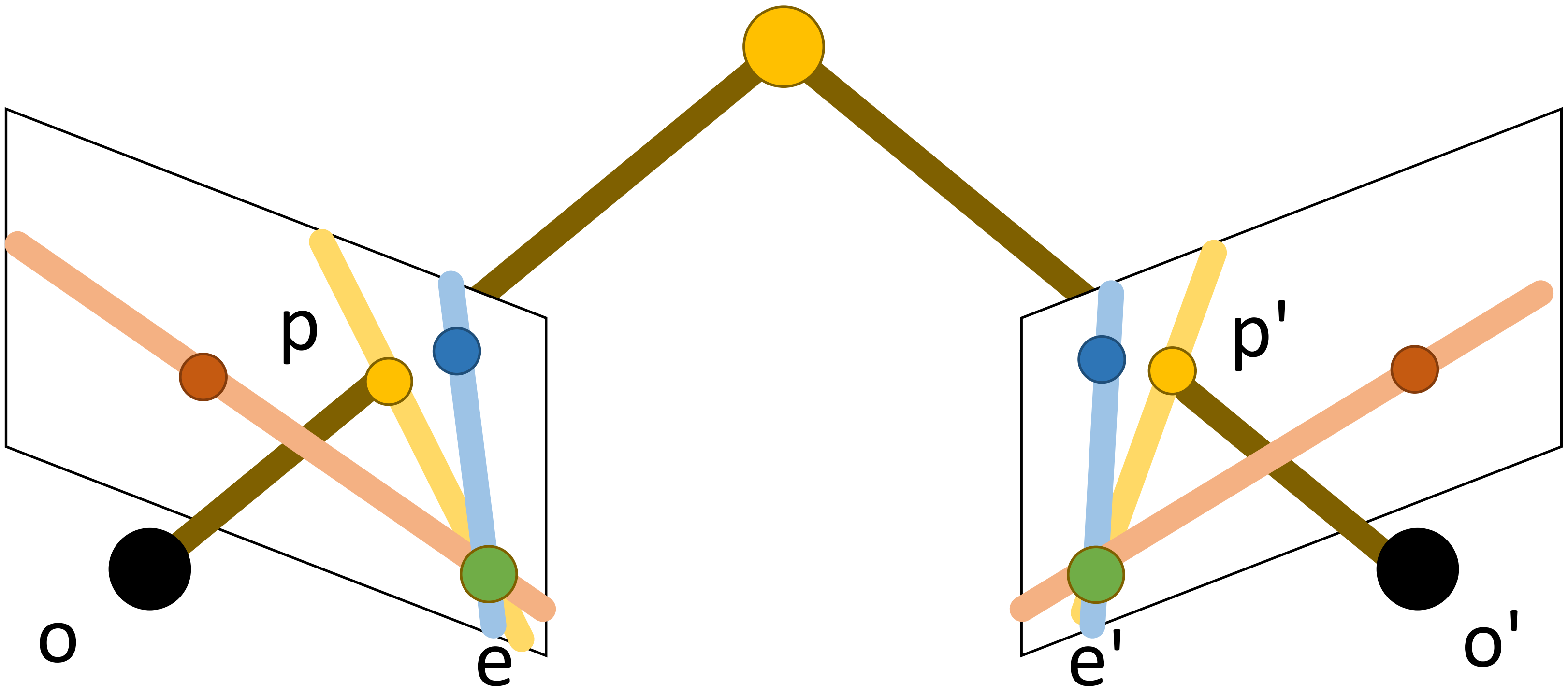
Example: Converging Cameras



Epipoles finite, maybe in image; epipolar lines converge

Slide credit: David Fouhey

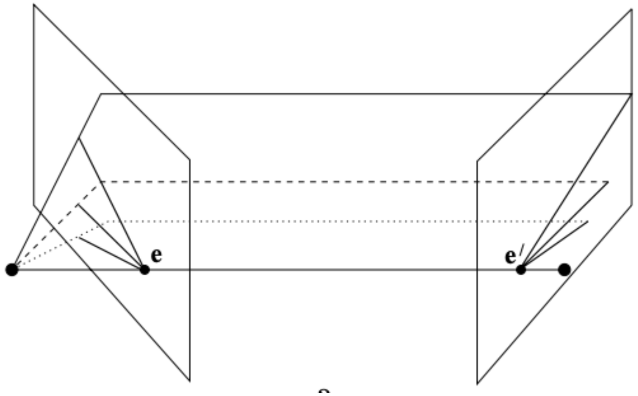
Example: Converging Cameras



Epipolar lines come in pairs:
given a point p, we can construct the epipolar line for p’.

Slide credit: David Fouhey

Example 1: Converging Cameras



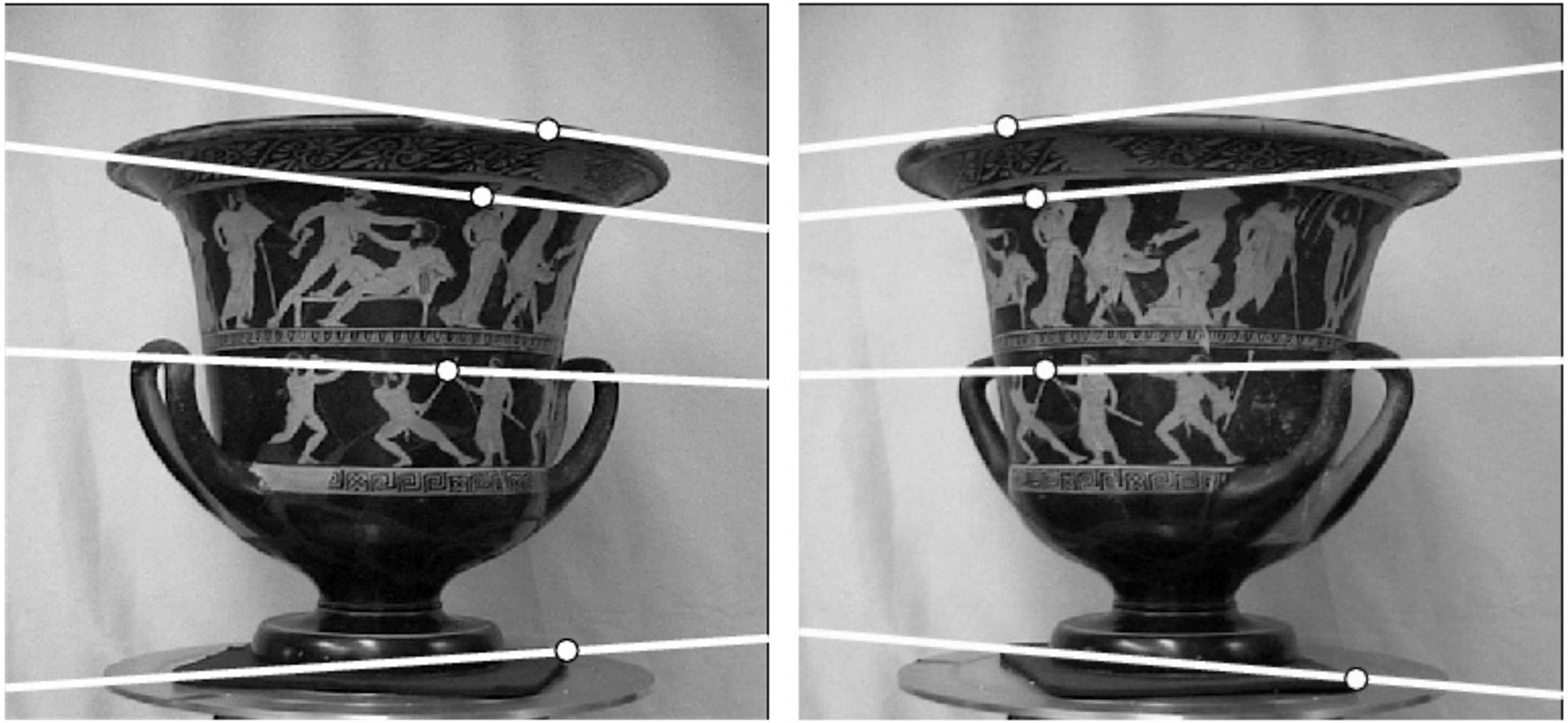
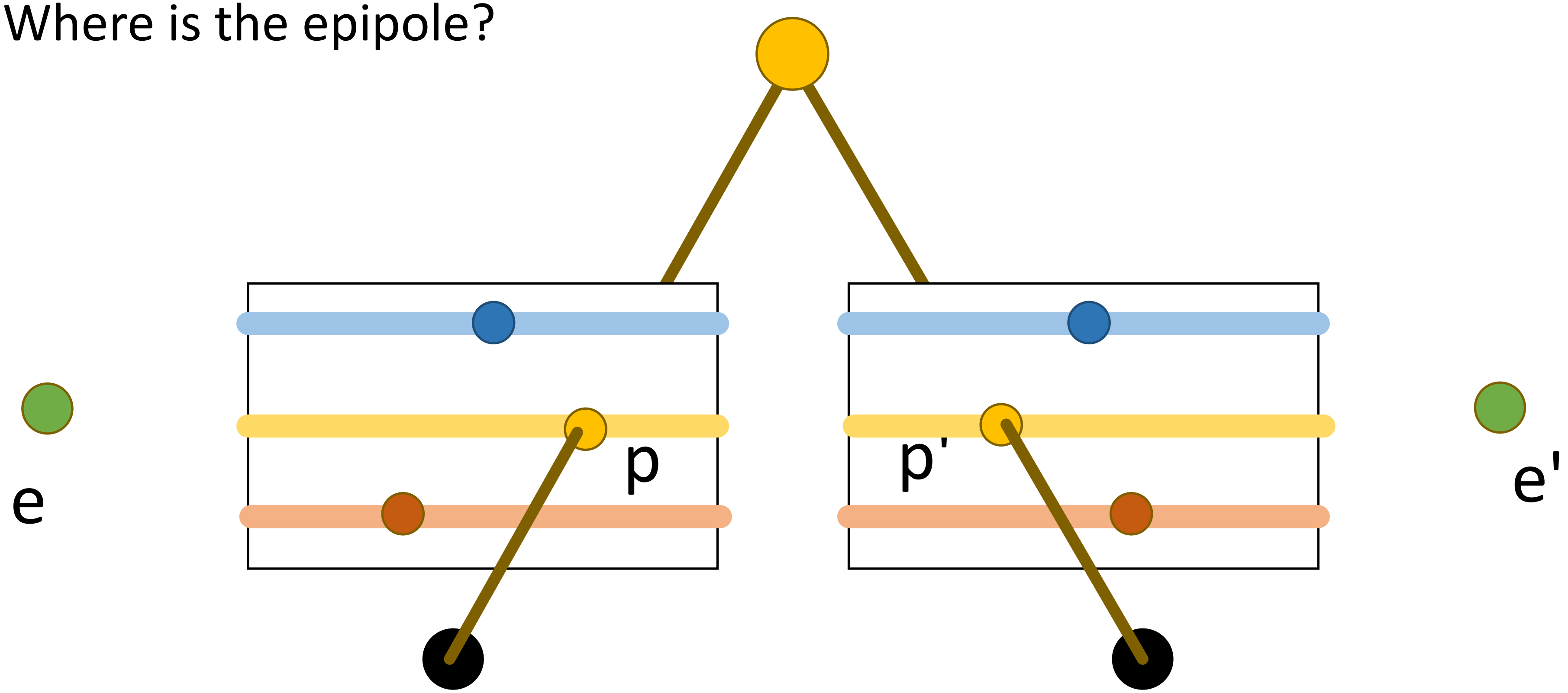


Image Credit: Hartley & Zisserman

Example: Parallel to Image Plane

Where is the epipole?



Epipoles *infinitely* far away, epipolar lines parallel

Slide credit: David Fouhey

Example: Forward Motion

Image Credit: Hartley & Zisserman

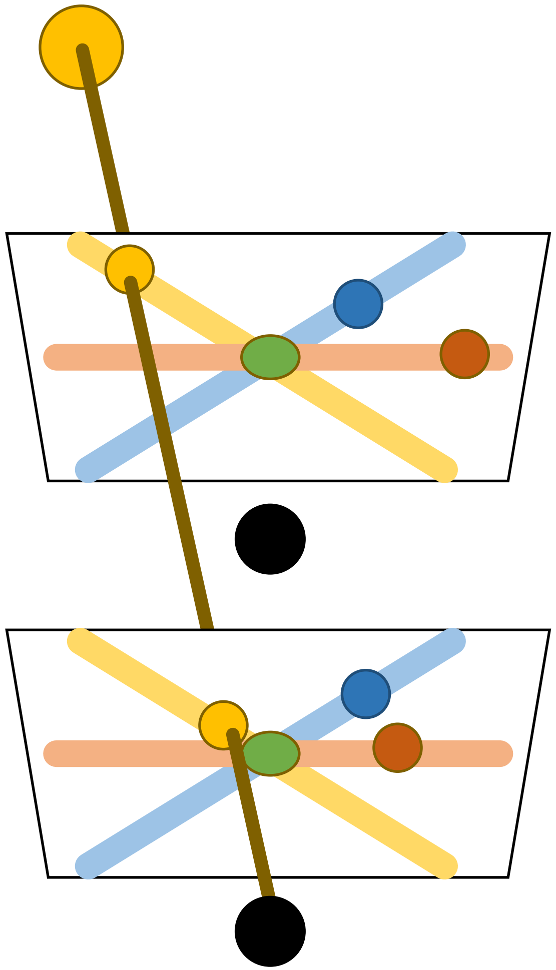
Example: Forward Motion

Image Credit: Hartley & Zisserman

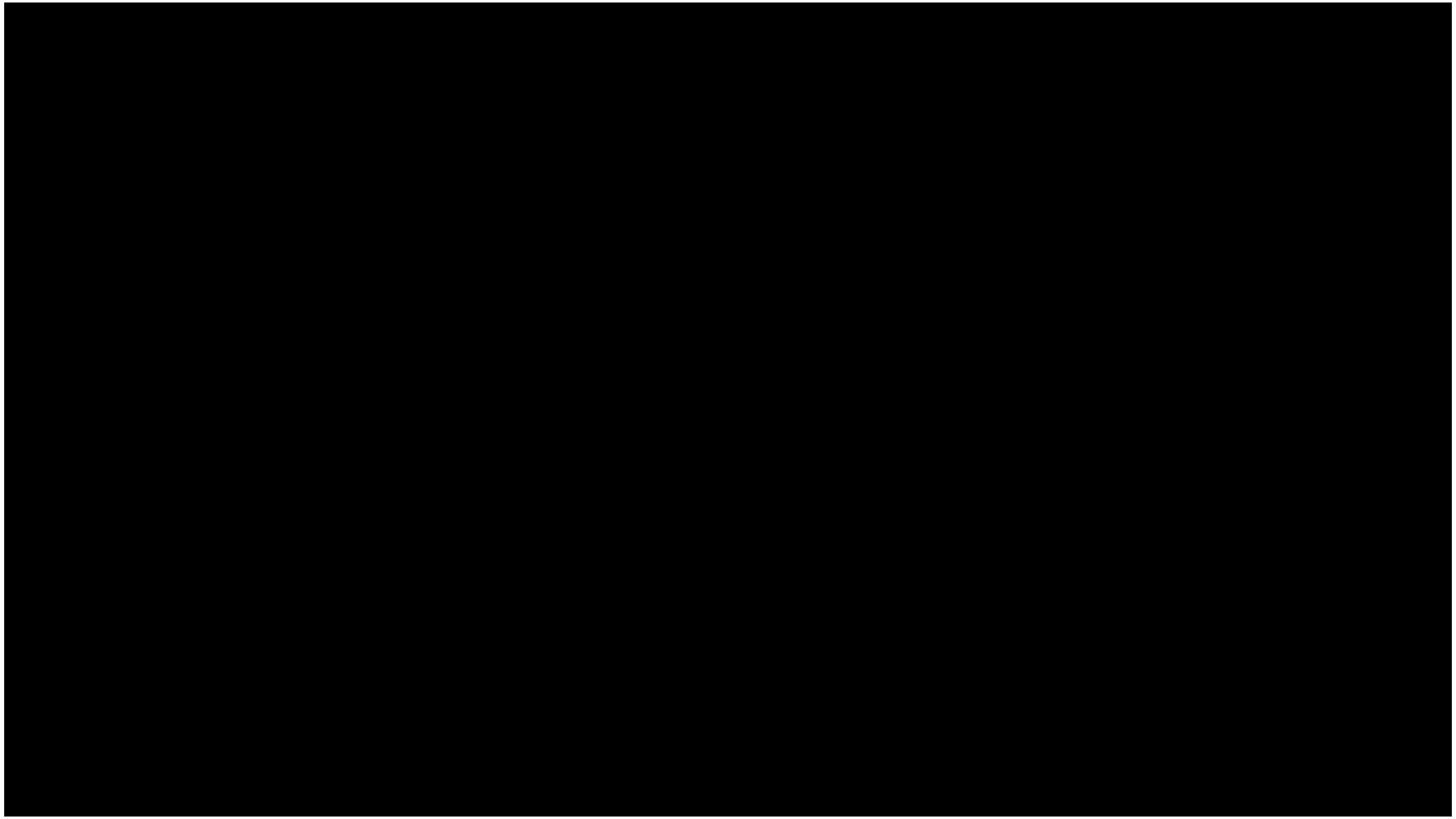
Example: Forward Motion

Epipole is focus of expansion / principal point of the camera.

Epipolar lines go out from principal point



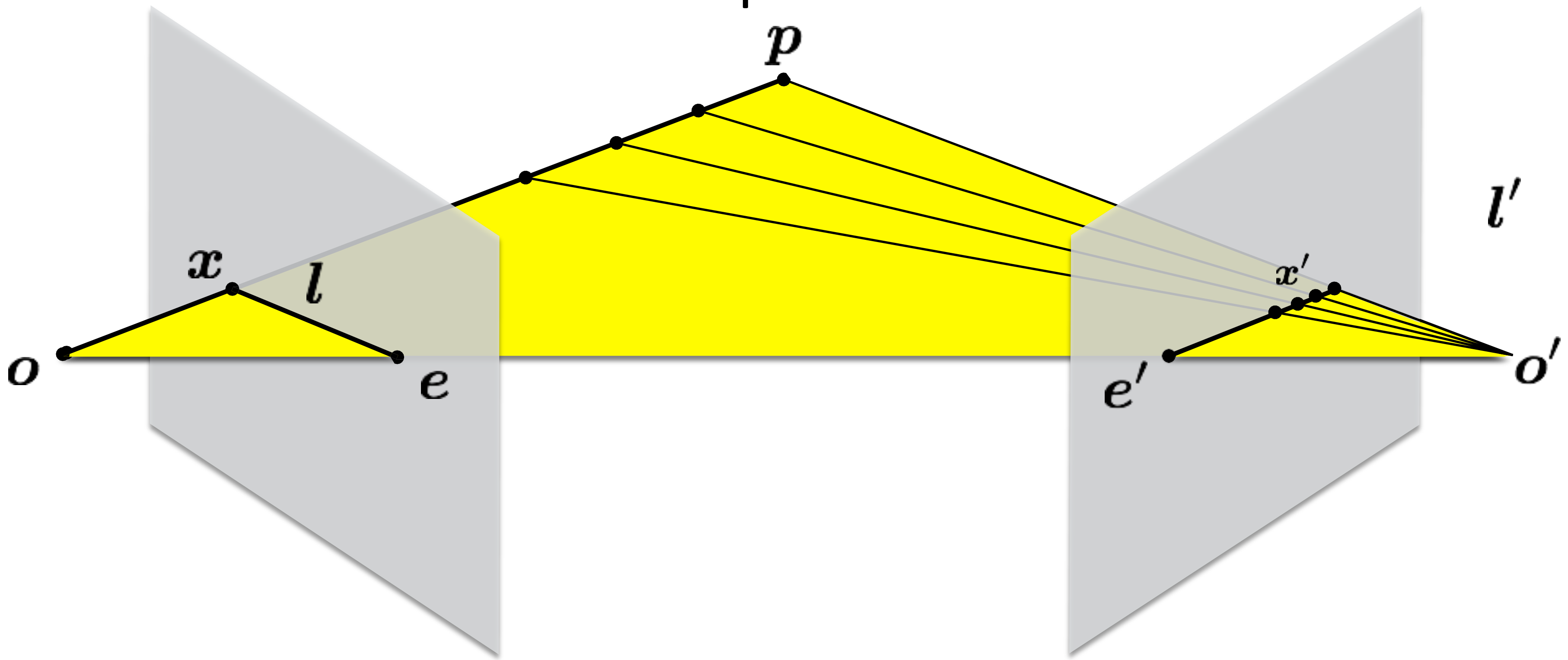
Motion perpendicular to image plane



http://vimeo.com/48425421

Slide credit: David Fouhey

Recap Time!



The point **x** (left image) maps to a ___________ in the right image

The baseline connects the ___________ and ____________

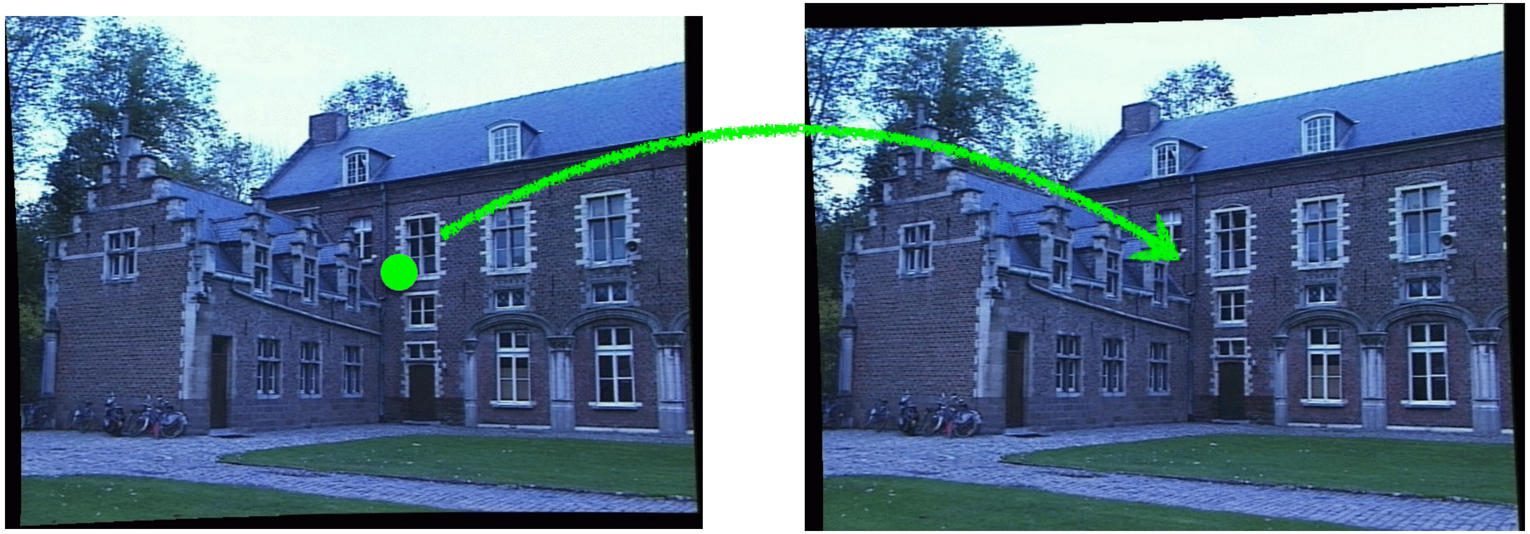
An epipolar line (left image) maps to a __________ in the right image

An epipole **e** is a projection of the______________ on the image plane

All epipolar lines in an image intersect at the ______________

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image

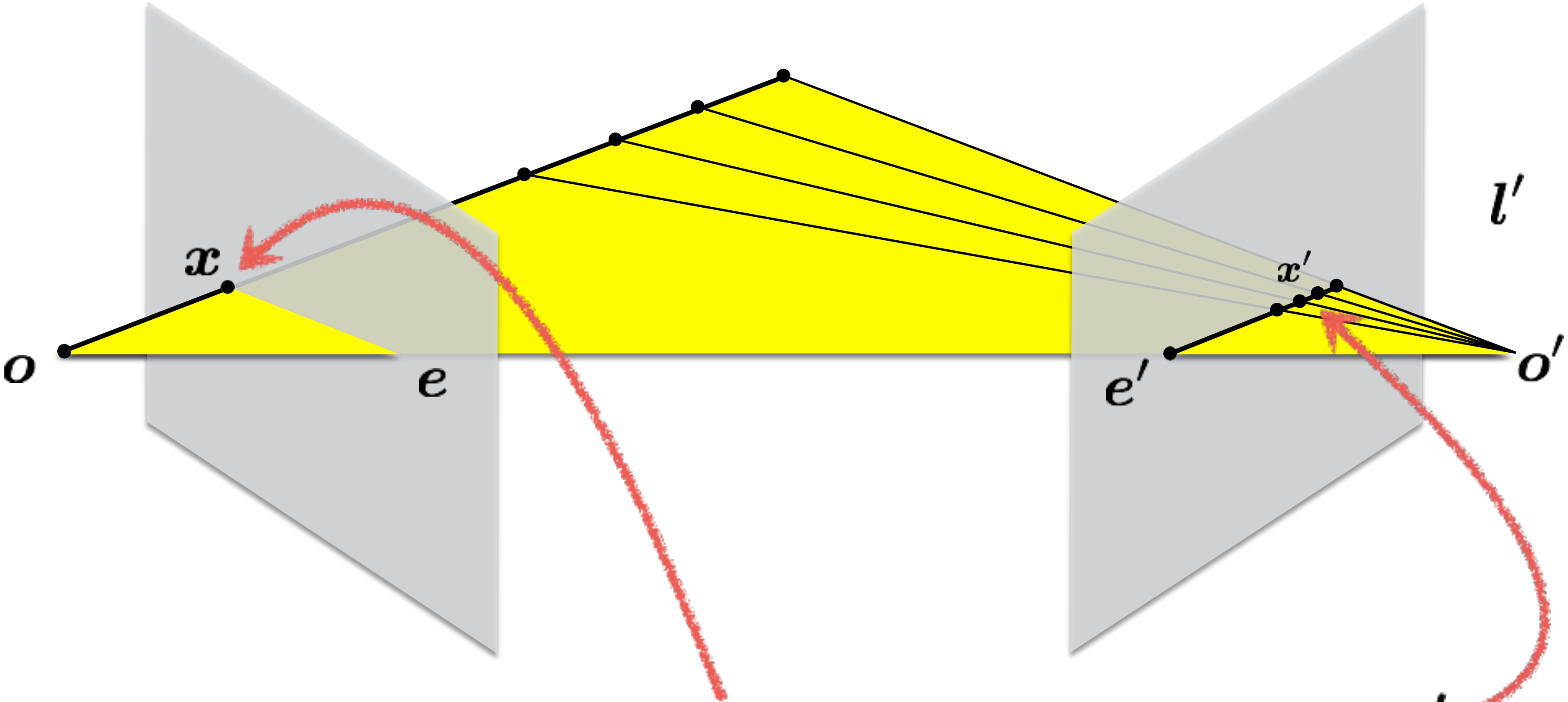


Left image

Right image

How would you do it?

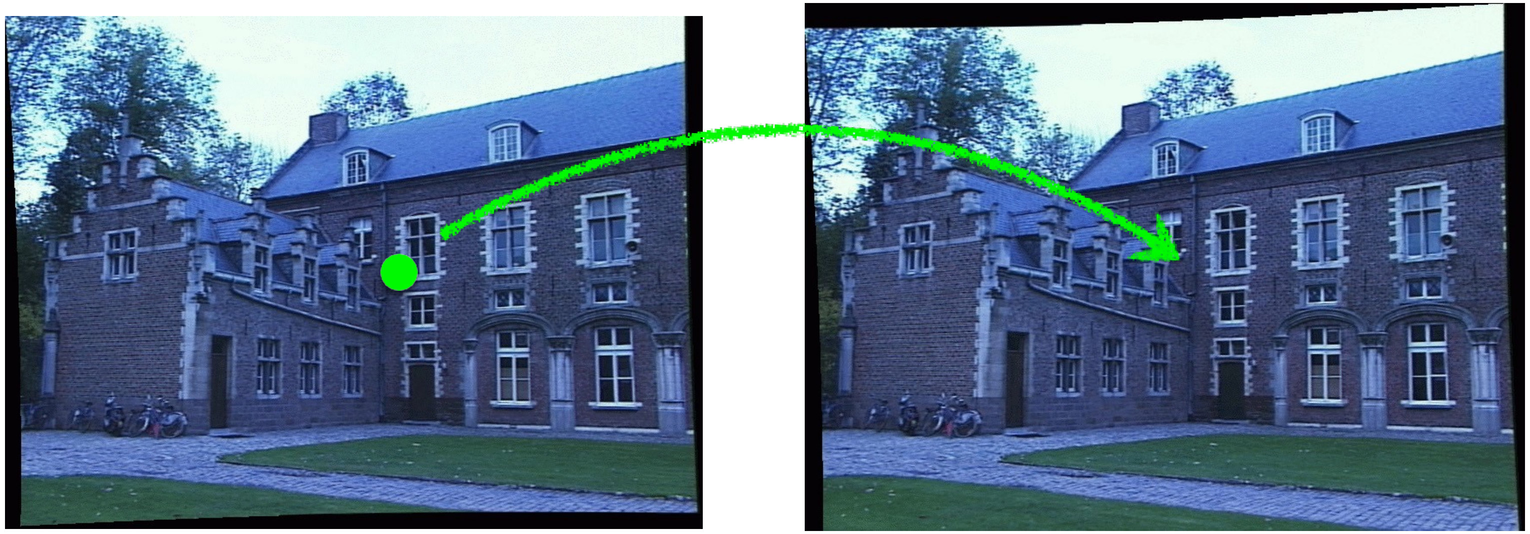
Epipolar constraint



Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l}'$

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



Left image

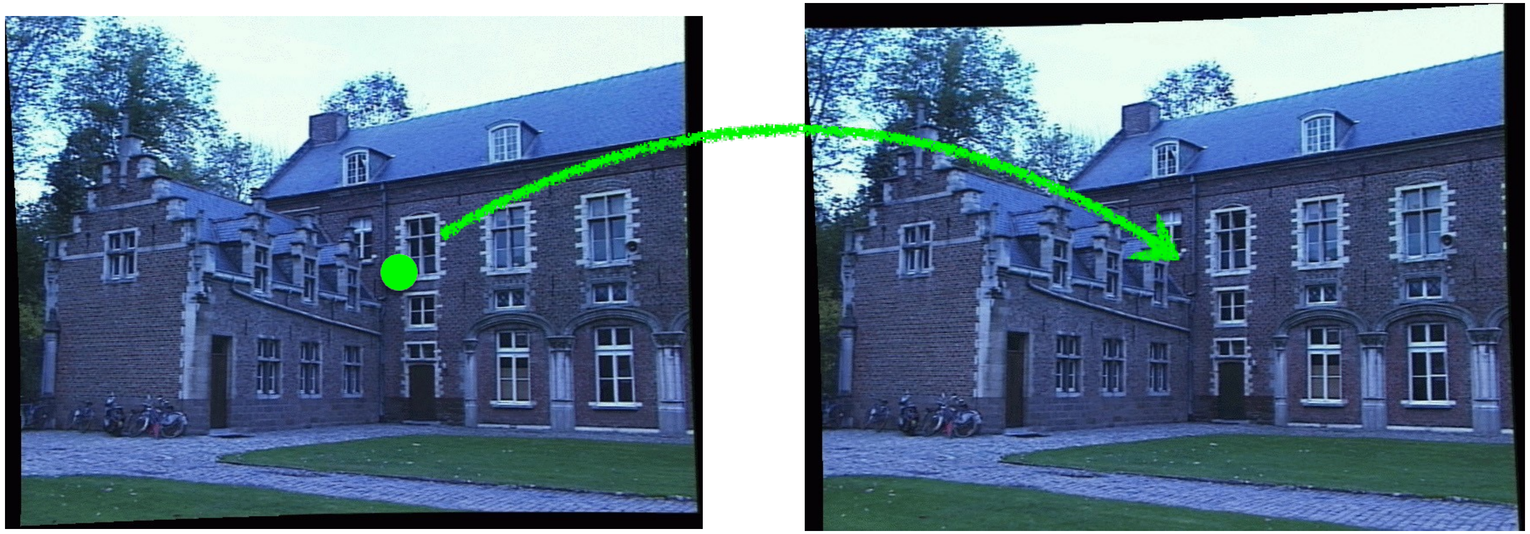
Right image

Want to avoid search over entire image

Epipolar constraint reduces search to a single line

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



Left image

Right image

Want to avoid search over entire image

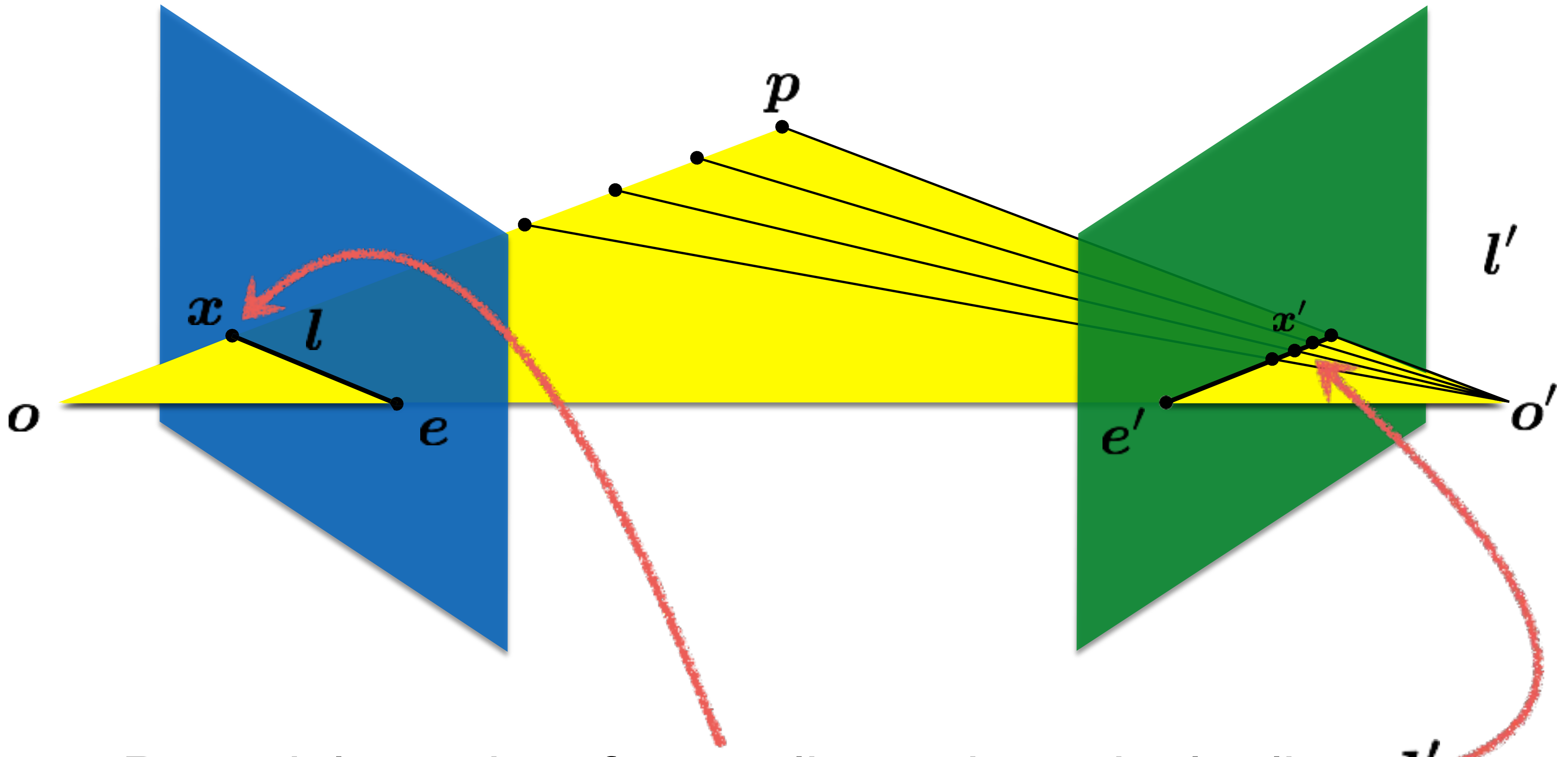
Epipolar constraint reduces search to a single line

How do you compute the epipolar line?

Today’s class

- Epipolar Geometry
- Essential Matrix
- Fundamental Matrix
- 8-point Algorithm
- Triangulation

Recall:Epipolar constraint

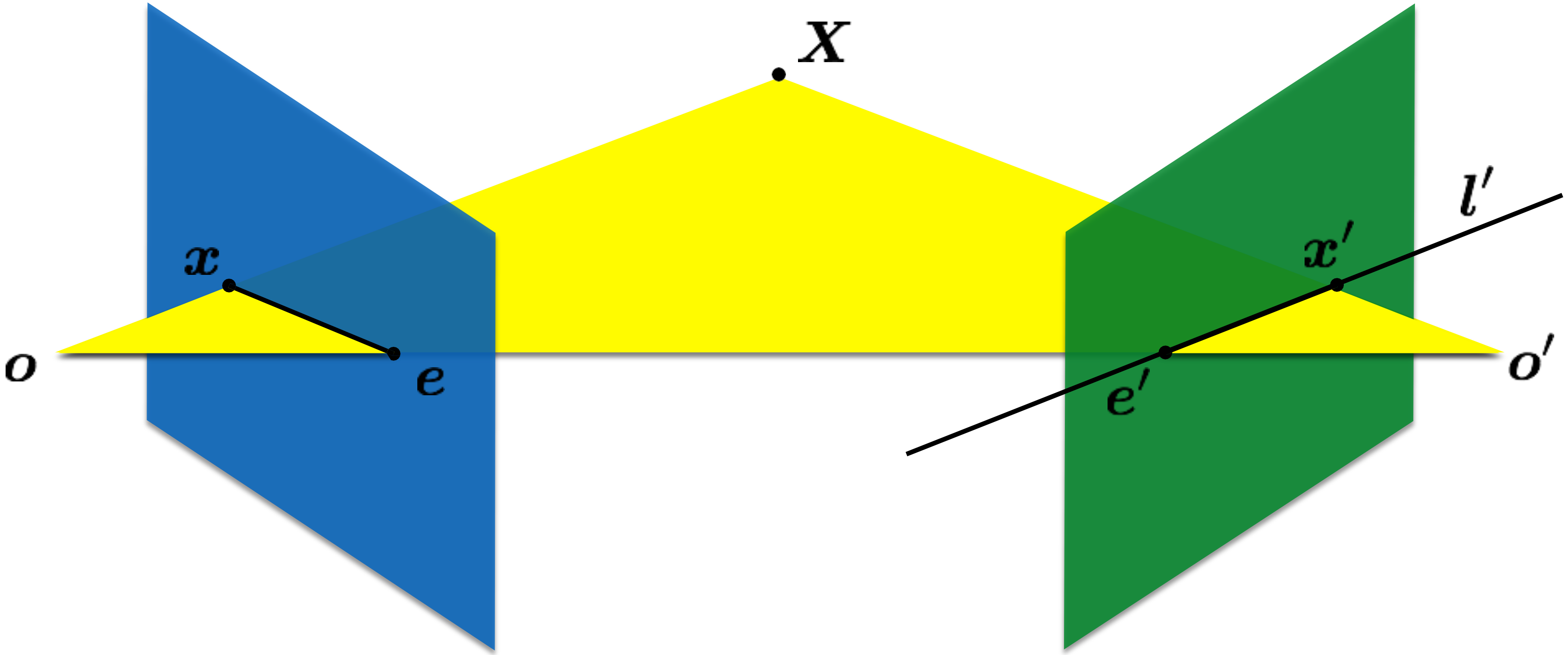


Potential matches for x lie on the epipolar line l'

Given a point in one image,
multiplying by the **essential matrix** will tell us
the **epipolar line** in the second view.

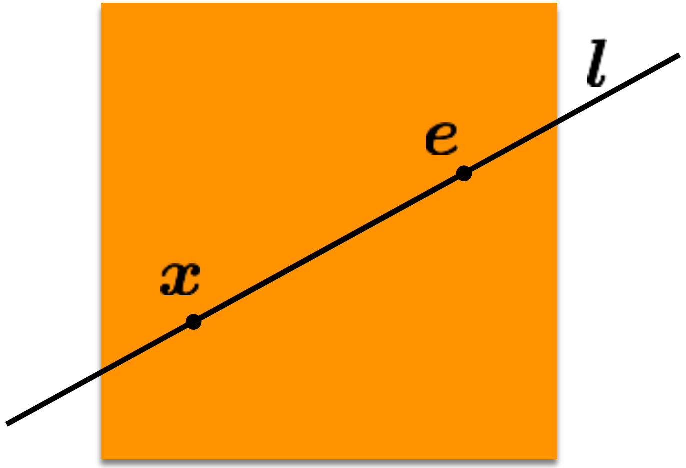
$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$

Essential matrix is 3x3 and encodes epipolar geometry.



Epipolar Line

$$ax + by + c = 0$$ in vector form $$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

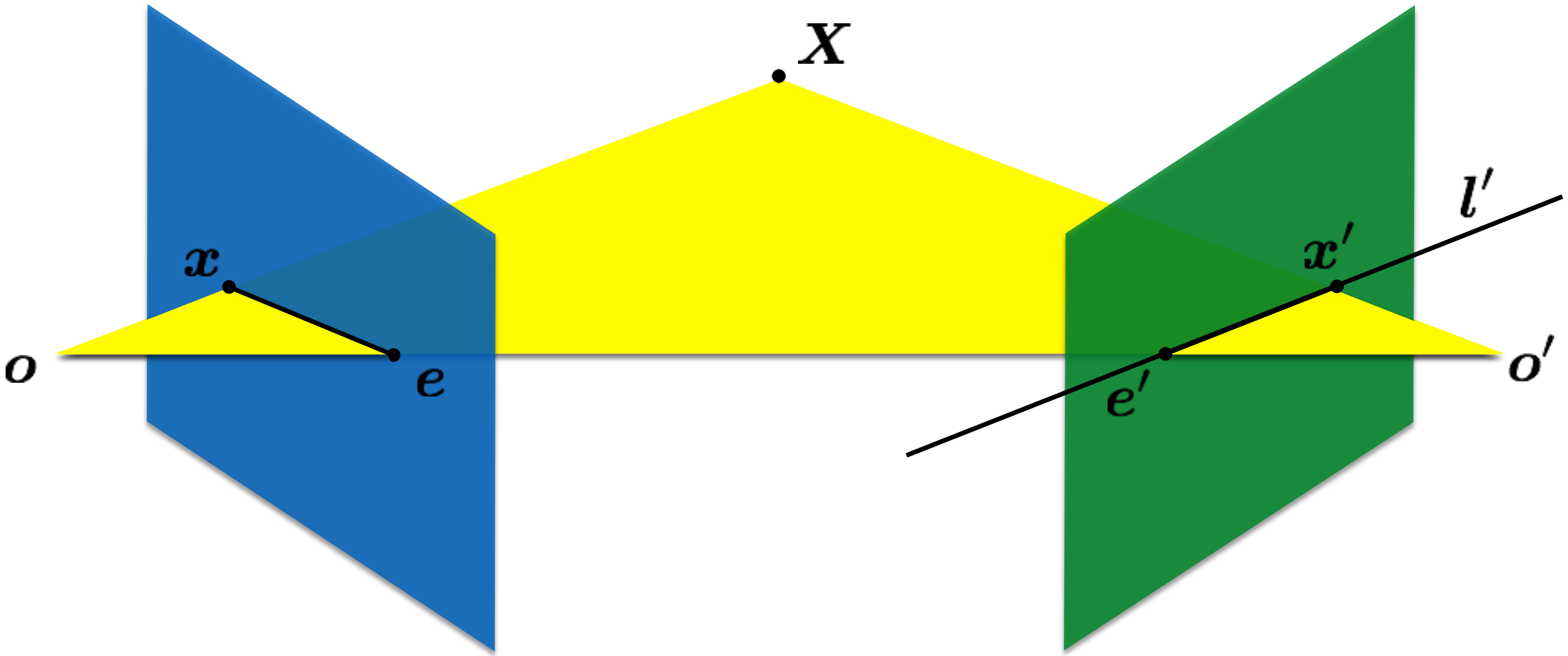


If the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

$$\boldsymbol{x}^\top \boldsymbol{l} = 0$$

So if $\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

$$\boxed{\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0}$$



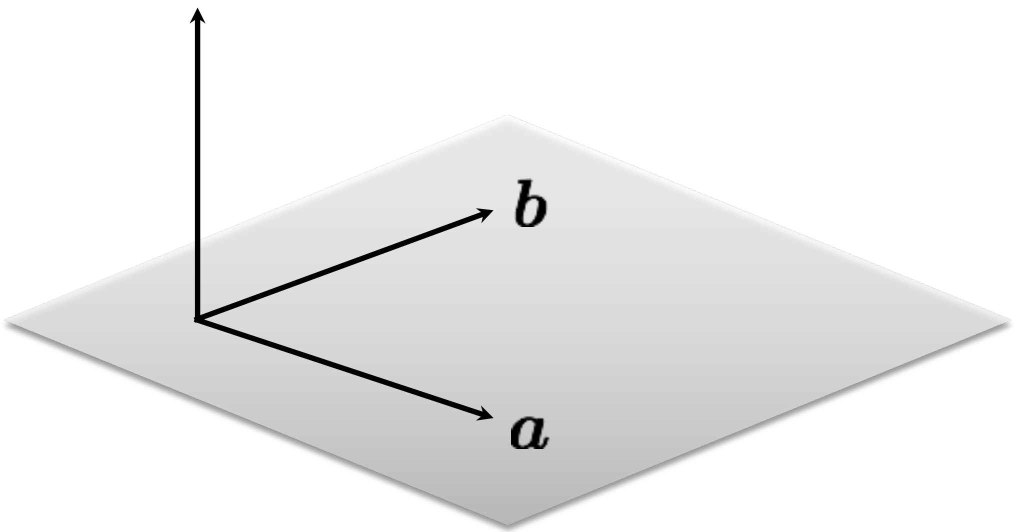
Where does the essential matrix come from?

Can we express essential matrix as function of camera parameters?

Linear algebra reminder: cross product

Vector (cross) product
takes two vectors and returns a vector perpendicular to both

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$



$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

cross product of two vectors in the same direction is zero vector

$$\boldsymbol{a} \times \boldsymbol{a} = 0$$

remember this!!!

$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

Linear algebra reminder: cross product

Cross product

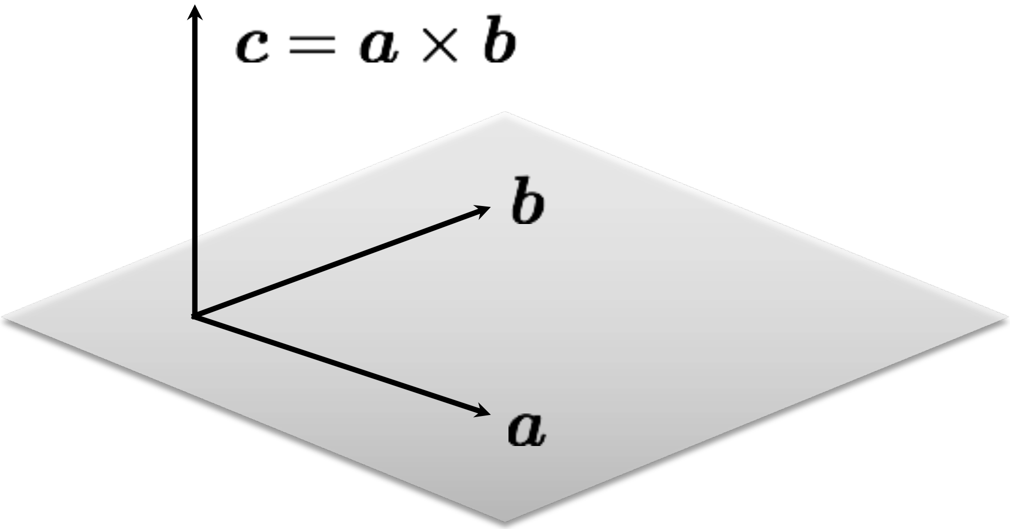
$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Can also be written as a matrix multiplication

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

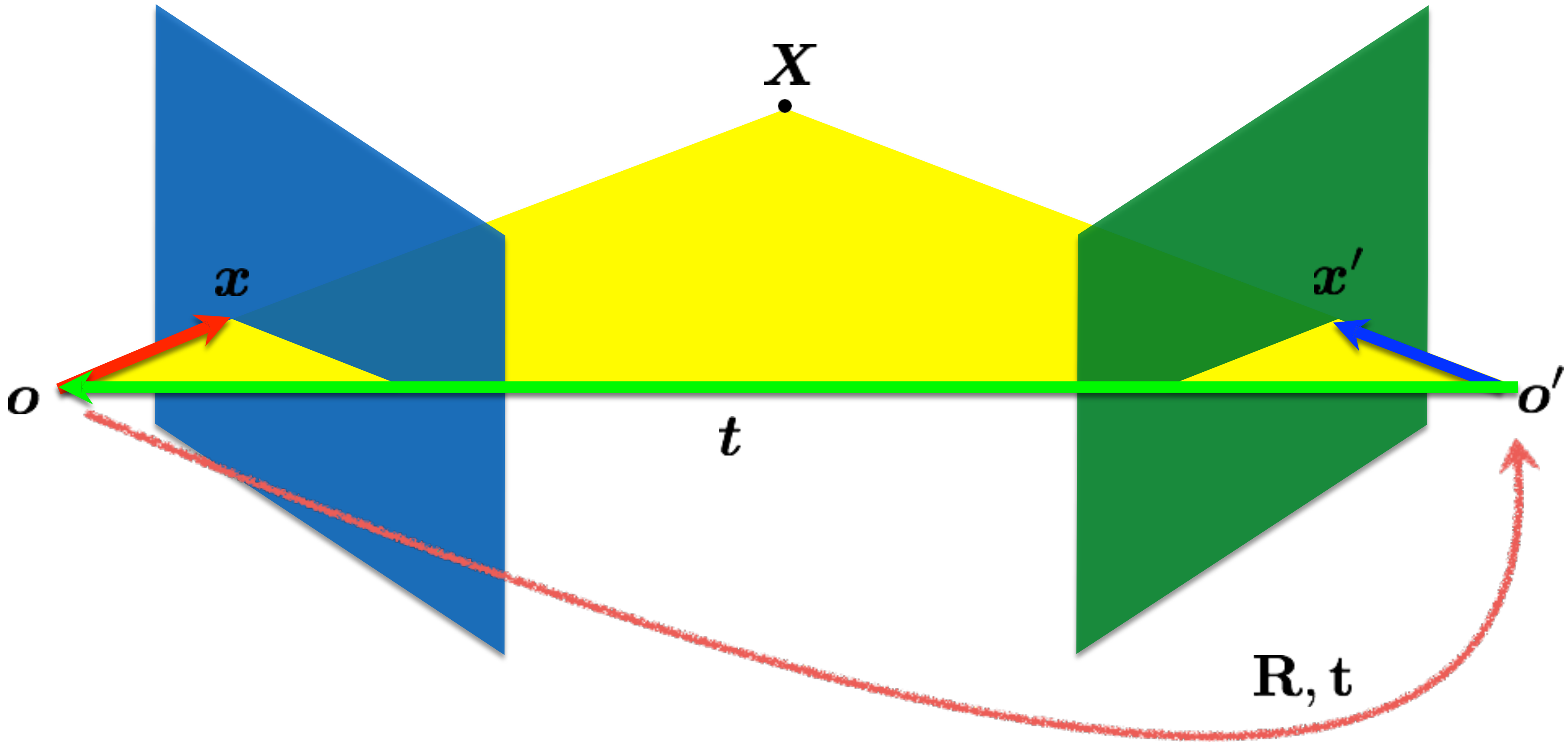
Skew symmetric

Compare with: dot product



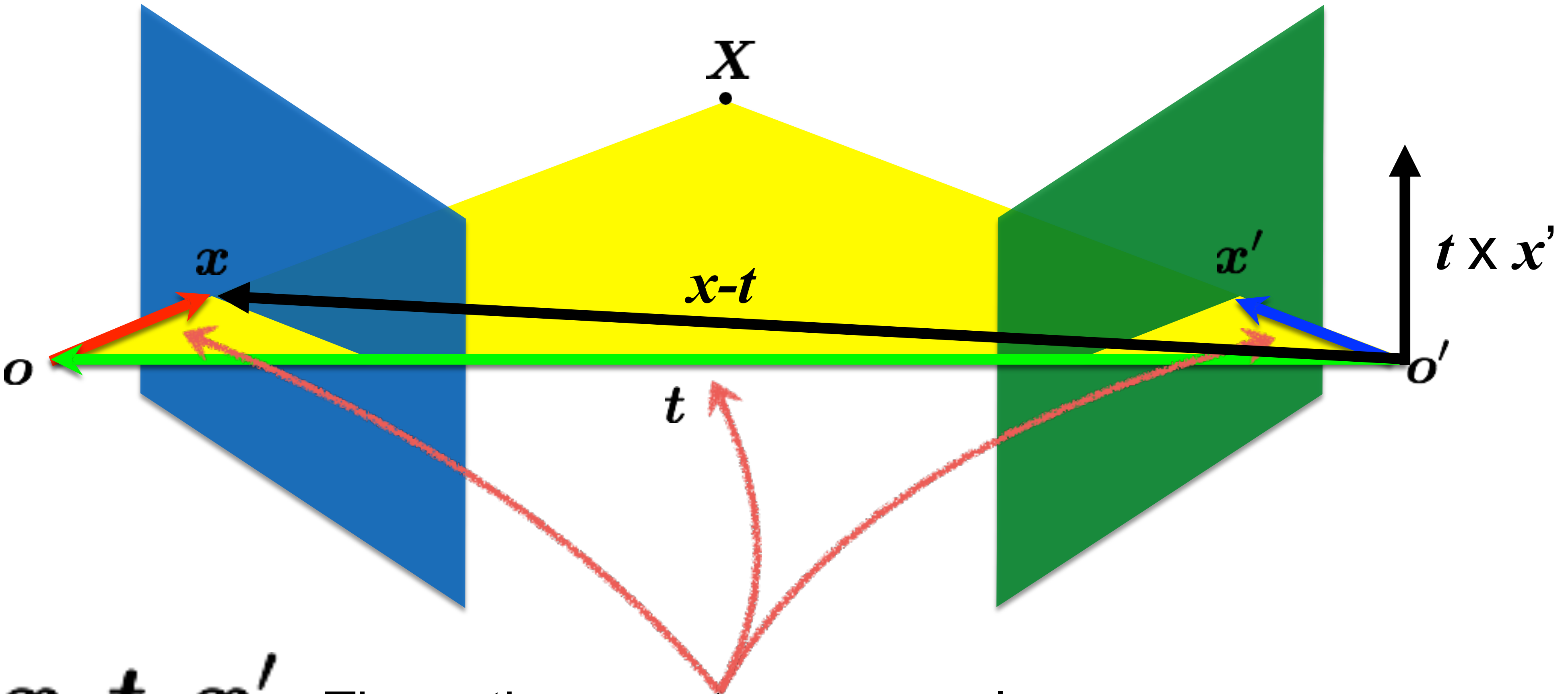
$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

dot product of two orthogonal vectors is (scalar) zero



$$x' = \mathbf{R}(x - t)$$

Camera-camera transform just like **world-camera** transform



$\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ These three vectors are coplanar

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}') = 0$$

dot product of orthogonal vectors

cross-product: vector orthogonal to plane

Putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

use skew-symmetric matrix to represent cross product

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

$$\boxed{\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = 0}$$

Essential Matrix
[Longuet-Higgins 1981]

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Skew symmetric

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times}$$

Longuet-Higgins Prize

The Longuet-Higgins Prize recognizes CVPR papers from ten years ago that have made a significant impact on computer vision research.

More information about this prize can be found here

2022	"Are We Ready for Autonomous Driving? The KITTI Vision Benchmark Suite"	A. Geiger, P. Lenz, R. Urtasun
2021	"Real-time human pose recognition in parts from single depth image"	J. Shotton, A. Fitzgibbon, M. Cook, T. Sharp, M. Finocchio, R. Moore, A. Kipman, A. Blake
2021	"Baby talk: Understanding and generating simple image descriptions"	G. Kulkarni, V. Premraj, S. Dhar, S. Li, Y. Choi, A. C. Berg, T. L. Berg
2020	"Secrets of Optical Flow Estimation and Their Principles"	D. Sun, S. Roth, M. Black
2019	"ImageNet: A large-scale hierarchical image database"	J. Deng, W. Dong, R. Socher, L.-J. Li, K. Li, L. Fei-Fei
2018	"A Discriminatively Trained, Multiscale, Deformable Part Model"	P. Felzenszwalb, D. McAllester, and D. Ramanan
2017	"Accurate, Dense, and Robust Multi-View Stereopsis"	Y. Furukawa, J. Ponce
2017	"Object Retrieval with Large Vocabularies and Fast Spatial Matching"	J. Philbin, O. Chum, M. Isard, J. Sivic, A. Zisserman
2016	"Beyond Bags of Features: Spatial Pyramid Matching for Recognizing Natural Scene Categories"	S. Lazebnik, C. Schmid, J. Ponce

Most coveted prize in Computer Vision! Test-of-the award!

Faculty @ UNC, Now at Meta

Faculty @ UNC, Now Faculty @ UIUC

Properties of the E matrix

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times}$$

Longuet-Higgins equation $\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = 0$

Epipolar lines

$\boldsymbol{x}^\top \boldsymbol{l} = 0$ $\boldsymbol{x}'^\top \boldsymbol{l}' = 0$

$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$ $\boldsymbol{l} = \mathbf{E}^T \boldsymbol{x}'$

Epipoles $\boldsymbol{e}'^\top \mathbf{E} = \mathbf{0}$ $\mathbf{E}\boldsymbol{e} = \mathbf{0}$

(2D points expressed in <u>camera</u> coordinate system)

Properties of the E matrix

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_{\times}}$$

- E has 5 degrees of freedom, why?
  - R has 3 degree of freedom
  - T has 3 degree of freedom
  - However since this is a projective transformation one can apply an arbitrary scale to E. Thus 1 degree of freedom less.

- E is rank 2, why?
  - $[t_x]$ is skew symmetric, hence rank 2.
  - Thus Det(E) = 0.

- E has 2 singular value both of which are equal.
  - $[t_x]$ a skew symmetric matrix has 2 equal singular values

2 possible notation

x' = R (x – t) $\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times}$

x' = Rx – Rt
= Rx + t' $\mathbf{E} = [\tilde{\mathbf{t}}]_\times \mathbf{R}$

Today's class

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- Essential Matrix
- Fundamental Matrix
- 8-point Algorithm
- Triangulation

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

In practice we have points in image coordinate, i.e. pixel values.

The essential matrix operates on image points expressed in **2D coordinates** in the camera coordinate system.

$$\hat{\boldsymbol{x}}' = \mathbf{K}'^{-1}\boldsymbol{x}'$$

$$\underbrace{\hat{\boldsymbol{x}}}_{\text{camera point}} = \mathbf{K}^{-1}\underbrace{\boldsymbol{x}}_{\text{image point}}$$

Writing out the epipolar constraint in terms of image coordinates

$$\boldsymbol{x}'^{\top}(\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1})\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

Fundamental Matrix

Properties of the E matrix

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times} \qquad \mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1} \qquad \mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_\times]\mathbf{R}\mathbf{K}^{-1}$$

Longuet-Higgins equation $\boldsymbol{x}'^\top \mathbf{E} \boldsymbol{x} = 0$

Epipolar lines

$\boldsymbol{x}^\top \boldsymbol{l} = 0$ $\qquad$ $\boldsymbol{x}'^\top \boldsymbol{l}' = 0$

$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$ $\qquad$ $\boldsymbol{l} = \mathbf{E}^T \boldsymbol{x}'$

Epipoles $\boldsymbol{e}'^\top \mathbf{E} = \mathbf{0}$ $\qquad$ $\mathbf{E}\boldsymbol{e} = \mathbf{0}$

(2D points expressed in image coordinate system)

Properties of the F matrix

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times} \qquad \mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1} \qquad \mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_\times]\mathbf{R}\mathbf{K}^{-1}$$

- F has 7 degrees of freedom, why?
  - F is 3x3, has 8 degrees of freedom, since it is a projective transformation.
  - F is rank 2. So 1 less degree of freedom.

- F is rank 2, why?
  - Same reason as E
  - $[t_x]$ is skew symmetric, hence rank 2.

- F has 2 singular value both of which are ~~equal.~~

Essential Matrix vs Homography

What's the difference between the essential matrix and a homography?

They are both 3 x 3 matrices but …

$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$$

Essential matrix maps a **point** to a **line**

- Rank 2
- 5 DoF

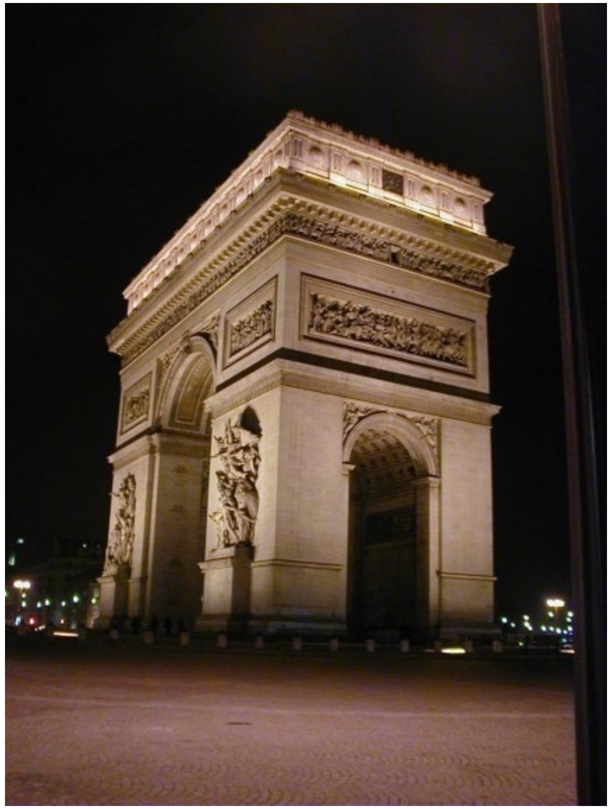
$$\boldsymbol{x}' = \mathbf{H}\boldsymbol{x}$$

Homography maps a **point** to a **point**

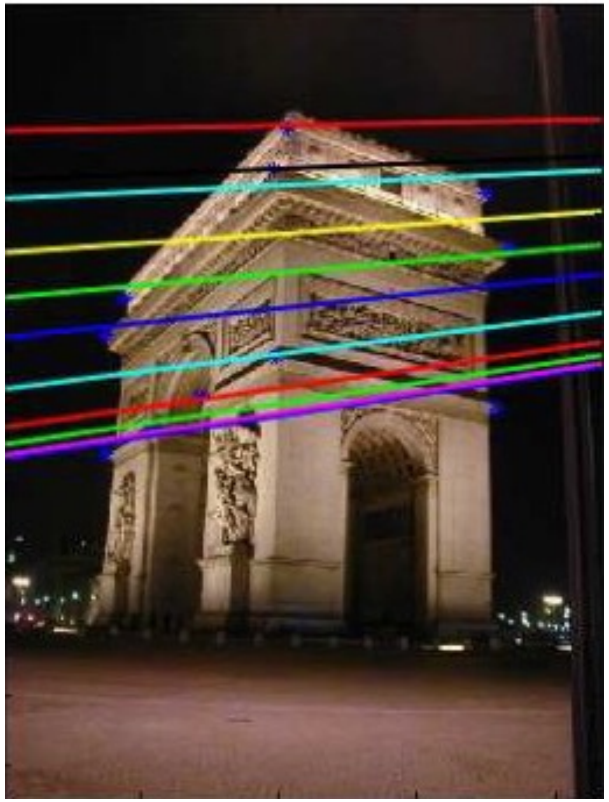
- Rank 3
- 8 DoF

Homography is a special case of the Essential/Fundamental matrix, for planar scenes

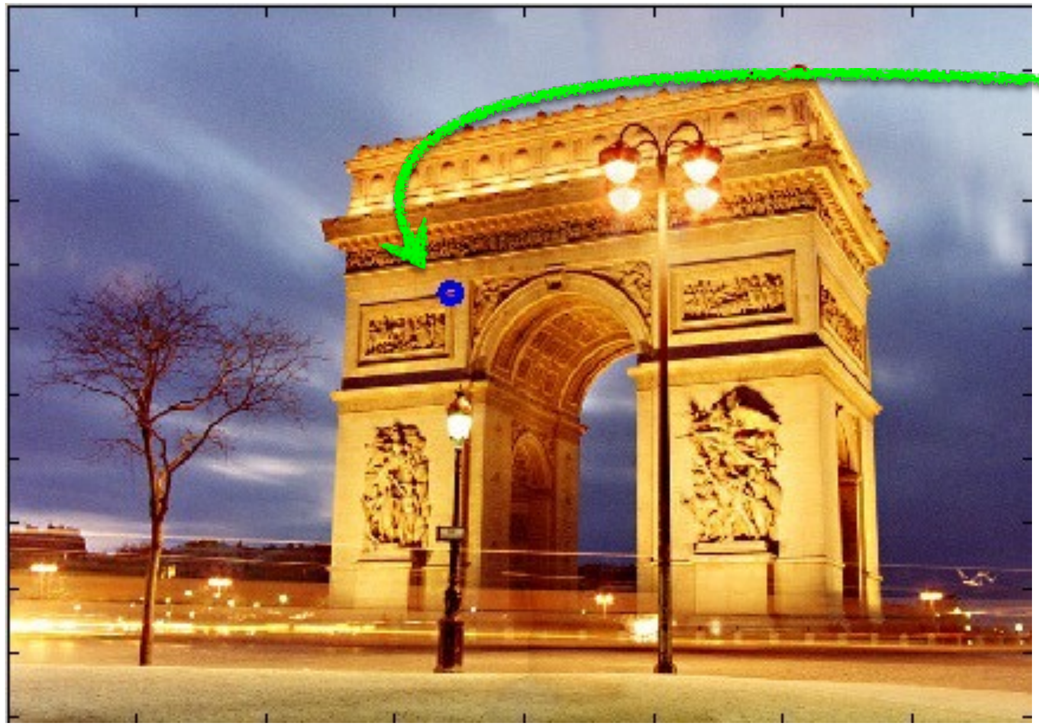
Example

epipolar lines

$$\mathbf{F} = \begin{bmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{bmatrix}$$

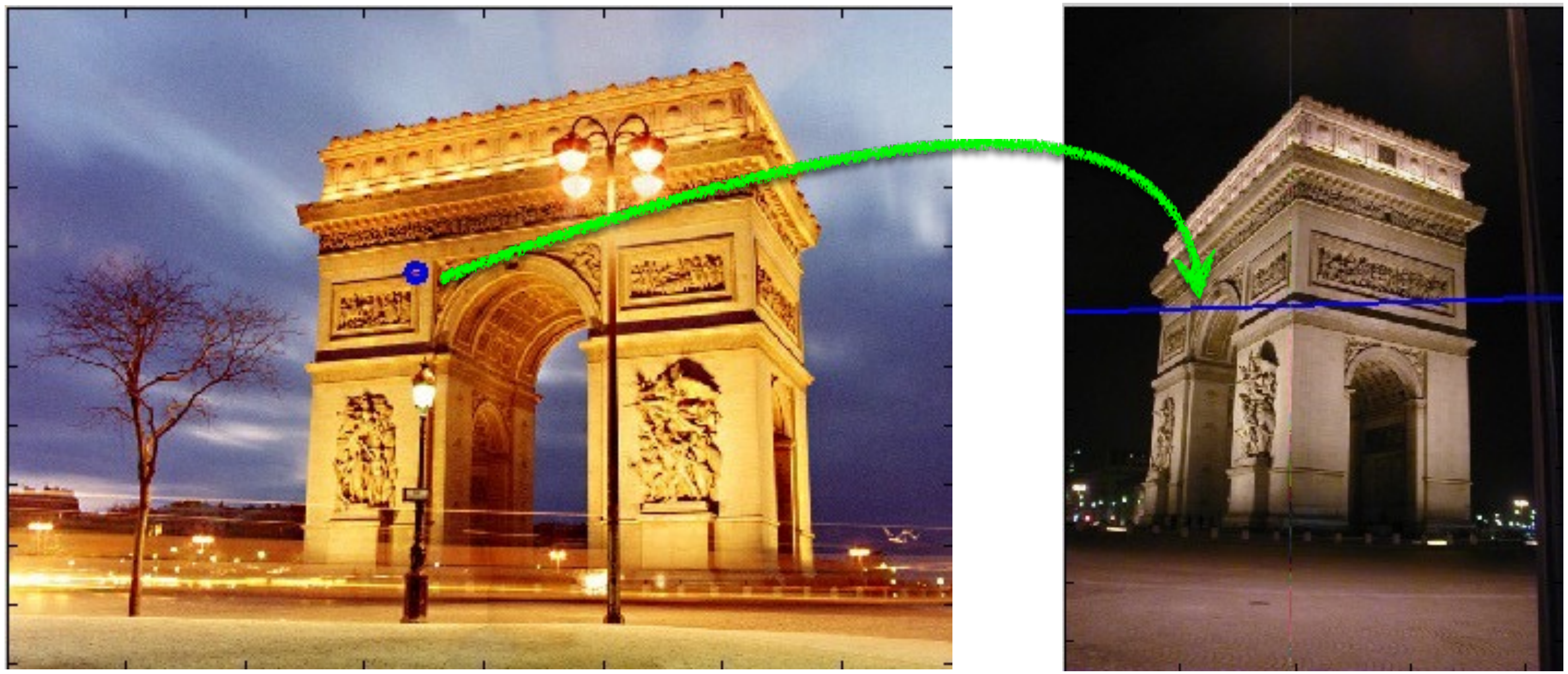


$$\boldsymbol{x} = \begin{bmatrix} 343.53 \\ 221.70 \\ 1.0 \end{bmatrix}$$

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$

$$
\begin{aligned}
\boldsymbol{l}' &= \mathbf{F}\boldsymbol{x} \\
&= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}
\end{aligned}
$$



Where is the epipole?

How would you compute it?

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

The epipole is in the right null space of **F**

How would you solve for the epipole?

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

The epipole is in the right null space of **F**

How would you solve for the epipole?

SVD!

SVDs are pretty useful, huh?

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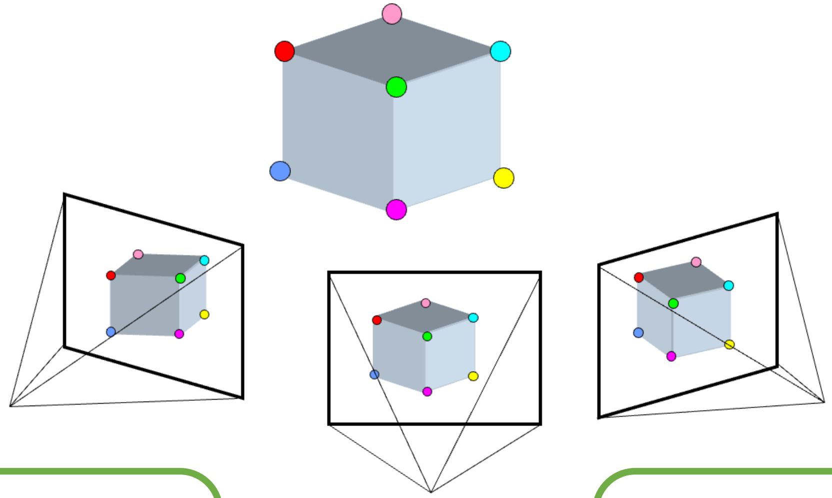
Big picture: 3 key components in 3D

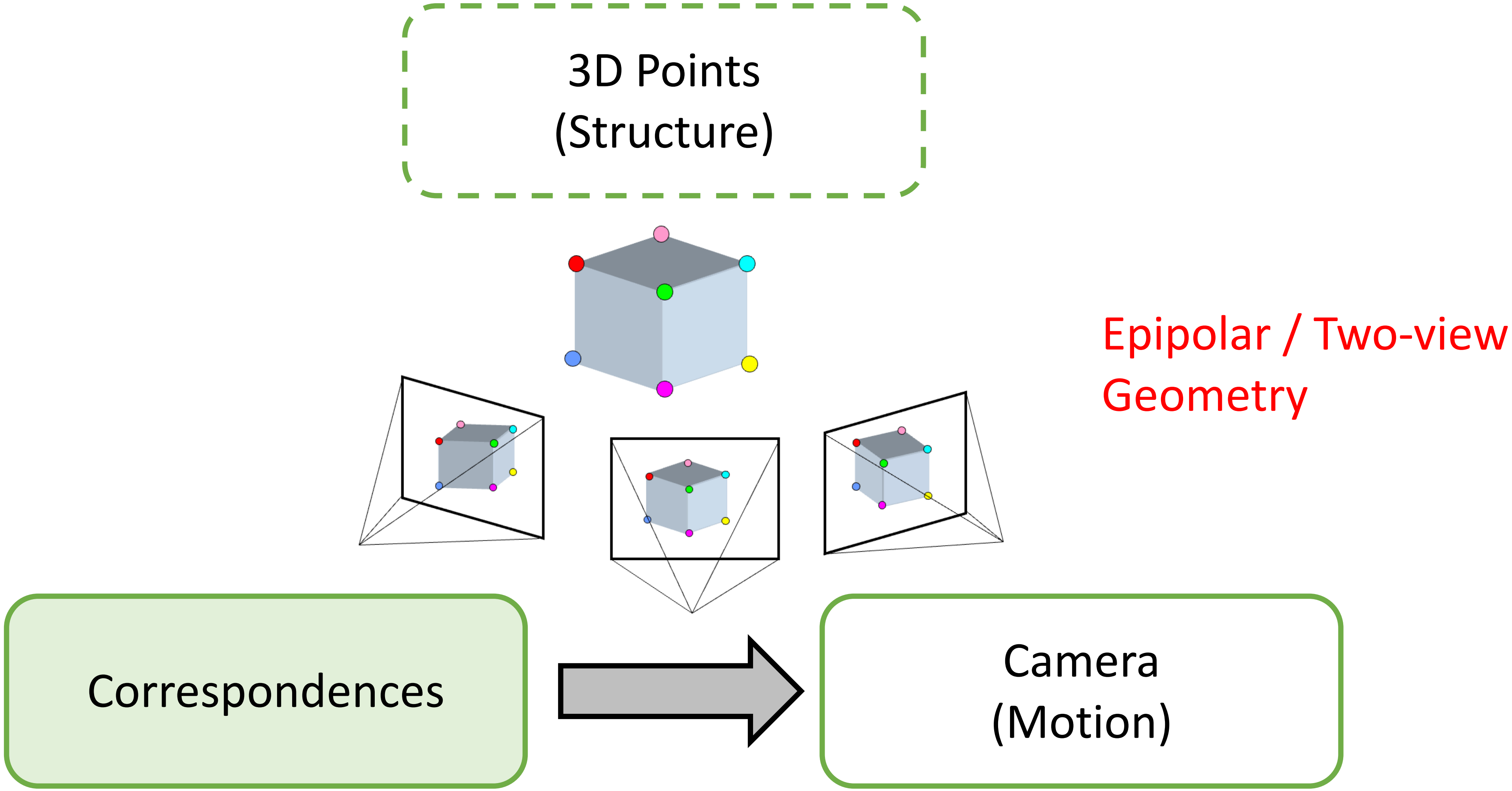
3D Points
(Structure)



Correspondences

Camera
(Motion)

Big picture: 3 key components in 3D



Estimating the fundamental matrix

Assume you have *M* matched *image* points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \qquad m = 1, \ldots, M$$

Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

How would you solve for the 3 x 3 ***F*** *matrix?*

Solve with SVD!

Set up a homogeneous linear system with 9 unknowns

$$\boldsymbol{x}_m'^{\top} \mathbf{F} \boldsymbol{x}_m = 0$$

$$\begin{bmatrix} x_m' & y_m' & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

How many equation do you get from one correspondence?

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

ONE correspondence gives you ONE equation

$$\begin{aligned} x_m x'_m f_1 + x_m y'_m f_2 + x_m f_3 + \\ y_m x'_m f_4 + y_m y'_m f_5 + y_m f_6 + \\ x'_m f_7 + y'_m f_8 + f_9 = 0 \end{aligned}$$

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

Set up a homogeneous linear system with 9 unknowns

Hence, the 8 point algorithm!

$$\begin{bmatrix} x_1x'_1 & x_1y'_1 & x_1 & y_1x'_1 & y_1y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_Mx'_M & x_My'_M & x_M & y_Mx'_M & y_My'_M & y_M & x'_M & y'_M & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{bmatrix} = \mathbf{0}$$

Note: This is different from the Homography estimation where each point pair contributes 2 equations.

We need at least 8 points

How many equations do you need?

How do you solve a homogeneous linear system?

$$\mathbf{A}\mathbf{X} = \mathbf{0}$$

8 x9 9x1

Total Least Squares

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

SVD!

Problem with eight-point algorithm

$$\begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & u & v \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -1$$

Problem with eight-point algorithm

250906.36	183269.57	921.81	200931.10	146766.13	738.21	272.19	198.81
2692.28	131633.03	176.27	6196.73	302975.59	405.71	15.27	746.79
416374.23	871684.30	935.47	408110.89	854384.92	916.90	445.10	931.81
191183.60	171759.40	410.27	416435.62	374125.90	893.65	465.99	418.65
48988.86	30401.76	57.89	298604.57	185309.58	352.87	846.22	525.15
164786.04	546559.67	813.17	1998.37	6628.15	9.86	202.65	672.14
116407.01	2727.75	138.89	169941.27	3982.21	202.77	838.12	19.64
135384.58	75411.13	198.72	411350.03	229127.78	603.79	681.28	379.48

$$\begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -1$$

- Poor numerical conditioning
- Can be fixed by rescaling the data

Problem with 8-point algorithm

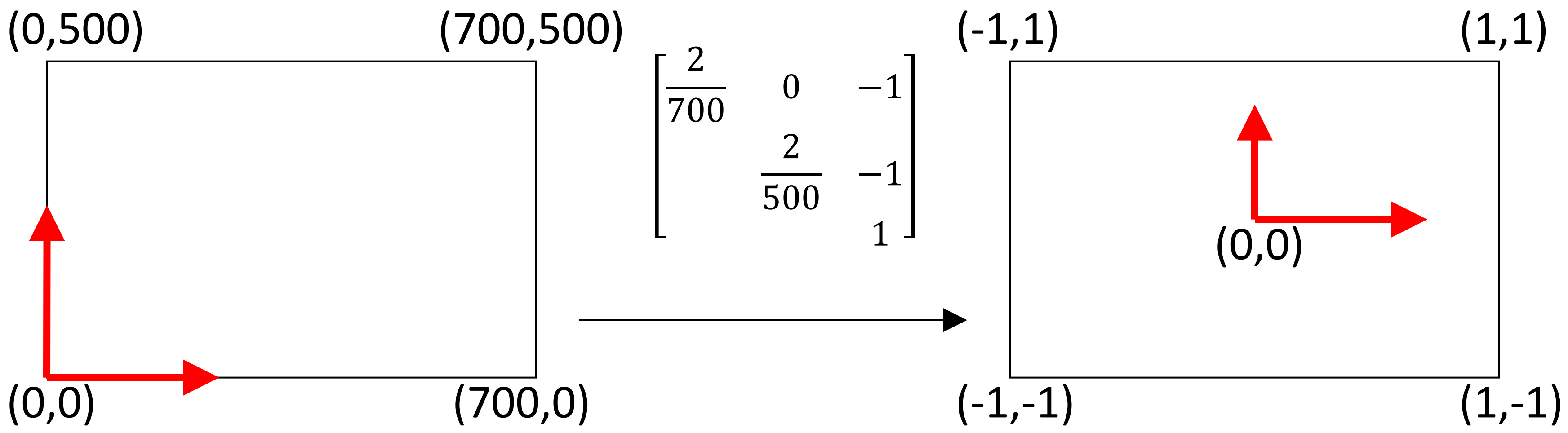
$$
\begin{array}{c}
\begin{bmatrix}
u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\
u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
u_nu_n' & v_nu_n' & u_n' & u_nv_n' & v_nv_n' & v_n' & u_n & v_n & 1
\end{bmatrix} \\
\begin{matrix} \sim 10000 & \sim 10000 & \sim 100 & \sim 10000 & \sim 10000 & \sim 100 & \sim 100 & \sim 100 & 1 \end{matrix}
\end{array}
\begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0
$$

Orders of magnitude difference
between column of data matrix
$\rightarrow$ least-squares yields poor results

Normalized 8-point algorithm

normalized least squares yields good results

Transform image to ~[-1,1]



$$\begin{bmatrix} \frac{2}{700} & 0 & -1 \\ & \frac{2}{500} & -1 \\ & & 1 \end{bmatrix}$$

Normalized 8-point algorithm

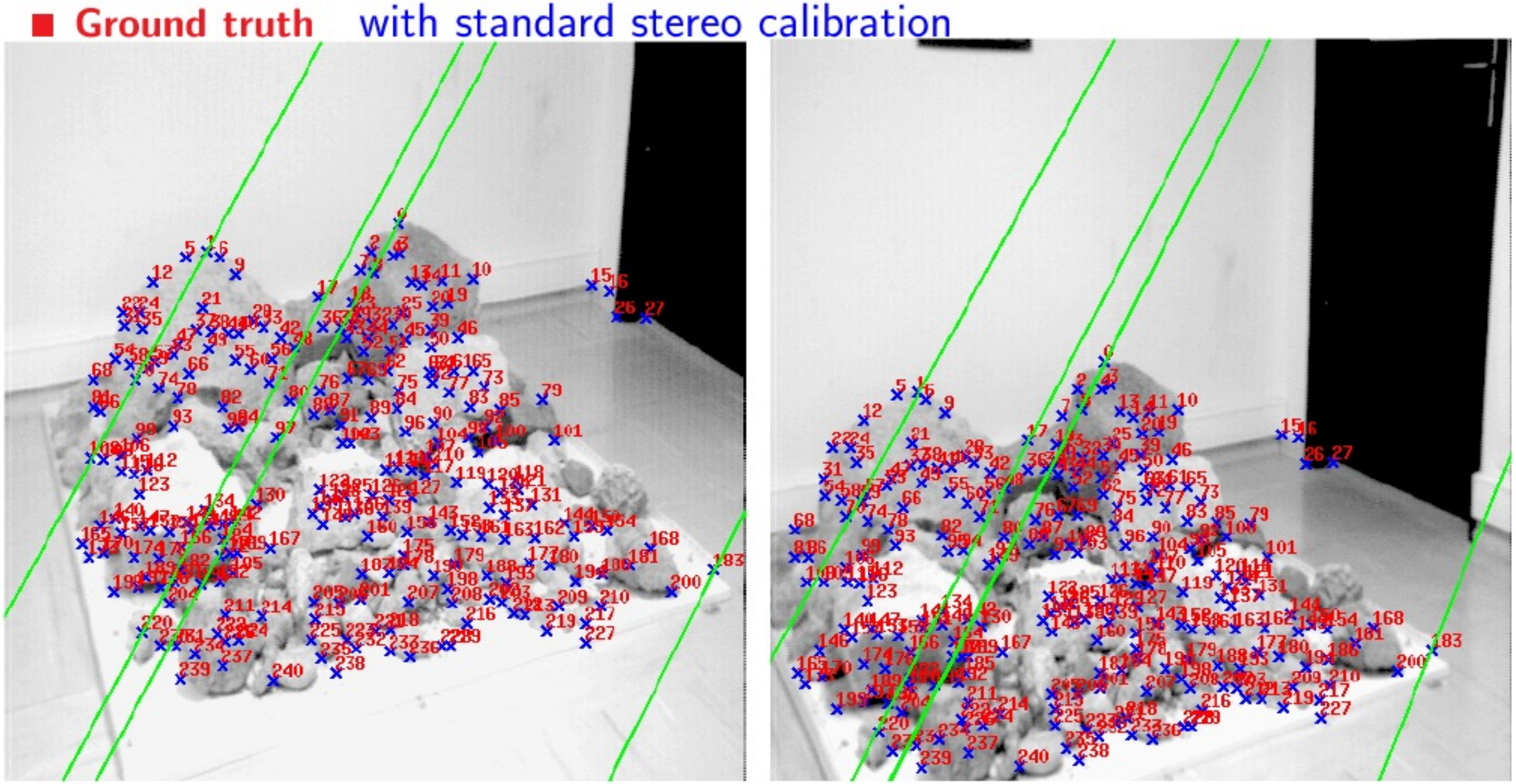
- Transform input by $\mathbf{\hat{x}_i} = \mathbf{Tx_i}$, $\mathbf{\hat{x}_i'} = \mathbf{Tx_i'}$
- Call 8-point on $\mathbf{\hat{x}_i}, \mathbf{\hat{x}_i'}$ to obtain $\mathbf{\hat{F}}$
- $\mathbf{F} = \mathbf{T}'^{\mathrm{T}}\mathbf{\hat{F}}\mathbf{T}$

$$\mathbf{x}'^{\mathrm{T}}\mathbf{F}\mathbf{x} = 0$$

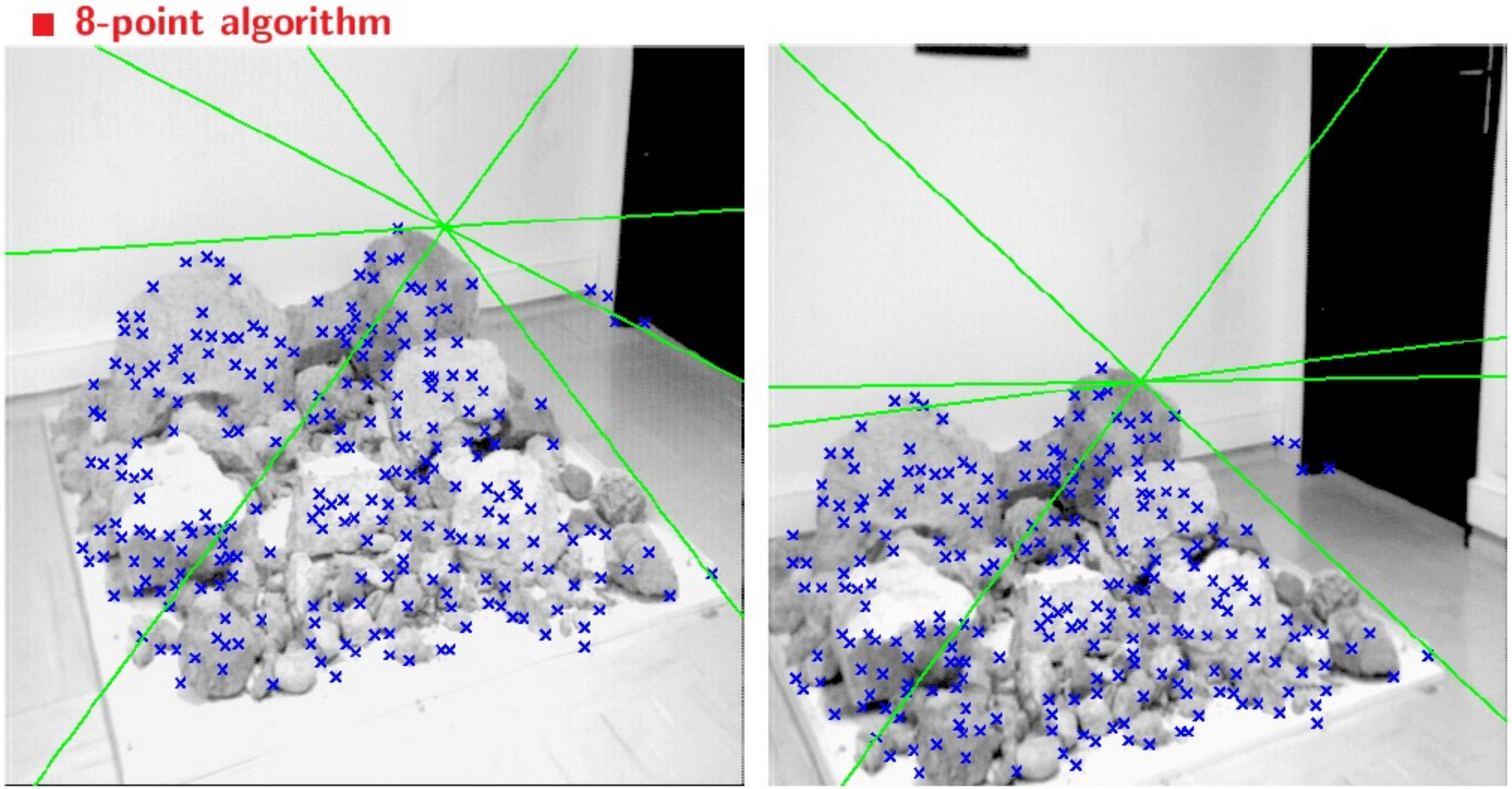
$$\underbrace{\boxed{\mathbf{\hat{x}}'^{\mathrm{T}}\mathbf{T}'^{-\mathrm{T}}}\,\mathbf{F}\,\boxed{\mathbf{T}^{-1}\mathbf{\hat{x}}}}_{\mathbf{\hat{F}}} = 0$$

Fundamental matrix of normalized camera coordinate

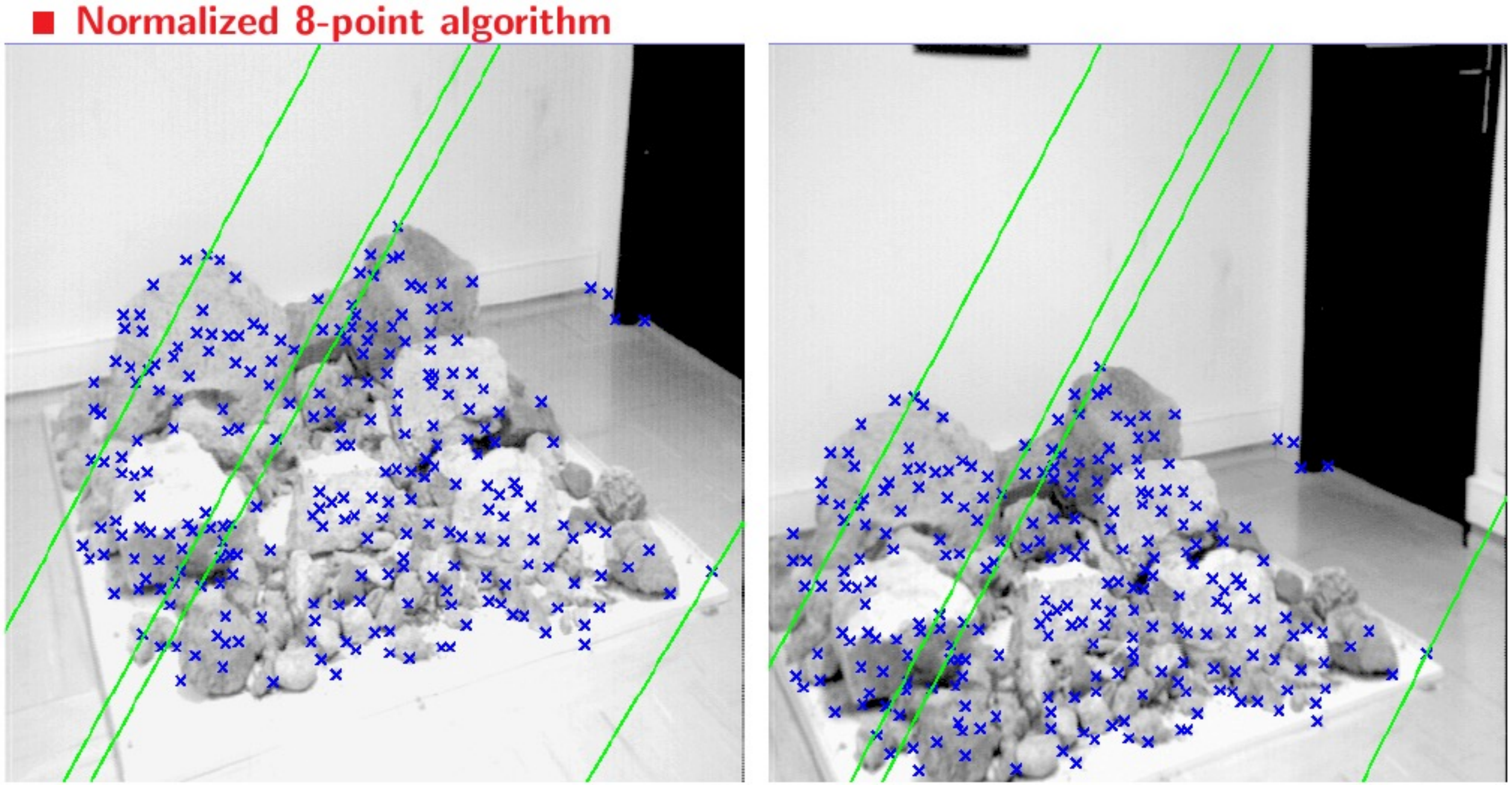
Results (ground truth)



Results (8 point algorithm)



Results (normalized 8-point algorithm)



Enforcing rank constraints

Problem: Given a matrix F, find the matrix F' of rank k that is closest to F,

$$\min_{\substack{F' \\ \text{rank}(F')=k}} \quad ||F - F'||^2$$

Solution: Compute the singular value decomposition of F,

$$F = U\Sigma V^T$$

Form a matrix Σ' by replacing all but the k largest singular values in Σ with 0.

Then the problem solution is the matrix F' formed as,

$$F' = U\Sigma' V^T$$

(Normalized) Eight-Point Algorithm

1. (Normalize points)
2. Construct the M x 9 matrix **A**
3. Find the SVD of **A**
4. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

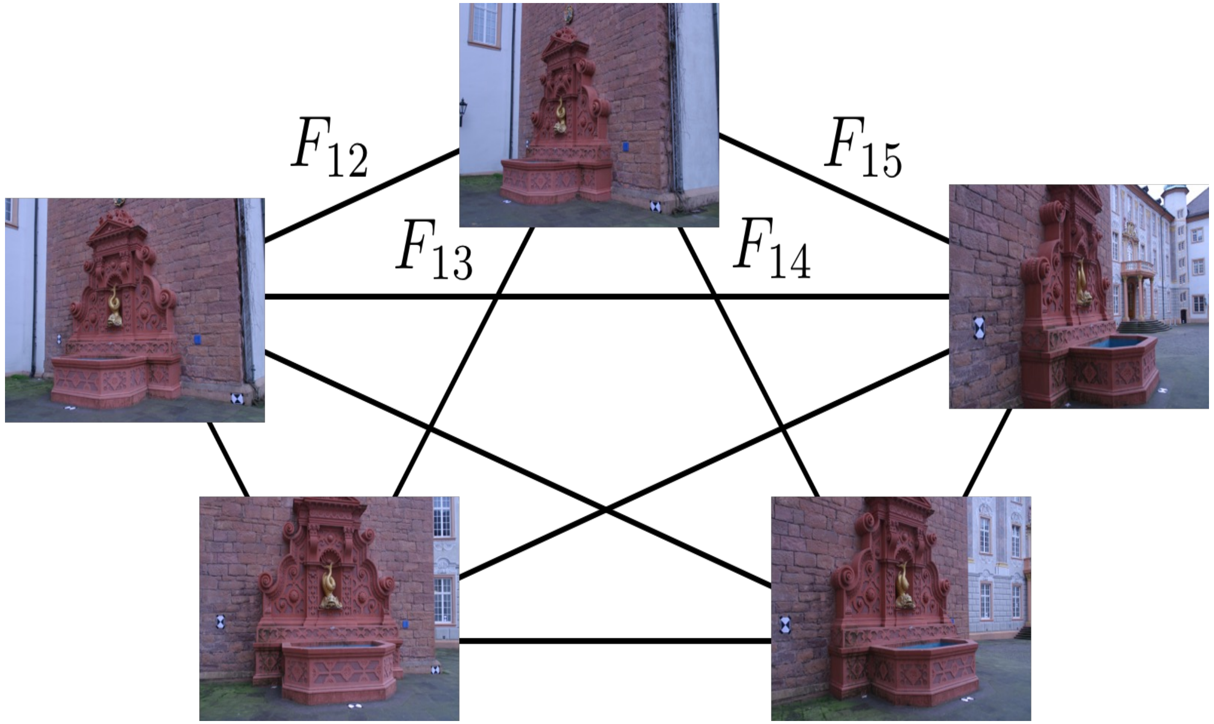
Fundamental -> Essential -> Rotation + Translation

- From normalized 8-pt algorithm we have F, s.t. rank(F)=2.
- Recover intrinsic camera matrix K and K' (find focal length of 2 cameras, often comes as a part of meta data).
- Recover Essential matrix E from $\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}$
- An ideal E is rank(2) and has 2 singular values that are equal, and is upto a scale.
  - An ideal E will have SVD E=U diag(1,1,0) V^T.
  - Project estimated E such that 2 singular values are 1.
- Decompose Essential matrix to obtain Rotation and Translation $\mathbf{E} = [\tilde{\mathbf{t}}]_\times \mathbf{R}$
  - 4 possible solutions -> only 1 case where reconstructed 3D pt is in front of both cameras.
  - See Results 9.18 & 9.19, pg 258-259 for the proof.

What about more than two views?

- The geometry of three views is described by a 3 x 3 x 3 tensor called the *trifocal tensor*
- The geometry of four views is described by a 3 x 3 x 3 x 3 tensor called the *quadrifocal tensor*
- After this it starts to get complicated…

“A New Rank Constraint on Multi-view Fundamental Matrices, and its Application to Camera Location Recovery”, Sengupta et. al. CVPR 2017.



Necessary but not sufficient

$$F = \begin{bmatrix} \mathbf{0} & F_{12} & F_{13} & F_{14} & F_{15} \\ F_{21} & \mathbf{0} & F_{23} & F_{24} & F_{25} \\ F_{31} & F_{32} & \mathbf{0} & F_{34} & F_{35} \\ F_{41} & F_{42} & F_{43} & \mathbf{0} & F_{45} \\ F_{51} & F_{52} & F_{53} & F_{54} & \mathbf{0} \end{bmatrix}$$

with $F = A + A^T$,
$rank(A) = 3$ and $rank(F) = 6$.

In case of all collinear cameras : $rank(A) \leq 2$ and $rank(F) \leq 4$

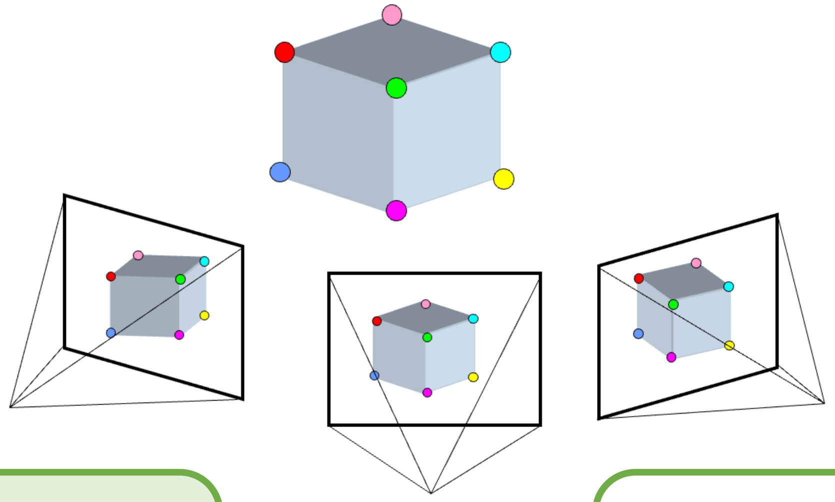
The Fundamental Matrix Song

Today’s class

- Epipolar Geometry
- Essential Matrix
- Fundamental Matrix
- 8-point Algorithm
- Triangulation

Big picture: 3 key components in 3D

3D Points
(Structure)



Estimate
Fundamental matrix

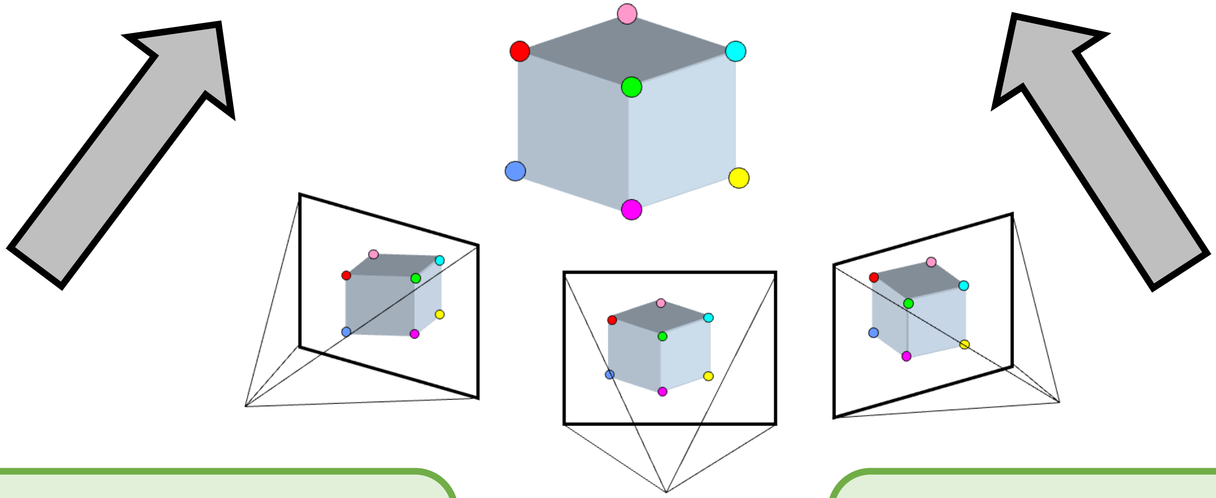
Correspondences

Camera
(Motion)

Big picture: 3 key components in 3D

3D Points
(Structure)

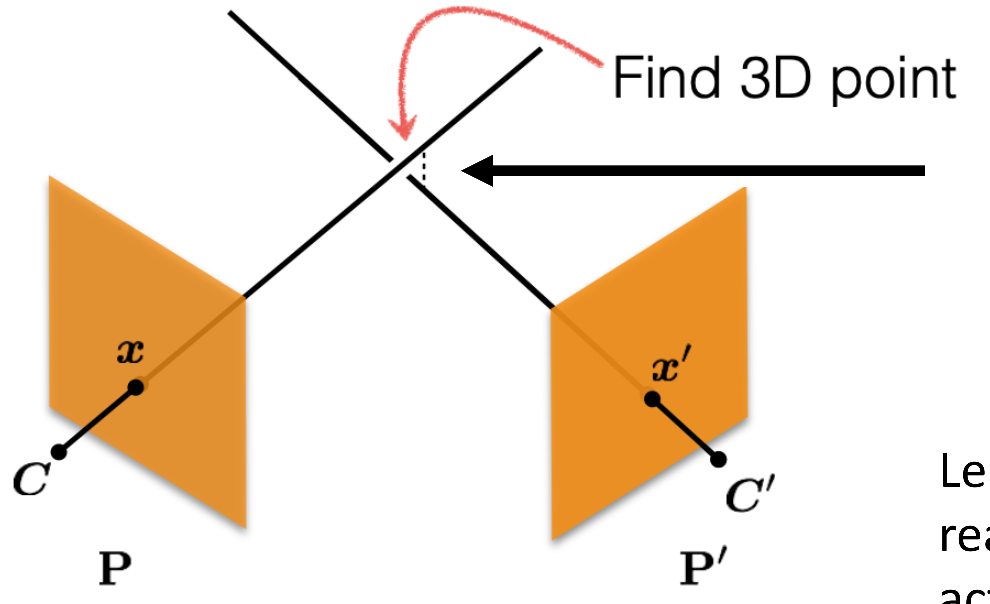
Triangulation



Correspondences

Camera
(Motion)

Triangulation Disclaimer: Noise



$$\mathbf{X} \text{ s.t.}$$

$$\mathbf{x} = \mathbf{P}\mathbf{X}, \ \mathbf{x}' = \mathbf{P}'\mathbf{X}$$

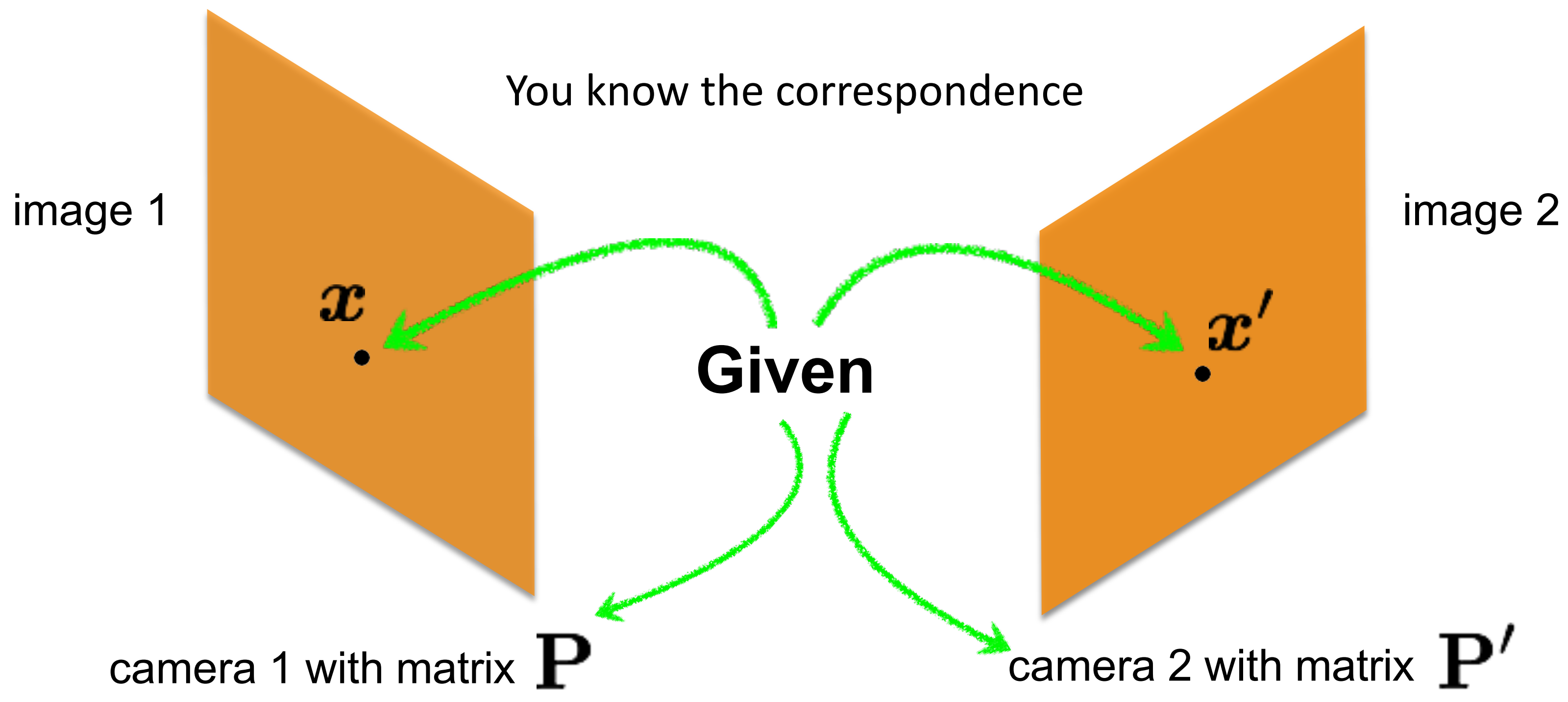
Ray's don't always intersect because of noise!!!

Least squares get you to a reasonable solution but it's not the actual geometric error (it's how far away the solution is from Ax = 0)

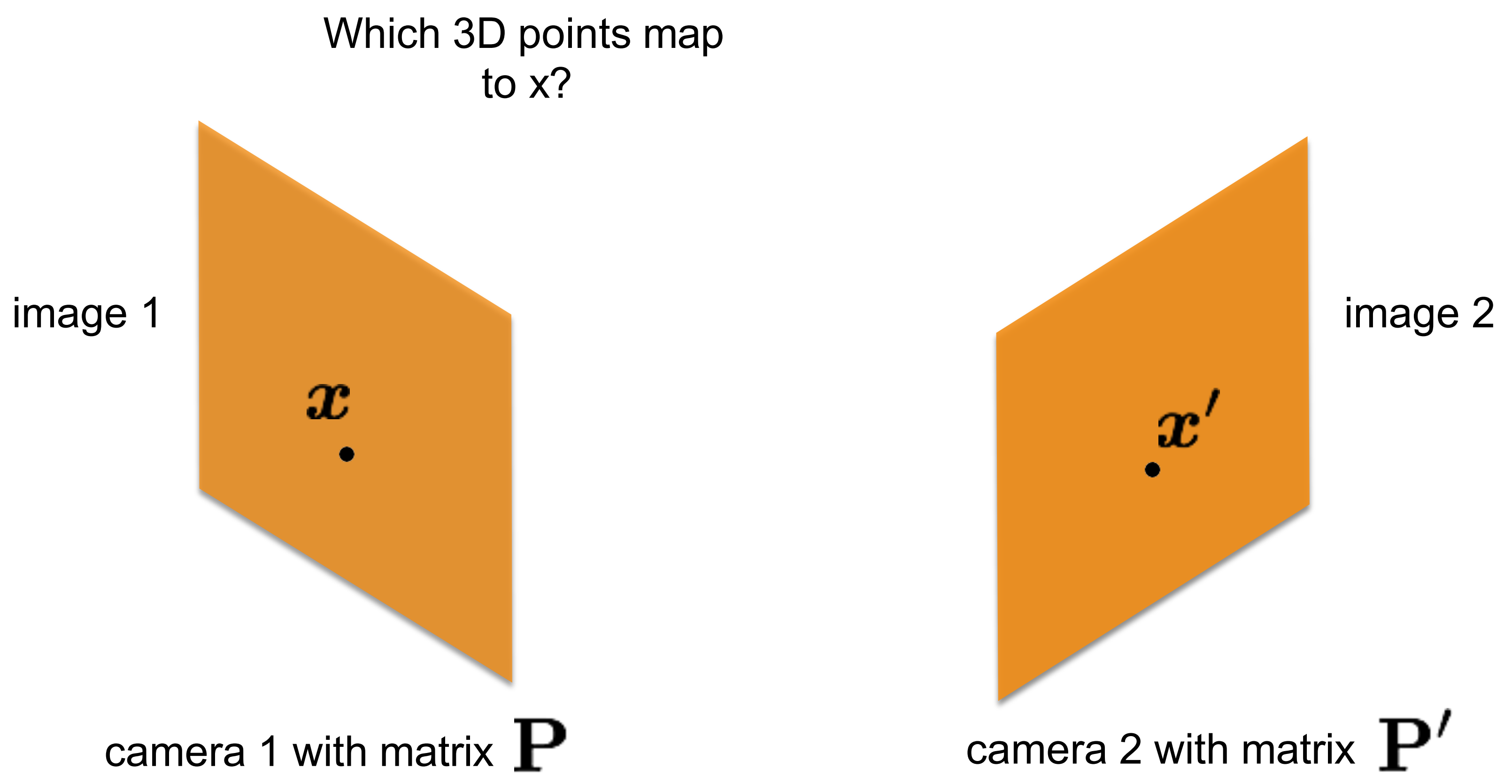
In practice with noise, you do non-linear least squares, or "bundle adjustment" (more than 2 image case, next lecture..)

Slide credit: Shubham Tulsiani

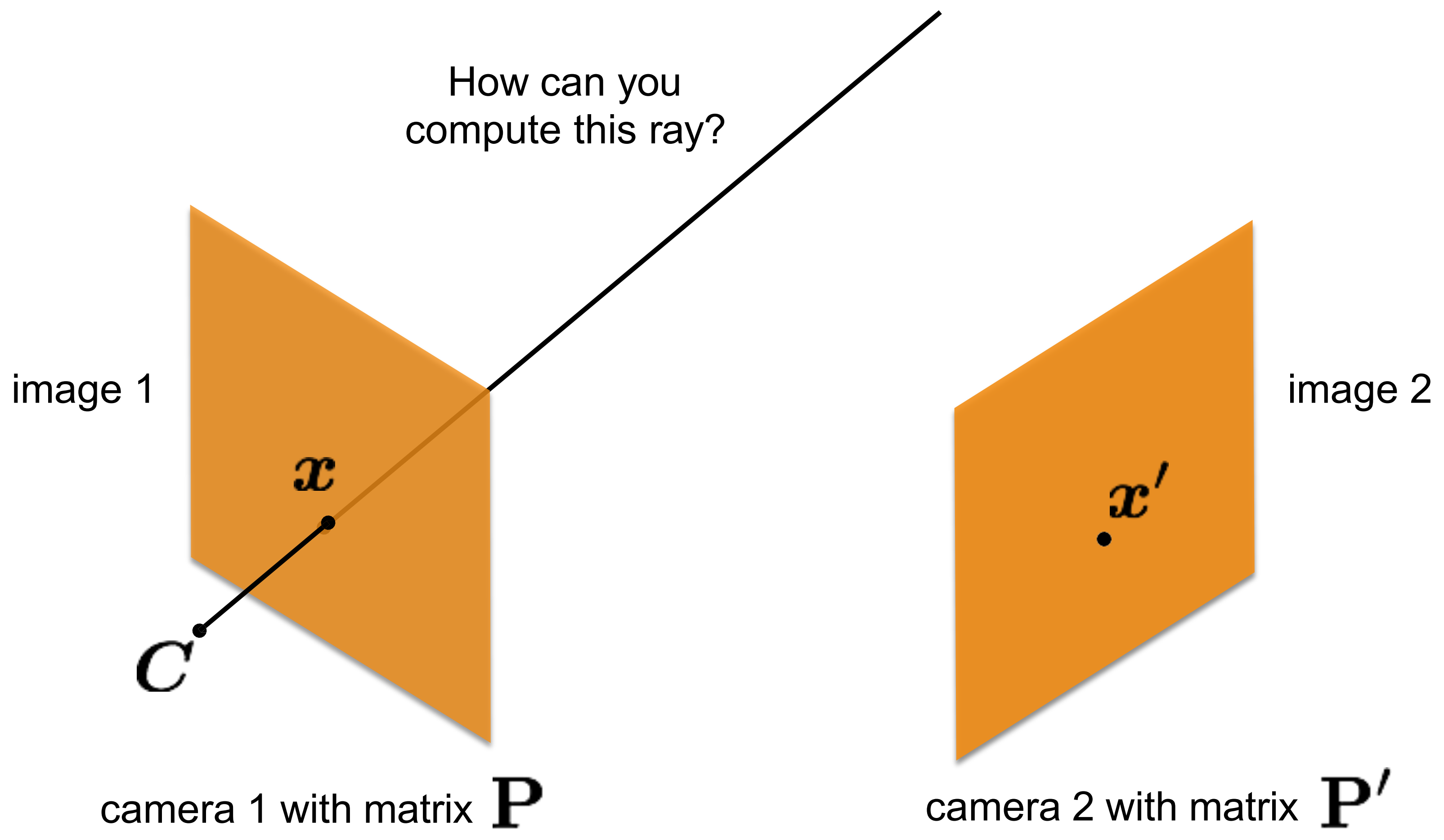
Triangulation



Triangulation



Triangulation



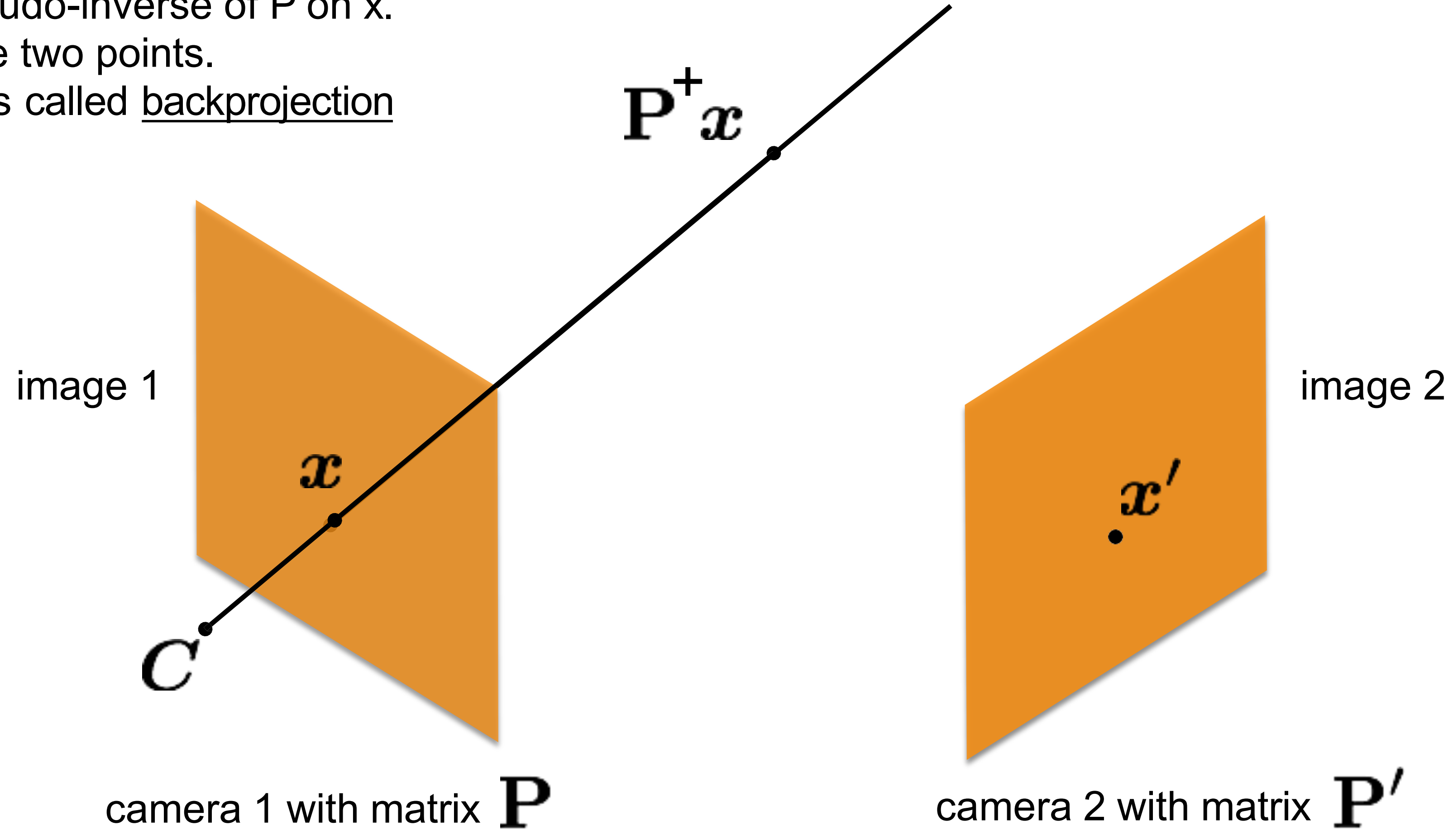
Triangulation

Create two points on the ray:

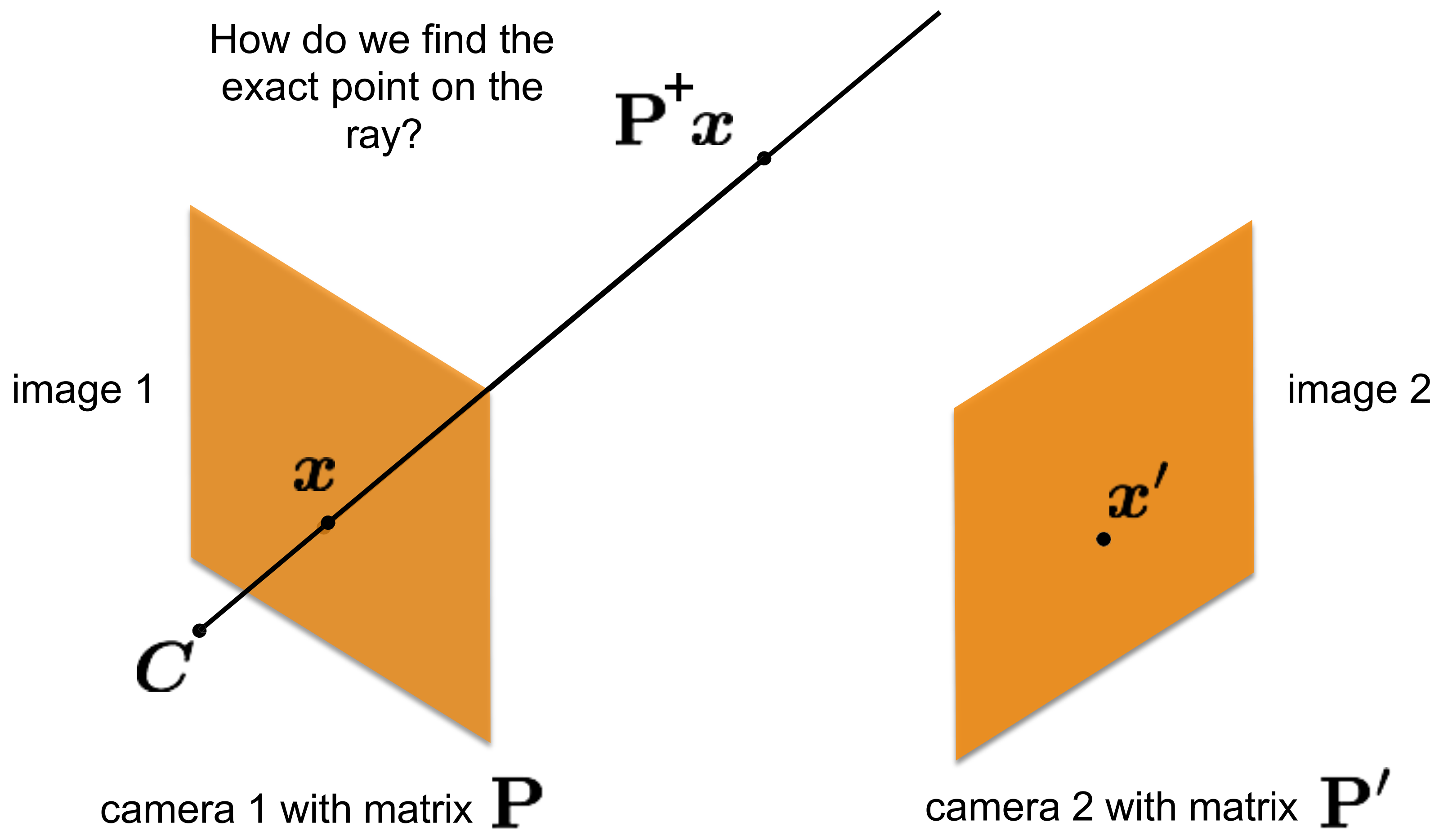
1) find the camera center; and
2) apply the pseudo-inverse of P on x.

Then connect the two points.

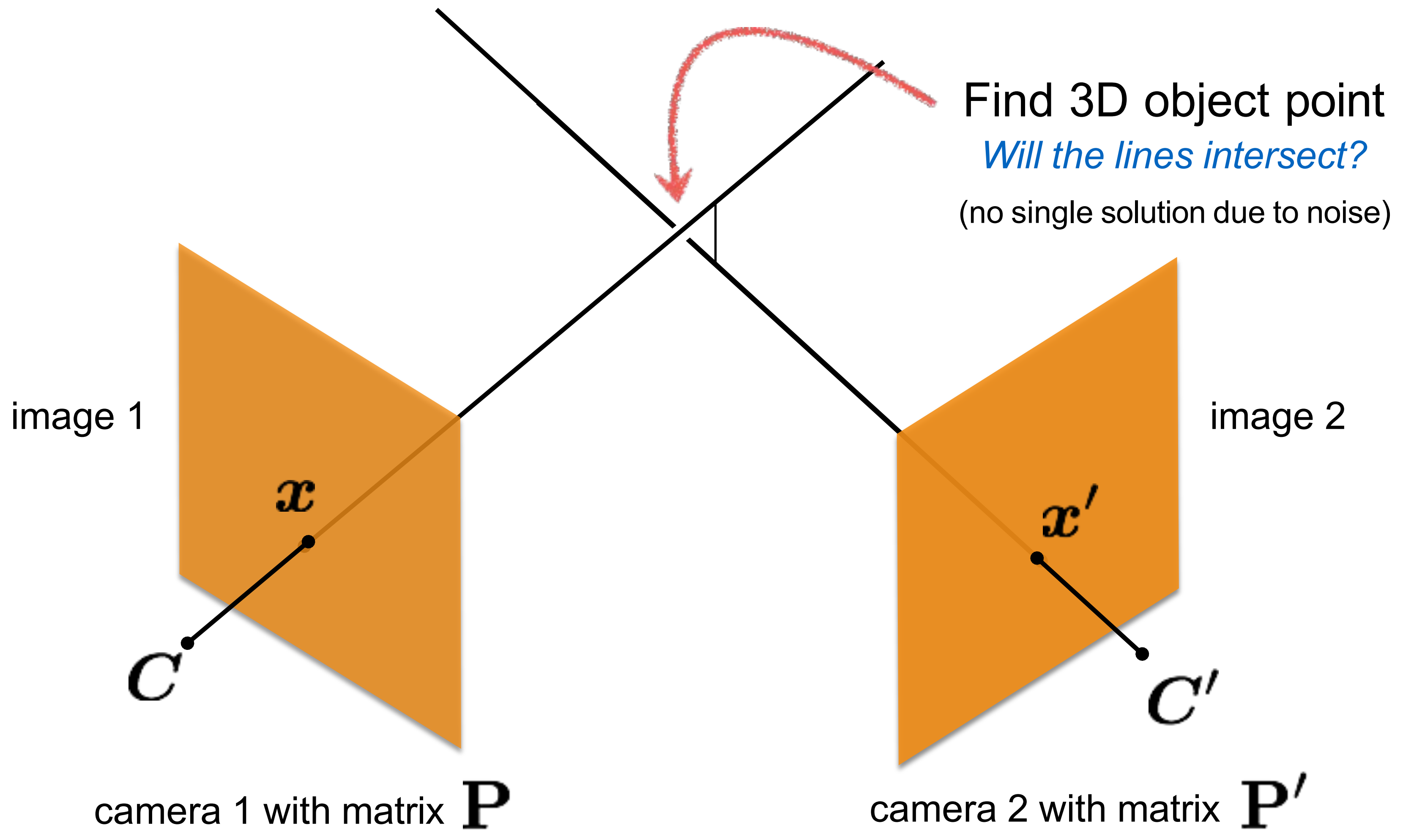
This procedure is called <u>backprojection</u>



Triangulation



Triangulation



Triangulation

Given a set of (noisy) matched points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

and camera matrices

$$\mathbf{P}, \mathbf{P}'$$

Estimate the 3D point

$$\mathbf{X}$$

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

This is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(heterogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

$$\mathbf{x} = \alpha \mathbf{P} \boldsymbol{X}$$

Same direction but differs by a scale factor

$$\mathbf{x} \times \mathbf{P} \boldsymbol{X} = \mathbf{0}$$

Cross product of two vectors of same direction is zero
(this equality removes the scale factor)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Do the same after first expanding out the camera matrix and points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} — & \boldsymbol{p}_1^\top & — \\ — & \boldsymbol{p}_2^\top & — \\ — & \boldsymbol{p}_3^\top & — \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you 2 equations

$$\begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Remove third row, and rearrange as system on unknowns

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}_i \boldsymbol{X} = \mathbf{0}$$

Now we can make a system of linear equations (two lines for each 2D point correspondence)

Concatenate the 2D points from both images

Two rows from camera one

Two rows from camera two

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

sanity check! dimensions?

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

SVD!

Slide Credits

- CS5670, Introduction to Computer Vision, **Cornell Tech, by Noah Snavely.**
- CS 194-26/294-26: Intro to Computer Vision and Computational Photography, **UC Berkeley, by Angjoo Kanazawa.**
- CS 16-385: Computer Vision, **CMU, by Matthew O'Toole**

Additional Reading

- Multiview Geometry, Hartley & Zisserman,
  - Chapter 9 (focus on topics discussed or mentioned in the slides).
  - Chapter 10.1, 10.2 (not discussed in class, no midterm ques, but imp to understand, practical importance.)
  - Chapter 11.1, 11.2
  - Chapter 12.1, 12.2, 12.3, 12.4 (no midterm ques, but imp to understand)